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The approximation of solutions for second order nonlinear oscillators using the polynomial least square method

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Abstract

In this paper, polynomial least square method (PLSM) is applied to find approximate solution for nonlinear oscillator differential equations. We illustrate that this method is very convenient and does not require linearization or small parameters. Comparisons are made between the results of PLSM and other methods in order to prove the accuracy of the PLSM method.

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1. Introduction

The evaluation of oscillations is essential in every physical system in which they occur. These oscillations are modeled by nonlinear oscillator differential equations and these equations have a great sensitivity to initial conditions. The study of nonlinear oscillator differential equations presents a large interest due to their large number of applications in diverse fields of engineering [17]. For these reasons the study of solutions of such equations is essential. Since in most of the cases the exact solutions cannot be found, approximate solutions must be computed. In order to find approximate solutions of these equations, many approximate analytical and numerical methods were proposed such as:

- homotopy perturbation method [12];
- harmonic balance method [2, 3, 21];
- adomian decomposition method [9];
- variational formulation method [13, 16];
- variational iteration methods [8, 14, 18];
- pseudo-spectral method [20];
- Rayleigh-Ritz method [11];
- parameter-expansion method [4, 22];

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- energy-balance method [5, 7, 14];
- amplitude-frequency formulation [4, 6];
- homotopy analysis method [19];
- max-min approach [4];
- optimal homotopy asymptotic method [10];

This paper considers the following general nonlinear oscillator differential equation:

$$u^{(2)}(t) + f(u^{(1)}(t), u(t), t) = 0, \tag{1.1}$$

subject to initial conditions:

$$u(0) = \alpha, \quad u'(0) = \beta. \tag{1.2}$$

Here, f is a nonlinear continuous function, $t \in R$, $\alpha, \beta \in R$. We applied the polynomial least square method (PLSM) to find approximate solutions for this second order nonlinear oscillator.

2. The polynomial least square method

We present the application of PLSM to the general problem (1.1)-(1.2).

For the problem (1.1)-(1.2) we consider the operator:

$$D(u) = u^{(2)}(t) + f(u^{(1)}(t), u(t), t).$$

If u_{app} is an approximate solution of the equation (1.1), the error obtained by replacing the exact solution u with the approximation u_{app} is given by the remainder:

$$R(t, u_{app}) = D(u_{app}(t)), \quad t \in [0, b].$$

We will find approximate polynomial solutions u_{app} of (1.1)-(1.2) on the $[0, b]$ interval, solutions which satisfy the following conditions:

$$|R(t, u_{app})| < \epsilon, \tag{2.1}$$

$$u_{app}(0) = \alpha, \quad u'_{app}(0) = \beta. \tag{2.2}$$

Definition 2.1. We call an *ϵ-approximate polynomial solution* of the problem (1.1)-(1.2) an approximate polynomial solution u_{app} satisfying the relations (2.1)-(2.2).

Definition 2.2. We call a *weak δ-approximate polynomial solution* of the problem (1.1)-(1.2) an approximate polynomial solution u_{app} satisfying the relation:

$$\int_0^b |R(t, u_{app})| dt \leqslant \delta$$

together with the initial conditions (2.2).

Definition 2.3. We consider the sequence of polynomials $P_m(t) = a_0 + a_1 t + ... + a_m t^m$, $a_i \in \mathbb{R}$, $i = 0, 1, ..., m$ satisfying the conditions:

$$P_m(0) = \alpha, \quad P'_m(0) = \beta.$$

We call the sequence of polynomials $P_m(t)$ *convergent to the solution of the problem* (1.1)-(1.2) if

$$\lim_{m \to \infty} D(P_m(t)) = 0.$$

We will find a weak ϵ-polynomial solution of the type:

$$\tilde{u}(t) = \sum_{k=0}^{m} c_k t^k, \tag{2.3}$$

where the constants $c_0, c_1, ..., c_m$ are calculated using the following steps:

