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Hermite–Hadamard type inequalities for (α, m)-HA and strongly (α, m)-HA convex functions

Chun-Ying He[a], Yan Wang[a], Bo-Yan Xi[a,*], Feng Qi[b,c]

[a] College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, China.
[b] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China.
[c] Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China.

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Abstract

In the paper, the authors define the concepts of (α, m)-harmonic-arithmetically convex functions and strongly (α, m)-harmonic-arithmetically convex functions, establish a new integral identity, and present some new Hermite–Hadamard type inequalities for (α, m)-harmonic-arithmetically convex functions and strongly (α, m)-harmonic-arithmetically convex functions.

Keywords: Strongly (α, m)-harmonic-arithmetically convex function, (α, m)-harmonic-arithmetically convex function, integral identity, Hermite–Hadamard type integral inequality.

2010 MSC: 26A51, 26D15, 41A55.

1. Introduction

The following definitions are well-known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex function if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2 ([15]). For $f : [0, b] \to \mathbb{R}$, $b > 0$, and $m \in (0, 1]$, if

$$f(tx + m(1-t)y) \leqslant tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m-convex function on $[0, b]$.

Definition 1.3 ([8]). For $f : [0, b] \to \mathbb{R}$, $b > 0$, and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1-t)y) \leqslant t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

*Corresponding author
Email addresses: hechunying9209@qq.com (Chun-Ying He), sella110@vip.qq.com (Yan Wang), baoyintu78@qq.com (Bo-Yan Xi), qifeng618@gmail.com, qifeng618@hotmail.com (Feng Qi)

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Definition 1.4 ([12]). For $f : [a, b] \to \mathbb{R}$, if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

is valid for all $x, y \in [a, b]$, $t \in [0, 1]$, and $c \geqslant 0$, then we say that $f(x)$ is a strongly convex function on $[a, b]$.

Definition 1.5 ([18]). Let $f : (0, b] \to \mathbb{R}$ and $m \in (0, 1]$ be a constant. If

$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \leqslant tf(x) + m(1-t)f(y), \tag{1.1}$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$, then f is said to be an m-harmonic-arithmetically convex function or, simply speaking, an m-HA-convex function. If the inequality (1.1) reverses, then f is said to be an m-harmonic-arithmetically concave function or, simply speaking, an m-HA-concave function.

Study of convex functions and the Hermite–Hadamard type integral inequalities have always been a very active research topic. We recall the following results.

Theorem 1.6 ([3, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.$$

Theorem 1.7 ([11, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ and $q \geqslant 1$, then*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}.$$

Theorem 1.8 ([4]). *Let $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be m-convex and $m \in (0, 1]$. If $f \in L_1([a, b])$ for $0 \leqslant a < b < \infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \min\left\{\frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2}\right\}.$$

Theorem 1.9 ([2, Theorem 2.2]). *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. If $0 \leqslant a < b < \infty$ and $f \in L_1([a, b])$, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \frac{f(x) + mf(x/m)}{2}\,dx \leqslant \frac{m+1}{4}\left[\frac{f(a) + f(b)}{2} + m\frac{f(a/m) + f(b/m)}{2}\right].$$

Theorem 1.10 ([7, Theorem 3.1]). *Let $I \supseteq \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \leqslant a < b < \infty$. If $|f'|^q$ is (α, m)-convex on $[a, b]$ for some given numbers $m, \alpha \in (0, 1]$, and $q \geqslant 1$, then*

$$\begin{aligned}&\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leqslant \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\min\left\{\left[v_1|f'(a)|^q + v_2 m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}, \left[v_2 m\left|f'\left(\frac{a}{m}\right)\right|^q + v_1|f'(b)|^q\right]^{1/q}\right\},\end{aligned}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right) \quad \text{and} \quad v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2 + \alpha + 2}{2} - \frac{1}{2^\alpha}\right).$$

For more results in this topic, please refer to the papers [1, 5, 6, 13, 14, 16, 17, 19, 20] and closely-related references therein.

The main purpose of this paper is to introduce the concept of "(α, m)-HA-convex functions" and "strongly (α, m)-HA-convex functions" and to establish some new Hermite–Hadamard type inequalities for these classes of functions.

2. Two definitions and a lemma

Now we introduce the concept of (α, m)-HA-convex functions and strongly (α, m)-HA-convex functions.

