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# Existence and uniqueness of positive solutions for singular fractional differential systems with coupled integral boundary conditions

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## Abstract

In this paper, we study the existence and uniqueness of positive solutions for a class of singular fractional differential systems with coupled integral boundary conditions. By using the properties of the Green function, the mixed monotone method and the fixed point theory in cones, we obtain the existence and uniqueness results for the problem. The results obtained herein generalize and improve some known results including singular and non-singular cases. 

Keywords: Singular fractional differential equations, Riemann-Stieltjes integral boundary value problem, positive solution, fixed point theorem in cone.
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## 1. Introduction

In this article, we consider the existence and uniqueness of positive solutions for a class of singular fractional differential systems with coupled integral boundary conditions as follows

$$\begin{cases} D_{0+}^{\alpha}u(t)+p_1(t)f_1(t,u(t),v(t))+q_1(t)g_1(t,u(t),v(t))=0, & t\in(0,1),\\ D_{0+}^{\beta}v(t)+p_2(t)f_2(t,u(t),v(t))+q_2(t)g_2(t,u(t),v(t))=0, & t\in(0,1),\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \ \ u(1)=\displaystyle\int_0^1 a(s)v(s)dA(s), & \\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, \ \ v(1)=\displaystyle\int_0^1 b(s)u(s)dB(s), & \end{cases} \tag{1.1}$$

where $\alpha,\beta\in\mathbb{R}, n-1<\alpha\leqslant n, m-1<\beta\leqslant m, n,m\in\mathbb{N},\ n,m\geqslant 2$, $D_{0+}^{\alpha}$ and $D_{0+}^{\beta}$ denote the Riemann-Liouville derivatives of orders $\alpha$ and $\beta$, respectively. $p_i,q_i\in C((0,1),[0,\infty))$, $a,b\in C([0,1],[0,\infty))$,

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$f_i \in C((0,1)\times(0,\infty)\times(0,\infty),[0,\infty))$, $g_i \in C((0,1)\times(0,\infty)\times(0,\infty),[0,\infty))$ and $f_i(t,x,y)$ may be singular at $t=0,1$ and $y=0$, and $g_i(t,x,y)$ may be singular at $t=0,1$ and $x=0$ $(i=1,2)$. $\int_0^1 a(s)v(s)dA(s)$, $\int_0^1 b(s)u(s)dB(s)$ denote the Riemann-Stieltjes integral with a signed measure, that is, $A,B:[0,1]\to[0,\infty)$ are functions of boundary variation. By a positive solution of BVP (1.1), we mean a pair of functions $(u,v)\in C[0,1]\times C[0,1]$ satisfying BVP (1.1) with $u(t)>0$ and $v(t)>0$ for all $t\in(0,1]$.

In recent years, boundary value problems for a coupled system of nonlinear differential equations have gained its popularity and importance due to its various applications in heat conduction, chemical engineering, underground water flow, thermo-elasticity and plasma physics. There have appeared some results for the existence of solutions or positive solutions of boundary value problems for a coupled system of nonlinear fractional differential equations, see [1, 2, 4, 6, 7, 9, 11–13, 16, 18–23, 25] and the references therein. Most of the results show that the equations have either single or multiple positive solutions.

In [3], Cui et al. investigated the following singular problem

$$\begin{cases} -x''(t)=f(t,x(t),y(t)), \quad t\in(0,1),\\ -y''(t)=g(t,x(t),y(t)), \quad t\in(0,1),\\ x(0)=\displaystyle\int_0^1 y(t)d\alpha(t), \quad y(0)=\int_0^1 x(t)d\beta(t),\\ x(1)=y(1)=0, \end{cases}$$

where $\int_0^1 y(t)d\alpha(t)$ and $\int_0^1 x(t)d\beta(t)$ denote the Riemann-Stieltjes integrals of $y$ and $x$ with respect to $\alpha$ and $\beta$, respectively; $f\in C((0,1)\times[0,\infty)\times(0,\infty),[0,\infty))$, $g\in C((0,1)\times(0,\infty)\times[0,\infty),[0,\infty))$ and $f(t,x,y)$ is nondecreasing in $x$ and nonincreasing in $y$ and may be singular at $t=0,1$ and $y=0$, while $g(t,x,y)$ is nonincreasing in $x$ and nondecreasing in $y$ and may be singular at $t=0,1$ and $x=0$.