- By substituting the approximate solution (2.3) in the equation (1.1) we obtain the following expression:
$$\mathfrak{R}(t, c_0, c_1, ..., c_m) = R(t, \tilde{u}) = \tilde{u}^{(2)}(t) + f(\tilde{u}^{(1)}(t), \tilde{u}(t), t). \tag{2.4}$$
If we could find the constants $c_0^0, c_1^0, ..., c_m^0$ such that $\mathfrak{R}(t, c_0^0, c_1^0, ..., c_m^0) = 0$ for any $t \in [0, b]$ and the equivalents of (1.2):
$$\tilde{u}(0) = \alpha, \quad \tilde{u}'(0) = \beta \tag{2.5}$$
are also satisfied, then by substituting $c_0^0, c_1^0, ..., c_m^0$ in (2.3) we obtain the exact solution of (1.1)-(1.2). In general, this situation is rarely encountered in polynomial approximation methods.

- Next we attach to the problem (1.1)-(1.2) the following real functional:
$$J(c_2, c_3, ..., c_m) = \int_0^b \mathfrak{R}^2(t, c_0, c_1, ..., c_m)dt, \tag{2.6}$$
where c_0, c_1 are computed as functions of $c_2, c_3, ..., c_m$ by using the initial conditions (2.5).

- We compute the values of $c_2^0, c_3^0, ..., c_m^0$ as the values which give the minimum of the functional (2.6) and the values of c_0^0, c_1^0 again as functions of $c_2^0, c_3^0, ..., c_m^0$ by using the initial conditions.

- Using the constants $c_0^0, c_1^0, ..., c_m^0$ thus determined, we consider the polynomial:
$$T_m(t) = \sum_{k=0}^{m} c_k^0\, x^k. \tag{2.7}$$

The following convergence theorem holds.

Theorem 2.4. *If the sequence of polynomials $P_m(t)$ converges to the solution of the problem (1.1)-(1.2), then the sequence of polynomials $T_m(t)$ from (2.7) satisfies the property:*
$$\lim_{m\to\infty} \int_0^b R^2(t, T_m)dt = 0.$$
Moreover, for all $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $m > m_0$, it follows that $T_m(t)$ is a weak ϵ-approximate polynomial solution of the problem (1.1)-(1.2).

Proof. Based on the way the coefficients of polynomial $T_m(t)$ are computed and taking into account the relations (2.4)-(2.7), the following inequality holds:
$$0 \leqslant \int_0^b R^2(t, T_m(t))dt \leqslant \int_0^b R^2(t, P_m(t))dt, \quad \forall m \in \mathbb{N}.$$
It follows that
$$0 \leqslant \lim_{m\to\infty} \int_0^b R^2(t, T_m(t))dt \leqslant \lim_{m\to\infty} \int_0^b R^2(t, P_m(t))dt = 0.$$
We obtain
$$\lim_{m\to\infty} \int_0^b R^2(t, T_m(t))dt = 0.$$
From this limit we obtain that for all $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $m > m_0$, it

follows that $T_m(t)$ is a weak ϵ-approximate polynomial solution of the problem (1.1)-(1.2) q.e.d. □

Remark 2.5. Any ϵ-approximate polynomial solution of the problem (1.1)-(1.2) is also a weak $\epsilon^2 \cdot b$-approximate polynomial solution, but the opposite is not always true. It follows that the set of weak approximate solutions of the problem (1.1)-(1.2) also contains the approximate solutions of the problem.

Taking into account the above remark, in order to find ϵ-approximate polynomial solutions of the problem (1.1)-(1.2) by PLSM we will first determine weak approximate polynomial solutions, $\tilde{u}_{app}$. If $|R(t, \tilde{u}_{app})| < \epsilon$, then $\tilde{u}_{app}$ is also an ϵ-approximate polynomial solution of the problem.

3. Applications

3.1. Application 1: Van der Pol oscillator

The first application is the Van der Pol oscillator [1]:

$$u''(t) + u'(t) + u(t) + u^2(t) \cdot u'(t) = 2 \cdot \cos(t) - \cos^3(t), \tag{3.1}$$

with the initial conditions $u(0) = 0,\ u'(0) = 1$.