Definition 2.1. For $f : (0, b^*] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1]^2$, a function f is said to be (α, m)-HA-convex on I, if

$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \leqslant t^\alpha f(x) + m(1 - t^\alpha) f(y),$$

for all $x, y \in (0, b^*]$ and $t \in [0, 1]$.

Definition 2.2. For $f : (0, b^*] \to \mathbb{R}$, $(\alpha, m) \in (0, 1]^2$, and $c \geqslant 0$, a function f is said to be strongly (α, m)-HA-convex on I, if

$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) \leqslant t^\alpha f(x) + m(1 - t^\alpha) f(y) - ct(1-t)\left(x^{-1} - y^{-1}\right)^2,$$

for all $x, y \in (0, b^*]$ and $t \in [0, 1]$.

Remark 2.3. Let $f(x) = \frac{1}{x^2}$ for $x \in \mathbb{R}_+ = (0, \infty)$ and let $m = \alpha = 0.3$, $c = 0.05$. Then

$$f\left(\left(\frac{t}{x} + m\frac{1-t}{y}\right)^{-1}\right) = \frac{[ty + m(1-t)x]^2}{(xy)^2} \leqslant \frac{ty^2 + (1-t)(mx)^2}{(xy)^2}$$

and

$$t^\alpha y^2 + m(1 - t^\alpha)x^2 - ct(1-t)(y-x)^2 - (ty^2 + (1-t)(mx)^2) \geqslant 0,$$

for all $x, y > 0$ and $t \in [0, 1]$. So f is a strongly $(0.3, 0.3)$-HA-convex function on $\mathbb{R}_+$.

Remark 2.4. When $\alpha = 1$ and $m = 1$, the above Definition 2.2 becomes [10, Definition 1.2] which should be modified slightly in order that $tx + (1-t)y \neq 0$ for all $t \in [0, 1]$ and x, y on an interval.

To establish some new Hermite–Hadamard type inequalities for strongly (α, m)-HA-convex functions, we need the following lemma.

Lemma 2.5. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a, b])$, then*

$$\begin{aligned}
&\frac{f(a) + f(H(a,b)) + f(b)}{3} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \\
&= \frac{b-a}{4ab}\int_0^1 \left(\frac{1}{3} - t\right)\Big\{\left[ta^{-1} + (1-t)[H(a,b)]^{-1}\right]^{-2} f'\left(\left[ta^{-1} + (1-t)[H(a,b)]^{-1}\right]^{-1}\right) \\
&\quad - \left[tb^{-1} + (1-t)[H(a,b)]^{-1}\right]^{-2} f'\left(\left[tb^{-1} + (1-t)[H(a,b)]^{-1}\right]^{-1}\right)\Big\}\,dt,
\end{aligned}$$

where $H(a, b) = \frac{2ab}{a+b}$.

Proof. Let $x = \left(ta^{-1} + (1-t)[H(a,b)]^{-1}\right)^{-1}$ for $t \in [0, 1]$. Then

$$\begin{aligned}
&\int_0^1 \left(\frac{1}{3} - t\right)\left(ta^{-1} + (1-t)[H(a,b)]^{-1}\right)^{-2} f'\left(\left(ta^{-1} + (1-t)[H(a,b)]^{-1}\right)^{-1}\right) dt \\
&= \frac{2ab}{b-a}\left(\frac{2}{3}f(a) + \frac{1}{3}f(H(a,b))\right) - \left(\frac{2ab}{b-a}\right)^2 \int_a^{H(a,b)} \frac{f(x)}{x^2}\,dx.
\end{aligned}$$

Similarly, letting $x = \left(tb^{-1} + (1-t)[H(a,b)]^{-1}\right)^{-1}$ for $t \in [0, 1]$ gives

$$\int_0^1 \left(\frac{1}{3}-t\right)\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2} f'\left((tb^{-1}+(1-t)[H(a,b)]^{-1})^{-1}\right) dt$$
$$= -\frac{2ab}{b-a}\left(\frac{2}{3}f(b)+\frac{1}{3}f(H(a,b))\right)+\left(\frac{2ab}{b-a}\right)^2\int_{H(a,b)}^{b}\frac{f(x)}{x^2}dx.$$

Adding these two equalities leads to Lemma 2.5. □

3. Some new integral inequalities of the Hermite–Hadamard type

In this section, integral inequalities of the Hermite–Hadamard type related to strongly (α, m)-HA-convex function are discussed.