In [21], Wang et al. considered the following singular fractional differential system with coupled boundary conditions

$$\begin{cases} D_{0+}^{\alpha_1}u(t)+f(t,u(t),v(t))=0,\\ D_{0+}^{\alpha_2}v(t)+g(t,u(t),v(t))=0, \quad t\in(0,1),\\ u(0)=u'(0)=\cdots=u^{(n-2)}=0, \quad u(1)=\mu_1\displaystyle\int_0^1 v(s)dA_1(s),\\ v(0)=v'(0)=\cdots=v^{(n-2)}=0, \quad v(1)=\mu_2\displaystyle\int_0^1 u(s)dA_2(s), \end{cases}$$

where $n-1<\alpha_i\leqslant n, n\geqslant 2$, and $D_{0+}^{\alpha_i}$ is the standard Riemann-Liouvill derivative. $f\in C((0,1)\times[0,\infty)\times(0,\infty),[0,\infty))$, $g\in C((0,1)\times(0,\infty)\times[0,\infty),[0,\infty))$ and $f(t,x,y)$ is nondecreasing in $x$ and nonincreasing in $y$ and may be singular at $t=0,1$ and $y=0$, while $g(t,x,y)$ is nonincreasing in $x$ and nondecreasing in $y$ and may be singular at $t=0,1$ and $x=0$. By using the Guo-Krasnosel'skii fixed point theorem, they obtained the existence of a positive solution and the uniqueness of the positive solution under the condition $\alpha_1=\alpha_2$.

In [8], Henderson and Luca studied the system of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,v(t))=0, \quad 0<t<1, \quad n-1<\alpha\leqslant n,\\ D_{0+}^{\beta}v(t)+g(t,u(t))=0, \quad 0<t<1, \quad m-1<\beta\leqslant m, \end{cases}$$

with the integral boundary conditions

$$\begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \quad u(1)=\displaystyle\int_0^1 u(s)dH(s),\\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, \quad v(1)=\displaystyle\int_0^1 v(s)dK(s), \end{cases}$$

where $n, m \in \mathbb{N}$, $n,\ m \geqslant 2$, $D_{0+}^{\alpha}$ and $D_{0+}^{\beta}$ denote the Riemann-Liouville derivatives of orders $\alpha$ and $\beta$ respectively and $f, g : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous and $f(t,0) = g(t,0) = 0$ for all $t \in [0,1]$. They obtained the existence and multiplicity of positive solutions for the above BVP by using the Guo-Krasnosel'skii fixed point theorem and some theorems from the fixed point index theory, but they did not discuss the uniqueness of positive solutions.

Motivated by the above mentioned work, the purpose of this article is to investigate the existence and uniqueness of positive solutions for singular fractional differential systems with coupled integral boundary conditions under certain conditions on the functions $f_i$ and $g_i$ $(i = 1,2)$. The main new features presented in this paper are as follows. Firstly, we divided the functions of the BVP into $f_i$ and $g_i$ so that the boundary value problem has a more general form. Secondly, $D_{0+}^{\alpha}$ and $D_{0+}^{\beta}$ denote the Riemann-Liouville derivatives of orders $\alpha$ and $\beta$ in which $\alpha \in (n-1,n], \beta \in (m-1,m], n,m \in \mathbb{N}$. Thirdly, if $dA(s) = dB(s) = ds$ or $h(s)ds$, then BVP (1.1) reduces to a multi-point boundary value problem as a special case. Fourthly, the nonlinearity is allowed to be singular in regard to time and space variable elements. In particular, for any $\alpha, \beta \in (0,+\infty)$, we obtain the existence and uniqueness of positive solutions for singular fractional differential systems (1.1). The results obtained herein generalize and improve some known results including singular and non-singular cases.