The exact solution of this equation is $u(t) = \sin(t)$. Using the polynomial least square method we performed the following 7-th degree approximate polynomial solution of equation (3.1):

$$\begin{aligned} P_1(t) = {} & t - 3.18483 \cdot 10^{-7} \cdot t^2 - 0.166662 \cdot t^3 - 0.0000216909 \cdot t^4 + 0.0083825 \cdot t^5 - 0.0000549143 \cdot t^6 \\ & - 0.000172381 \cdot t^7. \end{aligned}$$

Table 1 presents the comparison between the absolute errors (as the difference in absolute value between the approximate solution and the exact solution) corresponding to the approximate solution obtained using the homotopy perturbation method (HPM), to the approximate solution obtained by variational iteration method (VIM) [1] and to our approximate solution obtained by PLSM.

Table 1: Comparison of absolute errors of the approximate solutions for Application 3.1.

t	Exact sol.	HPM error	VIM error	PLSM sol.	PLSM error
0	0	0	0	0	0
0.1	$9.983\ 10^{-2}$	$0.550\ 10^{-9}$	$0.237\ 10^{-7}$	$9.983\ 10^{-2}$	$0.454\ 10^{-9}$
0.2	$1.986\ 10^{-1}$	$0.704\ 10^{-7}$	$0.172\ 10^{-5}$	$1.986\ 10^{-1}$	$0.788\ 10^{-9}$
0.3	$2.955\ 10^{-1}$	$0.116\ 10^{-5}$	$0.191\ 10^{-4}$	$2.955\ 10^{-1}$	$0.114\ 10^{-8}$
0.4	$3.894\ 10^{-1}$	$0.824\ 10^{-5}$	$0.104\ 10^{-3}$	$3.894\ 10^{-1}$	$0.351\ 10^{-9}$
0.5	$4.794\ 10^{-1}$	$0.371\ 10^{-4}$	$0.387\ 10^{-3}$	$4.794\ 10^{-1}$	$0.145\ 10^{-8}$
0.6	$5.646\ 10^{-1}$	$0.125\ 10^{-3}$	$0.112\ 10^{-2}$	$5.646\ 10^{-1}$	$0.501\ 10^{-9}$
0.7	$6.442\ 10^{-1}$	$0.343\ 10^{-3}$	$0.271\ 10^{-2}$	$6.442\ 10^{-1}$	$0.113\ 10^{-8}$
0.8	$7.173\ 10^{-1}$	$0.812\ 10^{-3}$	$0.581\ 10^{-2}$	$7.173\ 10^{-1}$	$0.951\ 10^{-9}$
0.9	$7.833\ 10^{-1}$	$0.171\ 10^{-2}$	$0.113\ 10^{-1}$	$7.833\ 10^{-1}$	$0.411\ 10^{-9}$
1	$8.414\ 10^{-1}$	$0.328\ 10^{-2}$	$0.202\ 10^{-1}$	$8.414\ 10^{-1}$	$0.277\ 10^{-10}$

It is easy to see that the approximate solution given by PLSM is much closer to the exact solution than the previous ones: the approximate solution given by HPM, and the approximate solution given by VIM [1].

3.2. *Application 2: nonlinear oscillator*

We consider the nonlinear oscillator differential equation [1]:

$$u''(t) - u(t) + u^2(t) + (u'(t))^2 - 1 = 0$$

with the initial conditions $u(0) = 2,\ u'(0) = 0$.

The exact solution of the above equation is $u(t) = 1 + \cos(t)$. Using the steps described in the previous section we perform the following 7-th degree approximate polynomial solution by PLSM:

$$P_2(t) = 2 - 0.500001 \cdot t^2 + 9.12188 \cdot 10^{-6} \cdot t^3 + 0.04162 \cdot t^4 + 0.000114194 \cdot t^5 - 0.00153451 \cdot t^6 + 0.0000941646 \cdot t^7.$$

Table 2 presents the comparison between the absolute errors corresponding to the approximate solution obtain by HPM, to the approximate solution given by VIM [1] and our approximate solution obtained by PLSM.