Theorem 3.1. *Let* $f:(0,b^*]\to\mathbb{R}$ *be differentiable on* $(0,b^*]$, $a,b\in(0,b^*]$ *with* $a<b$, *and* $f'\in L_1([a,b])$. *If* $|f'|$ *is strongly* (α,m)*-HA-convex on* $(0,b]$ *for some constant* $c\geqslant 0$ *and* $(\alpha,m)\in(0,1]^2$, *then*

$$\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leqslant \frac{b-a}{4ab}\Big\{S(a,b;\alpha)|f'(a)|+m\big[S(a,b;0)-S(a,b;\alpha)\big]\big|f'(mH(a,b))\big|$$
$$-c\big[S(a,b;1)-S(a,b;2)\big]\big[a^{-1}-[mH(a,b)]^{-1}\big]^2+S(b,a;\alpha)|f'(b)|$$
$$+m\big[S(b,a;0)-S(b,a;\alpha)\big]\big|f'(mH(a,b))\big|-c\big[S(b,a;1)-S(b,a;2)\big]\big[[mH(a,b)]^{-1}-b^{-1}\big]^2\Big\},$$

where

$$S(a,b;\alpha)=\frac{[H(a,b)]^2}{\alpha+2}{}_2F_1\left(2,\alpha+2;\alpha+3;\frac{a-H(a,b)}{a}\right)$$
$$+\frac{2[H(a,b)]^2}{3^{\alpha+2}(\alpha+1)(\alpha+2)}{}_2F_1\left(2,\alpha+1;\alpha+3;\frac{a-H(a,b)}{3a}\right)$$
$$-\frac{[H(a,b)]^2}{3(\alpha+1)}{}_2F_1\left(2,\alpha+1;\alpha+2;\frac{a-H(a,b)}{a}\right)$$

and ${}_2F_1(c,d;e;z)$ *is the hypergeometric function defined by*

$${}_2F_1(c,d;e;z)=\frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)}\int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c}\,dt, \tag{3.1}$$

for $e>d>0$, $|z|<1$, *and* $c\in\mathbb{R}$.

Proof. From Lemma 2.5 and the strongly (α,m)-HA-convexity of $|f'|$, we have

$$\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leqslant\frac{b-a}{4ab}\left[\int_0^1\left|\frac{1}{3}-t\right|\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left|f'\left((ta^{-1}+(1-t)[H(a,b)]^{-1})^{-1}\right)\right|dt\right.$$
$$\left.+\int_0^1\left|\frac{1}{3}-t\right|\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left|f'\left((tb^{-1}+(1-t)[H(a,b)]^{-1})^{-1}\right)\right|dt\right]$$
$$\leqslant\frac{b-a}{4ab}\left\{\int_0^1\left|\frac{1}{3}-t\right|\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\Big[t^\alpha|f'(a)|\right. \tag{3.2}$$
$$+m(1-t^\alpha)\left|f'(mH(a,b))\right|-ct(1-t)\left(a^{-1}-[mH(a,b)]^{-1}\right)^2\Big]dt$$
$$+\int_0^1\left|\frac{1}{3}-t\right|\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\Big[t^\alpha|f'(b)|$$
$$\left.+m(1-t^\alpha)\left|f'(mH(a,b))\right|-ct(1-t)\left(b^{-1}-[mH(a,b)]^{-1}\right)^2\Big]dt\right\}.$$

Since

$$\int_0^1 \left|\frac{1}{3}-t\right| \left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2} dt = S(a,b;0), \tag{3.3}$$

$$\int_0^1 t(1-t) \left|\frac{1}{3}-t\right| \left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2} dt = S(a,b;1)-S(a,b;2),$$

$$\int_0^1 \left|\frac{1}{3}-t\right| \left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2} t^\alpha dt = S(a,b;\alpha), \tag{3.4}$$