The rest of the paper is organized as follows. In Section 2, we present the necessary definitions and properties to prove our main results, and obtain the corresponding Green function and some of its properties. In Section 3, we give the existence and uniqueness theorem for the positive solutions with respect to a cone for the BVP (1.1). In Section 4, as an application, an interesting example is presented to illustrate the main result. Conclusions are presented in Section 5.

## 2. Preliminaries and lemmas

For the convenience of the reader, we present some definitions and lemmas to be used in the proof of our main result. They also can be found in the literature [10, 14, 15, 17].

**Definition 2.1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

**Definition 2.2.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number $\alpha$, provided that the right hand side is pointwise defined on $(0,\infty)$.

**Lemma 2.3** ([10])**.** *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0,$$

*has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$, $C_i \in \mathbb{R}$ $(i = 1,2,\cdots,N)$ as the unique solution, where $N = [\alpha] + 1$.*

From the definition of the Riemann-Liouville derivative, we can obtain the statement.

**Lemma 2.4** ([10])**.** *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

*for some $C_i \in \mathbb{R}$ $(i = 1,2,\cdots,N)$, where $N = [\alpha] + 1$.*

In the following, we present the Green function of the fractional differential equation boundary value problem.

**Lemma 2.5.** *Let* $x, y \in C(0,1) \cap L^1(0,1)$ *be given functions. Then the boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + x(t) = 0, \quad 0 < t < 1, \ n-1 < \alpha \leqslant n, \\ D_{0+}^{\beta} v(t) + y(t) = 0, \quad 0 < t < 1, \ m-1 < \beta \leqslant m, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ u(1) = \displaystyle\int_0^1 a(s)v(s)dA(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ v(1) = \displaystyle\int_0^1 b(s)u(s)dB(s), \end{cases} \tag{2.1}$$

*where* $n, m \in \mathbb{N}$, $n$, $m \geqslant 2$, *is equivalent to*

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s)x(s)ds + \int_0^1 H_1(t,s)y(s)ds, \quad t \in [0,1], \\ v(t) = \displaystyle\int_0^1 G_2(t,s)y(s)ds + \int_0^1 H_2(t,s)x(s)ds, \quad t \in [0,1], \end{cases} \tag{2.2}$$

*where*

$$\begin{cases} G_1(t,s) = g_1(t,s) + \dfrac{\Delta_1}{\Delta} t^{\alpha-1} \displaystyle\int_0^1 g_1(\tau,s)b(\tau)dB(\tau), \\ H_1(t,s) = \dfrac{t^{\alpha-1}}{\Delta} \displaystyle\int_0^1 g_2(\tau,s)a(\tau)dA(\tau), \\ G_2(t,s) = g_2(t,s) + \dfrac{\Delta_2}{\Delta} t^{\beta-1} \displaystyle\int_0^1 g_2(\tau,s)a(\tau)dA(\tau), \\ H_2(t,s) = \dfrac{t^{\beta-1}}{\Delta} \displaystyle\int_0^1 g_1(\tau,s)b(\tau)dB(\tau), \end{cases} \tag{2.3}$$

*and*

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, \quad 0 \leqslant s \leqslant t \leqslant 1, \\ [t(1-s)]^{\alpha-1}, \quad 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.4}$$

$$g_2(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} [t(1-s)]^{\beta-1} - (t-s)^{\beta-1}, \quad 0 \leqslant s \leqslant t \leqslant 1, \\ [t(1-s)]^{\beta-1}, \quad 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.5}$$

*in which* $\Delta = 1 - \Delta_1\Delta_2 \neq 0$, *and* $\Delta_1 = \int_0^1 a(s)s^{\beta-1}dA(s), \Delta_2 = \int_0^1 b(s)s^{\alpha-1}dB(s)$.