Table 2: Comparison of absolute errors of the approximate solutions for Application 3.2.

t	Exact sol.	HPM error	VIM error	PLSM sol.	PLSM error
0	2	0	0	2	0
0.1	1.995	$0.833\ 10^{-5}$	$0.833\ 10^{-6}$	1.995	$0.776\ 10^{-9}$
0.2	1.980	$0.133\ 10^{-3}$	$0.133\ 10^{-3}$	1.980	$0.168\ 10^{-8}$
0.3	1.955	$0.676\ 10^{-3}$	$0.675\ 10^{-3}$	1.955	$0.217\ 10^{-8}$
0.4	1.921	$0.213\ 10^{-2}$	$0.213\ 10^{-2}$	1.921	$0.691\ 10^{-9}$
0.5	1.877	$0.523\ 10^{-2}$	$0.521\ 10^{-2}$	1.877	$0.251\ 10^{-8}$
0.6	1.825	$0.109\ 10^{-1}$	$0.108\ 10^{-1}$	1.825	$0.601\ 10^{-9}$
0.7	1.764	$0.202\ 10^{-1}$	$0.200\ 10^{-1}$	1.764	$0.221\ 10^{-8}$
0.8	1.696	$0.345\ 10^{-1}$	$0.341\ 10^{-1}$	1.696	$0.167\ 10^{-8}$
0.9	1.621	$0.554\ 10^{-1}$	$0.547\ 10^{-1}$	1.621	$0.646\ 10^{-9}$
1	1.540	$0.847\ 10^{-1}$	$0.834\ 10^{-1}$	1.540	$0.205\ 10^{-9}$

3.3. *Application 3: unforced Duffing oscillator*

Consider the unforced Duffing oscillator differential equation [15]:

$$u''(t) + (t) - \frac{1}{6} \cdot u^3(t) = 0$$

with the initial conditions $u(0) = 0,\ u'(0) = 1.6376$.

The 7-th degree approximate polynomial solution by PLSM is:

$$P_3(t) = 1.6376 \cdot t - 0.0000944879 \cdot t^2 - 0.271593 \cdot t^3 - 0.00677573 \cdot t^4 + 0.0665547 \cdot t^5 - 0.0198777 \cdot t^6 + 0.000794326 \cdot t^7.$$

The comparison (for Application 3.3) between the absolute errors (as the difference in absolute value between the approximate solution and the numerical solution) corresponding to the approximate solution obtained using the Haar wavelet method (HWM) from [15], and to our approximate solution obtained by PLSM, is given in Table 3.

Table 3: Comparison of absolute errors of the approximate solutions for Application 3.3.

t	Numerical sol.	HWM error	PLSM sol.	PLSM error
0.0156	0.0255	$0.18\ 10^{-5}$	0.0255	$0146\ 10^{-7}$
0.1094	0.1787	$0.6\ 10^{-5}$	0.1787	$0.134\ 10^{-6}$
0.2344	0.3803	$0.9\ 10^{-5}$	0.3803	$0.264\ 10^{-6}$
0.3594	0.5761	$0.3\ 10^{-4}$	0.5761	$0.464\ 10^{-7}$
0.4844	0.7635	$0.69\ 10^{-4}$	0.7635	$0.347\ 10^{-6}$
0.6094	0.9401	$0.131\ 10^{-3}$	0.9401	$0.5\ 10^{-7}$
0.7344	1.1042	$0.23\ 10^{-3}$	1.1042	$0.284\ 10^{-6}$
0.8594	1.2546	$0.32\ 10^{-3}$	1.2546	$0.494\ 10^{-7}$
0.9844	1.3906	$0.4\ 10^{-3}$	1.3906	$0.196\ 10^{-7}$

3.4. *Application 4: forced Duffing oscillator*

Consider the forced Duffing oscillator differential equation [15]:

$$u''(t) + u(t) - \frac{1}{6} \cdot u^3(t) = 2 \cdot \sin(t)$$

with the initial conditions $u(0) = 0,\ u'(0) = -2.7676$.