substituting equality (3.3) to (3.4) into the inequality (3.2) yields

$$\begin{aligned}
&\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \\
&\leqslant \frac{b-a}{4ab}\left\{\int_0^1 \left|\frac{1}{3}-t\right| \left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left[t^\alpha|f'(a)|\right.\right. \\
&\quad \left.+m(1-t^\alpha)\left|f'(mH(a,b))\right| - ct(1-t)\left(a^{-1}-[mH(a,b)]^{-1}\right)^2\right]dt \\
&\quad +\int_0^1 \left|\frac{1}{3}-t\right| \left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left[t^\alpha|f'(b)|\right. \\
&\quad \left.\left.+m(1-t^\alpha)\left|f'(mH(a,b))\right| - ct(1-t)\left(b^{-1}-[mH(a,b)]^{-1}\right)^2\right]dt\right\} \\
&= \frac{b-a}{4ab}\Big\{S(a,b;\alpha)|f'(a)| + m\big[S(a,b;0)-S(a,b;\alpha)\big]\left|f'(mH(a,b))\right| \\
&\quad - c\big[S(a,b;1)-S(a,b;2)\big]\left[a^{-1}-[mH(a,b)]^{-1}\right]^2 \\
&\quad + S(b,a;\alpha)|f'(b)| + m\big[S(b,a;0)-S(b,a;\alpha)\big]\left|f'(mH(a,b))\right| \\
&\quad - c\big[S(b,a;1)-S(b,a;2)\big]\left[[mH(a,b)]^{-1}-b^{-1}\right]^2\Big\}.
\end{aligned}$$

Theorem 3.1 is thus proved. □

Theorem 3.2. *Let* $f:(0,b^*]\to\mathbb{R}$ *be a differentiable function on* $(0,b^*]$, $a,b\in(0,b^*]$ *with* $a<b$, *and* $f'\in L_1([a,b])$. *If* $|f'|$ *is strongly* (α,m)*-HA-convex on* $(0,b]$ *for some constant* $c\geqslant 0$ *and* $(\alpha,m)\in(0,1]^2$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \\
&\leqslant \frac{b-a}{4ab}\left\{Q(a,b;\alpha)|f'(a)| + m\frac{29a^2+16[H(a,b)]^2-162Q(a,b;\alpha)}{162}\left|f'(mH(a,b))\right|\right. \\
&\quad - c\frac{116a^2+69[H(a,b)]^2}{4860}\left[a^{-1}-[mH(a,b)]^{-1}\right]^2 + Q(b,a;\alpha)|f'(b)| \\
&\quad \left.+m\frac{29b^2+16[H(a,b)]^2-162Q(b,a;\alpha)}{162}\left|f'(mH(a,b))\right| - c\frac{116b^2+69[H(a,b)]^2}{4860}\left[b^{-1}-[mH(a,b)]^{-1}\right]^2\right\},
\end{aligned}$$

where

$$Q(a,b;\alpha)=\frac{(\alpha+1)\left[2\times 3^{\alpha+2}\alpha+3^{\alpha+3}+2\right]a^2+2\left[\left(3^{\alpha+2}+2\right)\alpha+8\right][H(a,b)]^2}{3^{\alpha+3}(\alpha+1)(\alpha+2)(\alpha+3)}.$$

Proof. From Lemma 2.5, we have

$$\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right|$$

$$\leqslant \frac{b-a}{4ab}\left[\int_0^1\left|\frac{1}{3}-t\right|\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left|f'\left(\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}\right)\right|dt\right.$$
$$\left.+\int_0^1\left|\frac{1}{3}-t\right|\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left|f'\left(\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}\right)\right|dt\right].$$

By the GA-inequality, we have

$$\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\leqslant ta^2+(1-t)[H(a,b)]^2,$$