*Proof.* By Lemmas 2.3 and 2.4, the solution of the system (2.1) is

$$\begin{cases} u(t) = -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}x(s)ds + c_1 t^{\alpha-1} + \cdots + c_n t^{\alpha-n}, \ t \in [0,1], \\ v(t) = -\dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1}y(s)ds + d_1 t^{\beta-1} + \cdots + d_m t^{\beta-m}, \ t \in [0,1], \end{cases} \tag{2.6}$$

where $c_i, d_j \in \mathbb{R}$ $(i = 1,2,3,\cdots,n; j = 1,2,3,\cdots,m)$. By using the conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$ and $v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0$, we obtain $c_2 = c_3 = \cdots = c_n = 0$ and $d_2 = d_3 = \cdots = d_m = 0$. Then by (2.6) we conclude

$$\begin{cases} u(t) = c_1 t^{\alpha-1} - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}x(s)ds, \ t \in [0,1], \\ v(t) = d_1 t^{\beta-1} - \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1}y(s)ds, \ t \in [0,1]. \end{cases} \tag{2.7}$$

Combining (2.7) with the conditions $u(1)=\int_0^1 a(s)v(s)dA(s)$ and $v(1)=\int_0^1 b(s)u(s)dB(s)$, we deduce

$$\begin{cases} c_1-\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1(1-s)^{\alpha-1}x(s)ds=\int_0^1 a(s)[d_1s^{\beta-1}-\dfrac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}y(\tau)d\tau]dA(s),\\ d_1-\dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^1(1-s)^{\beta-1}y(s)ds=\int_0^1 b(s)[c_1s^{\alpha-1}-\dfrac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}x(\tau)d\tau]dB(s),\end{cases}$$

or equivalently

$$\begin{cases} c_1-d_1\displaystyle\int_0^1 a(s)s^{\beta-1}dA(s)=\dfrac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds-\dfrac{1}{\Gamma(\beta)}\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds,\\ d_1-c_1\displaystyle\int_0^1 b(s)s^{\alpha-1}dB(s)=\dfrac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}y(s)ds-\dfrac{1}{\Gamma(\alpha)}\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds.\end{cases}\tag{2.8}$$

The above system in the unknowns $c_1$ and $d_1$ has the determinant

$$\begin{aligned}\Delta=&\begin{vmatrix}1 & -\int_0^1 a(s)s^{\beta-1}dA(s)\\ -\int_0^1 b(s)s^{\alpha-1}dB(s) & 1\end{vmatrix}\\ =&1-\left(\int_0^1 a(s)s^{\beta-1}dA(s)\right)\left(\int_0^1 b(s)s^{\alpha-1}dB(s)\right)\\ =&1-\Delta_1\Delta_2.\end{aligned}\tag{2.9}$$

So by (2.8) and (2.9) we obtain

$$\begin{aligned}c_1=&\frac{1}{\Delta}\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds-\frac{\Delta_1}{\Gamma(\alpha)}\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds\\ &-\frac{1}{\Gamma(\beta)}\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds+\frac{\Delta_1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}y(s)ds\Bigg],\end{aligned}\tag{2.10}$$

$$\begin{aligned}d_1=&\frac{1}{\Delta}\Bigg[\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}y(s)ds-\frac{\Delta_2}{\Gamma(\beta)}\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds\\ &-\frac{1}{\Gamma(\alpha)}\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds+\frac{\Delta_2}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds\Bigg].\end{aligned}\tag{2.11}$$