The 7-th degree approximate polynomial solution by PLSM is:

$$P_4(t) = -2.7676 \cdot t + 0.000608073 \cdot t^2 + 0.785244 \cdot t^3 + 0.0505135 \cdot t^4 - 0.363462 \cdot t^5 + 0.172764 \cdot t^6 - 0.0250286 \cdot t^7.$$

The comparison (for Application 3.4) between the absolute errors (as the difference in absolute value between the approximate solution and the numerical solution) corresponding to the approximate solution obtained using the Haar wavelet method (HWM) from [15], and to our approximate solution obtained by PLSM, is given in Table 4.

Table 4: Comparison of absolute errors of the approximate solutions for Application 3.4.

t	Numerical sol.	HWM error	PLSM sol.	PLSM error
0.0156	-0.0431	$0.1\ 10^{-5}$	-0.0431	$0.126\ 10^{-6}$
0.1094	-0.3017	$0.2\ 10^{-4}$	-0.3017	$0.51\ 10^{-6}$
0.2344	-0.6386	$0.1\ 10^{-3}$	-0.6386	$0.218\ 10^{-5}$
0.3594	-0.9591	$0.3\ 10^{-3}$	-0.9591	$0.33\ 10^{-6}$
0.4844	-1.2560	$0.4\ 10^{-3}$	-1.2560	$0.199\ 10^{-5}$
0.6094	-1.5241	$0.6\ 10^{-3}$	-1.5241	$0.256\ 10^{-6}$
0.7344	-1.7599	$0.7\ 10^{-3}$	-1.7599	$0.212\ 10^{-5}$
0.8594	-1.9615	$0.9\ 10^{-3}$	-1.9615	$0.142\ 10^{-6}$
0.9844	-2.1285	$0.15\ 10^{-2}$	-2.1285	$0.396\ 10^{-6}$

3.5. *Application 5: unforced Duffing-Van der Pol oscillator*

Consider the unforced Duffing-Van der Pol oscillator [15]:

$$u''(t) - 0.1 \cdot (1 - u^2(t)) \cdot u'(t) + u(t) + 0.01 \cdot u^3(t) = 0$$

with $u(0) = 2,\ u'(0) = 0$.

The 7-th degree approximate polynomial solution by PLSM is:

$$P_5(t) = 2 - 1.03995 \cdot t^2 + 0.103266 \cdot t^3 + 0.0932818 \cdot t^4 - 0.0653592 \cdot t^5 + 0.0207549 \cdot t^6 - 0.00145624 \cdot t^7.$$

The comparison (for Application 3.5) between the absolute errors (as the difference in absolute value between the approximate solution and the numerical solution) corresponding to the approximate solution obtained using the Haar Wavelet Method (HWM) from [15], and to our approximate solution obtained by PLSM, is given in Table 5.

Table 5: Comparison of absolute errors of the approximate solutions for Application 3.5.

t	Numerical sol.	HWM error	PLSM sol.	PLSM error
0.0156	1.9997	$0.1\ 10^{-5}$	1.9997	$0.167\ 10^{-7}$
0.1094	1.9877	$0.1\ 10^{-2}$	1.9877	$0.667\ 10^{-7}$
0.2344	1.9444	$0.24\ 10^{-2}$	1.9444	$0.119\ 10^{-6}$
0.3594	1.8716	$0.39\ 10^{-2}$	1.8716	$0.101\ 10^{-8}$
0.4844	1.7713	$0.6\ 10^{-2}$	1.7713	$0.148\ 10^{-6}$
0.6094	1.6455	$0.91\ 10^{-2}$	1.6455	$0.135\ 10^{-7}$
0.7344	1.4962	$0.143\ 10^{-1}$	1.4962	$0.155\ 10^{-6}$
0.8594	1.3255	$0.226\ 10^{-1}$	1.3255	$0.119\ 10^{-8}$
0.9844	1.1355	$0.356\ 10^{-1}$	1.1355	$0.698\ 10^{-8}$

3.6. *Application 6: forced Duffing-Van der Pol oscillator*

Consider the forced Duffing-Van der Pol oscillator [15]:

$$\varepsilon u''(t) + (\delta + \beta \cdot u^2(t)) \cdot u'(t) - \mu \cdot u(t) + \alpha \cdot u^3(t) = 0.5 \cdot \cos(0.79 \cdot t).$$