for $t\in[0,1]$. Using the strongly (α,m)-HA-convexity of $|f'|$ gives

$$\begin{aligned}
&\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\\
&\leqslant \frac{b-a}{4ab}\left\{\int_0^1\left|\frac{1}{3}-t\right|\left(ta^2+(1-t)[H(a,b)]^2\right)\Big[t^\alpha|f'(a)|\right.\\
&\quad+m\,(1-t^\alpha)\left|f'(mH(a,b))\right|-ct(1-t)\left(a^{-1}-[mH(a,b)]^{-1}\right)^2\Big]\,dt\\
&\quad+\int_0^1\left|\frac{1}{3}-t\right|\left(tb^2+(1-t)[H(a,b)]^2\right)\Big[t^\alpha|f'(b)|\\
&\quad\left.+m\,(1-t^\alpha)\left|f'(mH(a,b))\right|-ct(1-t)\left(b^{-1}-[mH(a,b)]^{-1}\right)^2\Big]\,dt\right\}\\
&=\frac{b-a}{4ab}\left\{Q(a,b;\alpha)|f'(a)|+m\frac{29a^2+16[H(a,b)]^2-162Q(a,b;\alpha)}{162}\left|f'(mH(a,b))\right|\right.\\
&\quad-c\frac{116a^2+69[H(a,b)]^2}{4860}\left[a^{-1}-[mH(a,b)]^{-1}\right]^2+Q(b,a;\alpha)|f'(b)|\\
&\quad+m\frac{29b^2+16[H(a,b)]^2-162Q(b,a;\alpha)}{162}\left|f'(mH(a,b))\right|\\
&\quad\left.-c\frac{116b^2+69[H(a,b)]^2}{4860}\left[b^{-1}-[mH(a,b)]^{-1}\right]^2\right\}.
\end{aligned}$$

Theorem 3.2 is thus proved. □

Corollary 3.3. *Under the assumptions of Theorem 3.2, if $\alpha=1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\\
&\leqslant\frac{b-a}{19440ab}\Big\{5\left[137a^2+37[H(a,b)]^2\right]|f'(a)|\\
&\quad+5\left[137b^2+37[H(a,b)]^2\right]|f'(b)|+5m\left[37(a^2+b^2)+118[H(a,b)]^2\right]\left|f'(mH(a,b))\right|\\
&\quad-c\left[116a^2+69[H(a,b)]^2\right]\left[a^{-1}-[mH(a,b)]^{-1}\right]^2-c\left[116b^2+69[H(a,b)]^2\right]\left[b^{-1}-[mH(a,b)]^{-1}\right]^2\Big\}.
\end{aligned}$$

Theorem 3.4. *Let $f:(0,b^*]\to\mathbb{R}$ be a differentiable function on $(0,b^*]$, $a,b\in(0,b^*]$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is strongly (α,m)-HA-convex on $(0,b]$ for some constant $c\geqslant0$, $(\alpha,m)\in(0,1]^2$, and $q>1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\\
&\leqslant\frac{b-a}{4ab}\left\{\left(S\left(a^{2q/(q-1)},[H(a,b)]^{2q/(q-1)}\right)\right)^{1-1/q}\left[\frac{3^\alpha(6\alpha+3)+2}{3^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q\right.\right.
\end{aligned}$$

$$+m\frac{3^{\alpha}(5\alpha^2+3\alpha+4)-4}{2\times 3^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'\big(mH(a,b)\big)\right|^q-\frac{37c}{972}\left(a^{-1}-[mH(a,b)]^{-1}\right)^2\Bigg]^{1/q}$$
$$+\left(S\left(b^{2q/(q-1)},[H(a,b)]^{2q/(q-1)}\right)\right)^{1-1/q}\Bigg[\frac{3^{\alpha}(6\alpha+3)+2}{3^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q$$
$$+m\frac{3^{\alpha}(5\alpha^2+3\alpha+4)-4}{2\times 3^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'\big(mH(a,b)\big)\right|^q-\frac{37c}{972}\left(b^{-1}-[mH(a,b)]^{-1}\right)^2\Bigg]^{1/q}\Bigg\},$$

where $S(u,v)$ *is defined by*

$$S(u,v)=\frac{2u^{1/3}L(v^{2/3},u^{2/3})-v^{2/3}L(v^{1/3},u^{1/3})+v-2u}{3(\ln v-\ln u)},$$

for $v\neq u$ *and* $L(u,v)$ *is the logarithmic mean*

$$L(u,v)=\begin{cases}\dfrac{v-u}{\ln v-\ln u}, & u\neq v,\\ u, & u=v.\end{cases}$$