Therefore, by combining (2.7) with (2.10) and (2.11), we deduce

$$\begin{aligned}u(t)=&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)ds+\frac{t^{\alpha-1}}{\Delta}\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds\\ &-\frac{\Delta_1}{\Gamma(\alpha)}\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds\\ &-\frac{1}{\Gamma(\beta)}\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds+\frac{\Delta_1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}y(s)ds\Bigg],\\ v(t)=&-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)ds+\frac{t^{\beta-1}}{\Delta}\Bigg[\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}y(s)ds\\ &-\frac{\Delta_2}{\Gamma(\beta)}\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds\\ &-\frac{1}{\Gamma(\alpha)}\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds+\frac{\Delta_2}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds\Bigg].\end{aligned}$$

We conclude

$$
\begin{aligned}
u(t) =& \frac{1}{\Gamma(\alpha)}\left[\int_0^t [t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]x(s)ds\right.\\
&\left.+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds-\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds\right]\\
&+\frac{t^{\alpha-1}}{\Delta\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds-\frac{\Delta_1}{\Delta\Gamma(\alpha)}t^{\alpha-1}\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds\\
&+\frac{t^{\alpha-1}}{\Delta\Gamma(\beta)}\left[\int_0^1\int_0^1 a(\tau)\tau^{\beta-1}(1-s)^{\beta-1}dA(\tau)y(s)ds\right.\\
&\left.-\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds\right]\\
=& \frac{1}{\Gamma(\alpha)}\left[\int_0^t [t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]x(s)ds\right.\\
&+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds-\frac{1}{\Delta}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds\\
&+\frac{\Delta_1\Delta_2}{\Delta}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds+\frac{1}{\Delta}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds\\
&\left.-\frac{\Delta_1}{\Delta}t^{\alpha-1}\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds\right]\\
&+\frac{t^{\alpha-1}}{\Delta\Gamma(\beta)}\left[\int_0^1\int_0^1 a(\tau)\tau^{\beta-1}(1-s)^{\beta-1}dA(\tau)y(s)ds-\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds\right]\\
=& \frac{1}{\Gamma(\alpha)}\left\{\int_0^t [t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]x(s)ds+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds+\frac{\Delta_1}{\Delta}t^{\alpha-1}\right.\\
&\left.\times\left[\int_0^1\int_0^1 b(\tau)(\tau)^{\alpha-1}(1-s)^{\alpha-1}dB(\tau)x(s)ds-\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds\right]\right\}\\
&+\frac{t^{\alpha-1}}{\Delta\Gamma(\beta)}\left[\int_0^1\int_0^1 a(\tau)\tau^{\beta-1}(1-s)^{\beta-1}dA(\tau)y(s)ds-\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds\right].
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
u(t) =& \frac{1}{\Gamma(\alpha)}\left\{\int_0^t [t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]x(s)ds+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds+\frac{\Delta_1}{\Delta}t^{\alpha-1}\right.\\
&\times\left[\int_0^1\int_0^s b(\tau)(\tau)^{\alpha-1}(1-s)^{\alpha-1}dB(\tau)x(s)ds+\int_0^1\int_s^1 b(\tau)(\tau)^{\alpha-1}(1-s)^{\alpha-1}dB(\tau)x(s)ds\right.\\
&\left.\left.-\int_0^1\int_s^1 b(\tau)(\tau-s)^{\alpha-1}dB(\tau)x(s)ds\right]\right\}\\
&+\frac{t^{\alpha-1}}{\Delta\Gamma(\beta)}\left[\int_0^1\int_0^s a(\tau)\tau^{\beta-1}(1-s)^{\beta-1}dA(\tau)y(s)ds\right.\\
&\left.+\int_s^1\int_0^1 a(\tau)\tau^{\beta-1}(1-s)^{\beta-1}dA(\tau)y(s)ds-\int_0^1\int_s^1 a(\tau)(\tau-s)^{\beta-1}dA(\tau)y(s)ds\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{\Gamma(\alpha)}\left\{\int_0^t [t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]x(s)ds+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}x(s)ds\right.\\
&+\frac{\Delta_1}{\Delta}t^{\alpha-1}\left[\int_0^1\int_0^s b(\tau)(\tau)^{\alpha-1}(1-s)^{\alpha-1}dB(\tau)x(s)ds\right.\\
&\left.\left.+\int_0^1\int_s^1 b(\tau)[\tau^{\alpha-1}(1-s)^{\alpha-1}-(\tau-s)^{\alpha-1}]dB(\tau)x(s)ds\right]\right\}\\
&+\frac{t^{\alpha-1}}{\Delta\Gamma(\beta)}\left[\int_0^1\int_0^s a(\tau)\tau^{\beta-1}(1-s)^{\beta-1}dA(\tau)y(s)ds\right.\\
&\left.+\int_0^1\int_s^1 a(\tau)[\tau^{\beta-1}(1-s)^{\beta-1}-(\tau-s)^{\beta-1}]dA(\tau)y(s)ds\right]\\
=&\int_0^1 g_1(t,s)x(s)ds+\frac{\Delta_1}{\Delta}t^{\alpha-1}\int_0^1\int_0^1 g_1(\tau,s)b(\tau)dB(\tau)x(s)ds\\
&+\frac{t^{\alpha-1}}{\Delta}\int_0^1\int_0^1 g_2(\tau,s)a(\tau)dA(\tau)y(s)ds\\
=&\int_0^1 G_1(t,s)x(s)ds+\int_0^1 H_1(t,s)y(s)ds.
\end{aligned}
$$