The choice of $\varepsilon = 1,\ \delta = -0.1,\ \beta = 0.1,\ \mu = -0.5,\ \alpha = 0.5$ with the initial conditions $u(0) = 1,\ u'(0) = 0$ leads to the 7-th degree approximate polynomial solution by PLSM:

$$P_6(t) = 1 - 0.249998 \cdot t^2 - 0.0000247707 \cdot t^3 + 0.028724 \cdot t^4 - 0.00115228 \cdot t^5 - 0.00539765 \cdot t^6 + 0.00137076 \cdot t^7.$$

Table 6 presents the HW solution, PLSM solution and the errors corresponding to our approximate solution given by PLSM.

3.7. *Application 7: unforced Van der Pol oscillator*

Consider the unforced Van der Pol oscillator [15]:

$$u''(t) - 0.05 \cdot (1 - u^2(t)) \cdot u'(t) + u(t) = 0$$

with $u(0) = 0,\ u'(0) = 0.5$.

The 7-th degree approximate polynomial solution computed with PLSM is:

$$P_7(t) = 0.5 \cdot t + 0.0125002 \cdot t^2 - 0.0831289 \cdot t^3 - 0.00257833 \cdot t^4 + 0.00403136 \cdot t^5 + 0.000400308 \cdot t^6 - 0.000173925 \cdot t^7.$$

Table 7 presents HW solution, PLSM solution and the absolute errors corresponding to the approximate solution given by PLSM.

Table 6: Comparison of absolute errors of the approximate solutions for Application 3.6.

t	Numerical solution	HWM solution	PLSM solution	PLSM error
0.0156	0.999939	0.999939	0.999939	$0.123\ 10^{-6}$
0.1094	0.997012	0.997014	0.997012	$0.126\ 10^{-6}$
0.2031	0.989735	0.989735	0.989736	$0.112\ 10^{-6}$
0.2969	0.978179	0.978191	0.978179	$0.101\ 10^{-6}$
0.3906	0.962498	0.962516	0.962498	$0.106\ 10^{-6}$
0.5156	0.935436	0.935526	0.935436	$0.111\ 10^{-6}$
0.6094	0.910784	0.911024	0.910784	$0.882\ 10^{-7}$
0.7031	0.882691	0.883181	0.882691	$0.471\ 10^{-7}$
0.7969	0.851338	0.852307	0.851338	$0.278\ 10^{-7}$
0.8906	0.817033	0.818731	0.817033	$0.285\ 10^{-7}$
0.9844	0.779942	0.782791	0.779942	$0.102\ 10^{-7}$

Table 7: Comparison of absolute errors of the approximate solutions for Application 3.7.

t	Numerical solution	HWM solution	PLSM solution	PLSM error
0	0	0	0	0
0.1	0.0500417	0.05004	0.0500417	$0.117\ 10^{-7}$
0.2	0.0998322	0.09983	0.0998322	$0.141\ 10^{-7}$
0.3	0.14887	0.14886	0.14887	$0.245\ 10^{-7}$
0.4	0.196656	0.19665	0.196656	$0.139\ 10^{-7}$
0.5	0.242704	0.24270	0.242704	$0.108\ 10^{-7}$
0.6	0.286537	0.28653	0.286537	$0.883\ 10^{-8}$
0.7	0.327703	0.32770	0.327703	$0.580\ 10^{-8}$
0.8	0.365772	0.36577	0.365772	$0.669\ 10^{-9}$
0.9	0.400343	0.40034	0.400343	$0.419\ 10^{-8}$
1	0.431051	0.43105	0.431051	$0.703\ 10^{-8}$

4. Conclusions

In this work the polynomial least squares method (PLSM) was successfully used to obtain analytical approximate polynomial solutions for second order nonlinear oscillators. The results obtained by using PLSM were compared to results obtained by using the homotopy perturbation method (HPM), the variational iteration method (VIM) or the Haar wavelet method (HWM). The calculations illustrate the accuracy of the presented method and show that PLSM is a straightforward and efficient method to compute approximate solutions for nonlinear oscillator differential equations.

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