Proof. From the GA-inequality, it follows that

$$\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2q/(q-1)}\leqslant a^{2tq/(q-1)}\big[H(a,b)\big]^{2(1-t)q/(q-1)},$$

for $t\in[0,1]$. Using Lemma 2.5, Hölder's integral inequality, and strongly (α,m)-HA-convexity of $|f'|^q$ gives

$$\left|\frac{f(a)+f(H(a,b))+f(b)}{3}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|$$
$$\leqslant\frac{b-a}{4ab}\Bigg\{\left(\int_0^1\left|\frac13-t\right|\left[a^{2tq/(q-1)}\big[H(a,b)\big]^{2(1-t)q/(q-1)}\right]dt\right)^{1-1/q}$$
$$\times\left(\int_0^1\left|\frac13-t\right|\left|f'\Big(\big(ta^{-1}+(1-t)[H(a,b)]^{-1}\big)^{-1}\Big)\right|^q dt\right)^{1/q}$$
$$+\left(\int_0^1\left|\frac13-t\right|\left[b^{2tq/(q-1)}\big[H(a,b)\big]^{2(1-t)q/(q-1)}\right]dt\right)^{1-1/q}$$
$$\times\left(\int_0^1\left|\frac13-t\right|\left|f'\Big(\big(tb^{-1}+(1-t)[H(a,b)]^{-1}\big)^{-1}\Big)\right|^q dt\right)^{1/q}\Bigg\}$$
$$\leqslant\frac{b-a}{4ab}\Bigg\{\left(S\left(a^{2q/(q-1)},[H(a,b)]^{2q/(q-1)}\right)\right)^{1-1/q}\Bigg[\int_0^1\left|\frac13-t\right|\Big(t^{\alpha}\left|f'(a)\right|^q$$
$$+m(1-t^{\alpha})\left|f'\big(mH(a,b)\big)\right|^q-ct(1-t)\left(a^{-1}-[mH(a,b)]^{-1}\right)^2\Big)dt\Bigg]^{1/q}$$
$$+\left(S\left(b^{2q/(q-1)},[H(a,b)]^{2q/(q-1)}\right)\right)^{1-1/q}\Bigg[\int_0^1\left|\frac13-t\right|\Big(t^{\alpha}\left|f'(b)\right|^q$$
$$+m(1-t^{\alpha})\left|f'\big(mH(a,b)\big)\right|^q-ct(1-t)\left(b^{-1}-[mH(a,b)]^{-1}\right)^2\Big)dt\Bigg]^{1/q}\Bigg\}$$
$$=\frac{b-a}{4ab}\Bigg\{\left(S\left(a^{2q/(q-1)},[H(a,b)]^{2q/(q-1)}\right)\right)^{1-1/q}\Bigg[\frac{3^{\alpha}(6\alpha+3)+2}{3^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q$$
$$+m\frac{3^{\alpha}(5\alpha^2+3\alpha+4)-4}{2\times 3^{\alpha+2}(\alpha+1)(\alpha+2)}\left|f'\big(mH(a,b)\big)\right|^q-\frac{37c}{972}\left(a^{-1}-[mH(a,b)]^{-1}\right)^2\Bigg]^{1/q}$$

$$+\left(S\left(b^{2q/(q-1)},[H(a,b)]^{2q/(q-1)}\right)\right)^{1-1/q}\left[\frac{3^{\alpha}(6\alpha+3)+2}{3^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(b)|^{q}\right.$$
$$\left.+m\frac{3^{\alpha}(5\alpha^2+3\alpha+4)-4}{2\times 3^{\alpha+2}(\alpha+1)(\alpha+2)}|f'(mH(a,b))|^{q}-\frac{37c}{972}\left(b^{-1}-[mH(a,b)]^{-1}\right)^2\right]^{1/q}\Bigg\}.$$

Theorem 3.4 is thus proved. □

Theorem 3.5. *Let $f:(0,b^*]\to\mathbb{R}$ be a differentiable function on $(0,b^*]$, $a,b\in(0,b^*]$ with $a<b$, and $f\in L_1([a,b])$. If f is strongly (α,m)-HA-convex on $(0,b]$ for some constant $c\geqslant 0$ and $(\alpha,m)\in(0,1]^2$, then*

$$f(H(a,b))\leqslant\frac{ab}{b-a}\int_a^b\frac{[f(x)+m(2^{\alpha}-1)f(mx)]}{2^{\alpha}x^2}\,dx-\frac{c\left[(1-m)^2(a^2+ab+b^2)+m(b-a)^2\right]}{12(mab)^2}\tag{3.5}$$

and

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\leqslant\min\left\{\frac{f(a)+m\alpha f(mb)}{\alpha+1},\frac{m\alpha f(ma)+f(b)}{\alpha+1}\right\}-\frac{c(mb-a)^2}{6(mab)^2}.\tag{3.6}$$