In a similar manner, we deduce

$$
\begin{aligned}
v(t)=&\int_0^1 g_2(t,s)y(s)ds+\frac{\Delta_2}{\Delta}t^{\beta-1}\int_0^1\int_0^1 g_2(\tau,s)a(\tau)dA(\tau)y(s)ds\\
&+\frac{t^{\beta-1}}{\Delta}\int_0^1\int_0^1 g_1(\tau,s)b(\tau)dB(\tau)x(s)ds\\
=&\int_0^1 G_2(t,s)y(s)ds+\int_0^1 H_2(t,s)x(s)ds.
\end{aligned}
$$

Therefore, we obtain the expression (2.2) for the solution of problem (2.1). □

**Lemma 2.6** ([24]). *The functions* $g_1$ *and* $g_2$ *given by* (2.4) *and* (2.5) *have the following properties:*

$$
\frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}\leqslant g_1(t,s)\leqslant\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}\left(\text{or }\ \frac{t^{\alpha-1}(1-t)^{\alpha-1}}{\Gamma(\alpha)}\right)\quad \forall t,s\in[0,1],
$$

$$
\frac{t^{\beta-1}(1-t)s(1-s)^{\beta-1}}{\Gamma(\beta)}\leqslant g_2(t,s)\leqslant\frac{s(1-s)^{\beta-1}}{\Gamma(\beta-1)}\left(\text{or }\ \frac{t^{\beta-1}(1-t)^{\beta-1}}{\Gamma(\beta)}\right),\quad \forall t,s\in[0,1].
$$

The following properties of the Green function play an important role in this paper.

**Lemma 2.7.** *The Green functions* $G_i(t,s), H_i(t,s)$ $(i=1,2)$ *defined by* (2.3) *have the following properties:*

(1) $G_i(t,s), H_i(t,s)$ *are continuous functions on* $[0,1]\times[0,1]$ *and* $G_i(t,s), H_i(t,s)\geqslant 0,\ s,t\in[0,1]$ $(i=1,2)$;

(2) $G_i(t,s)\leqslant k_1 s(1-s)^{\gamma_1}$ (or $k_1t^{\gamma_1}$), $H_i(t,s)\leqslant k_1 s(1-s)^{\gamma_1}$ (or $k_1t^{\gamma_1}$), $G_i(t,s)\geqslant k_2t^{\gamma_2}s(1-s)^{\gamma_2}$, $H_i(t,s)\geqslant k_2t^{\gamma_2}s(1-s)^{\gamma_2}$ $(i=1,2)$, *where*

$$
k_1=\max\left\{\frac{\Delta_1}{\Delta\Gamma(\alpha-1)}\int_0^1 b(\tau)dB(\tau)+\frac{1}{\Gamma(\alpha-1)},\ \frac{\Delta_2}{\Delta\Gamma(\beta-1)}\int_0^1 a(\tau)dA(\tau)+\frac{1}{\Gamma(\beta-1)},\right.
$$

$$\frac{1}{\Delta\Gamma(\alpha-1)}\int_0^1 b(\tau)dB(\tau),\ \frac{1}{\Delta\Gamma(\beta-1)}\int_0^1 a(\tau)dA(\tau)\Big\},$$

$$k_2=\min\Big\{\frac{\Delta_1}{\Delta\Gamma(\alpha)}\int_0^1\tau^{\alpha-1}(1-\tau)b(\tau)dB(\tau),\ \frac{\Delta_2}{\Delta\Gamma(\beta)}\int_0^1\tau^{\beta-1}(1-\tau)a(\tau)dA(\tau),$$
$$\frac{1}{\Delta\Gamma(\alpha)}\int_0^1\tau^{\alpha-1}(1-\tau)b(\tau)dB(\tau),\ \frac{1}{\Delta\Gamma(\beta)}\int_0^1\tau^{\beta-1}(1-\tau)a(\tau)dA(\tau)\Big\},$$

*and* $\gamma_1=\min\{\alpha-1,\beta-1\},\ \ \gamma_2=\max\{\alpha-1,\beta-1\}$.

*Proof.* For any $t,s\in[0,1]$, by (2.2), (2.4), (2.5) and Lemma 2.6, we get

$$\begin{aligned}G_1(t,s)=&g_1(t,s)+\frac{\Delta_1}{\Delta}t^{\alpha-1}\int_0^1 g_1(\tau,s)b(\tau)dB(\tau)\\ \leqslant&\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}+\frac{\Delta_1}{\Delta}t^{\alpha-1}\int_0^1 g_1(\tau,s)b(\tau)dB(\tau)\\ \leqslant&\frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}+\frac{\Delta_1 s(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha-1)}\int_0^1 b(\tau)dB(\tau)\\ =&\left(\frac{\Delta_1}{\Delta\Gamma(\alpha-1)}\int_0^1 b(\tau)dB(\tau)+\frac{1}{\Gamma(\alpha-1)}\right)s(1-s)^{\alpha-1},\end{aligned}$$

or

$$\begin{aligned}G_1(t,s)\leqslant&\frac{t^{\alpha-1}(1-t)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\Delta_1}{\Delta}t^{\alpha-1}\int_0^1\frac{t^{\alpha-1}(1-t)^{\alpha-1}}{\Gamma(\alpha)}b(\tau)dB(\tau)\\ \leqslant&\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{\Delta_1}{\Delta\Gamma(\alpha-1)}t^{\alpha-1}\int_0^1 b(\tau)dB(\tau)\\ \leqslant&k_1t^{\alpha-1}.\end{aligned}$$

In a similar way, we can get

$$\begin{aligned}G_2(t,s)=&g_2(t,s)+\frac{\Delta_2}{\Delta}t^{\beta-1}\int_0^1 g_2(\tau,s)a(\tau)dA(\tau)\\ \leqslant&\left(\frac{\Delta_2}{\Delta\Gamma(\beta-1)}\int_0^1 a(\tau)dA(\tau)+\frac{1}{\Gamma(\beta-1)}\right)s(1-s)^{\beta-1},\end{aligned}$$

or

$$G_2(t,s)\leqslant k_1t^{\beta-1}.$$

On the other hand, we have

$$\begin{aligned}G_1(t,s)=&g_1(t,s)+\frac{\Delta_1}{\Delta}t^{\alpha-1}\int_0^1 g_1(\tau,s)b(\tau)dB(\tau)\\ \geqslant&\frac{\Delta_1}{\Delta}t^{\alpha-1}\int_0^1\frac{\tau^{\alpha-1}(1-\tau)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}b(\tau)dB(\tau)\\ =&\frac{\Delta_1}{\Delta\Gamma(\alpha)}\int_0^1\tau^{\alpha-1}(1-\tau)b(\tau)dB(\tau)t^{\alpha-1}s(1-s)^{\alpha-1}.\end{aligned}$$