Proof. By the strongly (α,m)-HA-convexity of f, we have

$$\begin{aligned}f(H(a,b))&=\int_0^1 f\left(\frac{2}{ta^{-1}+(1-t)b^{-1}+tb^{-1}+(1-t)a^{-1}}\right)dt\\&\leqslant\frac{1}{2^{\alpha}}\int_0^1\left[f\left(\left(ta^{-1}+(1-t)b^{-1}\right)^{-1}\right)+m(2^{\alpha}-1)f\left(m\left(tb^{-1}+(1-t)a^{-1}\right)^{-1}\right)\right]dt\\&\quad-\frac{c\left\{\left[(1-m)^2+m\right](a^2+b^2)+\left[(1-m)^2-2m\right]ab\right\}}{12(mab)^2}.\end{aligned}\tag{3.7}$$

Letting $x=[ta^{-1}+(1-t)b^{-1}]^{-1}$ and $x=[tb^{-1}+(1-t)a^{-1}]^{-1}$ for $t\in[0,1]$ gives

$$\int_0^1 f\left(\left(ta^{-1}+(1-t)b^{-1}\right)^{-1}\right)dt=\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\tag{3.8}$$

and

$$\int_0^1 f\left(m\left(tb^{-1}+(1-t)a^{-1}\right)^{-1}\right)dt=\frac{ab}{b-a}\int_a^b\frac{f(mx)}{x^2}\,dx.\tag{3.9}$$

Putting equality (3.8) to (3.9) into the inequality (3.7), the inequality (3.5) is thus proved.

Letting $x=[ta^{-1}+(1-t)b^{-1}]^{-1}$ for $t\in[0,1]$ results in

$$\begin{aligned}\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx&=\int_0^1 f\left(\left(ta^{-1}+(1-t)b^{-1}\right)^{-1}\right)dt\\&\leqslant\int_0^1\left[t^{\alpha}f(a)+m(1-t^{\alpha})f(mb)-ct(1-t)\frac{(mb-a)^2}{(mab)^2}\right]dt\\&=\frac{f(a)+m\alpha f(mb)}{\alpha+1}-\frac{c(mb-a)^2}{6(mab)^2}.\end{aligned}$$

Thus, the inequality (3.6) is proved. The proof of Theorem 3.5 is completed. □

Corollary 3.6. *Under the assumptions of Theorem 3.5, if $\alpha=m=1$, then*

$$f(H(a,b))+\frac{c(b-a)^2}{12(ab)^2}\leqslant\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\leqslant\frac{f(a)+f(b)}{2}-\frac{c(b-a)^2}{6(ab)^2}.$$

Remark 3.7. Corollary 3.6 recovers a result in [9]. This result was also recited in [10, Theorem 2.1].

Theorem 3.8. *Let* $f : (0, b^*] \to \mathbb{R}_0$ *be a differentiable function on* $(0, b^*]$, $a, b \in (0, b^*]$ *with* $a < b$, *and* $f \in L_1([a, b])$. *If* f *is strongly* (α, m)*-HA-convex on* $(0, b]$ *for constant* $c \geqslant 0$ *and* $(\alpha, m) \in (0, 1]^2$, *then*

$$\begin{aligned}\frac{ab}{b-a}\int_a^b f(x)\,dx \leqslant \frac{1}{\alpha+1}\min\Big\{& b^2{}_2F_1\left(2,\alpha+1;\alpha+2;1-a^{-1}b\right)f(a)\\ &+m\alpha b^2{}_2F_1\left(1,\alpha+1;\alpha+2;1-a^{-1}b\right)f(mb), a^2{}_2F_1\left(2,\alpha+1;\alpha+2;1-b^{-1}a\right)f(b)\\ &+m\alpha a^2{}_2F_1\left(1,\alpha+1;\alpha+2;1-b^{-1}a\right)f(ma)\Big\}\\ &-\frac{c[(a+b)\ln(a^{-1}b)-2(b-a)](mb-a)^2}{m^2(b-a)^3},\end{aligned}$$

where ${}_2F_1(c, d; e; z)$ *is the hypergeometric function defined by* (3.1). *In particular, if* $\alpha = m = 1$,

$$\begin{aligned}\frac{ab}{b-a}\int_a^b f(x)\,dx \leqslant{}& \frac{a^2b\left[b\ln(a^{-1}b)-(b-a)\right]}{(b-a)^2}f(a)+\frac{ab^2\left[(b-a)-a\ln(a^{-1}b)\right]}{(b-a)^2}f(b)\\ &-\frac{c\left[(a+b)\ln(a^{-1}b)-2(b-a)\right]}{b-a}.\end{aligned}$$

Proof. Letting $x = [ta^{-1} + (1-t)b^{-1}]^{-1}$ for $t \in [0, 1]$, by the strongly (α, m)-HA-convexity of f, we have

$$\begin{aligned}\frac{ab}{b-a}\int_a^b f(x)\,dx &= \int_0^1\left[ta^{-1}+(1-t)b^{-1}\right]^{-2} f\left(\left(ta^{-1}+(1-t)b^{-1}\right)^{-1}\right)dt\\ &\leqslant \int_0^1\left[ta^{-1}+(1-t)b^{-1}\right]^{-2}\left[t^\alpha f(a)+m\left(1-t^\alpha\right)f(mb)-ct(1-t)(a^{-1}-(mb)^{-1})^2\right]dt\\ &= \frac{b^2}{\alpha+1}{}_2F_1\left(2,\alpha+1,\alpha+2,1-\frac{b}{a}\right)f(a)+\frac{m\alpha b^2}{\alpha+1}{}_2F_1\left(1,\alpha+1,\alpha+2,1-\frac{b}{a}\right)f(mb)\\ &\quad -c\frac{[(a+b)\ln(a^{-1}b)-2(b-a)](mb-a)^2}{m^2(b-a)^3}.\end{aligned}$$

The proof of Theorem 3.8 is thus completed. □

Theorem 3.9. *Let* $f : (0, b^*] \to \mathbb{R}_0$ *be a differentiable function on* $(0, b^*]$, $a, b \in (0, b^*]$ *with* $a < b$, *and* $f \in L_1([a, b])$. *If* f *is strongly* (α, m)*-HA-convex on* $(0, b]$ *for constant* $c \geqslant 0$ *and* $(\alpha, m) \in (0, 1]^2$, *then*

$$\frac{ab}{b-a}\int_a^b f(x)\,dx \leqslant \frac{(\alpha+1)a^2+b^2}{(\alpha+1)(\alpha+2)}f(a)+\frac{m\alpha\left((\alpha+1)a^2+(\alpha+3)b^2\right)}{2(\alpha+1)(\alpha+2)}f(mb)-c\frac{(a^2+b^2)(mb-a)^2}{12(mab)^2}.$$

Proof. Letting $x = [ta^{-1} + (1-t)b^{-1}]^{-1}$ for $t \in [0, 1]$, by the strongly (α, m)-HA-convexity of f, we have

$$\begin{aligned}\frac{ab}{b-a}\int_a^b f(x)\,dx &= \int_0^1\left(ta^{-1}+(1-t)b^{-1}\right)^{-2} f\left(\left(ta^{-1}+(1-t)b^{-1}\right)^{-1}\right)dt\\ &\leqslant \int_0^1\left[ta^2+(1-t)b^2\right]\left[t^\alpha f(a)+m\left(1-t^\alpha\right)f(mb)-ct(1-t)(a^{-1}-(mb)^{-1})^2\right]dt\\ &= \frac{(\alpha+1)a^2+b^2}{(\alpha+1)(\alpha+2)}f(a)+\frac{m\alpha\left((\alpha+1)a^2+(\alpha+3)b^2\right)}{2(\alpha+1)(\alpha+2)}f(mb)-c\frac{(a^2+b^2)(mb-a)^2}{12(mab)^2}.\end{aligned}$$

The proof of Theorem 3.9 is thus completed. □

Corollary 3.10. *Under the assumptions of Theorem* 3.9, *if* $\alpha = m = 1$, *then*

$$\frac{ab}{b-a}\int_a^b f(x)\,dx \leqslant \frac{(2a^2+b^2)f(a)+(a^2+2b^2)f(b)}{6}-c\frac{(a^2+b^2)(b-a)^2}{12(ab)^2}.$$

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