In a similar way, we get

$$G_2(t,s)\geqslant\frac{\Delta_2}{\Delta\Gamma(\beta)}\int_0^1\tau^{\beta-1}(1-\tau)a(\tau)dA(\tau)t^{\beta-1}s(1-s)^{\beta-1}.$$

In the same way, we obtain the other inequalities about $H_i(t,s)$ $(i=1,2)$, so we omit it. The proof is complete. □

For convenience in presentation, we present the assumptions to be used later in the following.

($H_0$) $A, B : [0,1] \to \mathbb{R}$ are functions of bounded variation and $\int_0^1 g_i(t,s)b(t)dB(t) > 0, \int_0^1 g_i(t,s)a(t)dA(t) > 0$ $(i=1,2)$ for all $s \in [0,1]$;

($H_1$) $f_i \in C((0,1)\times(0,\infty)\times(0,\infty),[0,\infty))$ may be singular at $t=0,1$ and $y=0$, $f_i(t,x,y)$ is nondecreasing in $x$ and nonincreasing in $y$, and there exist $\lambda_i, \mu_i \in [0,1)$ such that

$$c^{\lambda_i} f_i(t,x,y) \leqslant f_i(t,cx,y), \quad f_i(t,x,cy) \leqslant c^{-\mu_i} f_i(t,x,y), \quad \forall x,y>0,\ c\in(0,1),\ i=1,2.$$

($H_2$) $g_i \in C((0,1)\times(0,\infty)\times(0,\infty),[0,\infty))$ may be singular at $t=0,1$ and $x=0$, $g_i(t,x,y)$ is nonincreasing in $x$ and nondecreasing in $y$, and there exist $\xi_i, \eta_i \in [0,1)$ such that

$$c^{\xi_i} g_i(t,x,y) \leqslant g_i(t,x,cy), \qquad g_i(t,cx,y) \leqslant c^{-\eta_i} g_i(t,x,y), \quad \forall x,y>0,\ c\in(0,1),\ i=1,2.$$

($H_3$) $0 < \int_0^1 p_i(t) f_i(t,1,t^{\gamma_2})dt < \infty$, $0 < \int_0^1 q_i(t) g_i(t,t^{\gamma_2},1)dt < \infty$, $i=1,2$.

*Remark* 2.8.

(1) ($H_1$) implies that

$$f_i(t,cx,y) \leqslant c^{\lambda_i} f_i(t,x,y), \quad f_i(t,x,cy) \leqslant c^{\mu_i} f_i(t,x,y), \ \ \forall x,y>0, \ \ c>1, \ \ i=1,2;$$

(2) ($H_2$) implies that

$$g_i(t,x,cy) \leqslant c^{\xi_i} g_i(t,x,y), \quad g_i(t,x,y) \leqslant c^{\eta_i} g_i(t,cx,y), \ \ \forall x,y>0, \ \ c>1, \ \ i=1,2.$$

*Remark* 2.9. By ($H_1$), ($H_2$) and ($H_3$), we can get

$$0 < \int_0^1 p_i(t) f_i(t,t^{\gamma_2},1)dt < \infty, \quad 0 < \int_0^1 q_i(t) g_i(t,1,t^{\gamma_2})dt < \infty, \ \ i=1,2.$$

For our constructions, we shall consider the Banach space $E = C[0,1]$ equipped with the standard norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let $Q = \{u \in E | u(t) \geqslant 0, t \in [0,1]\}$, $Q$ is a cone of $E$. Similarly, for each $(x,y) \in E \times E$, we write $\|(x,y)\|_1 = \max\{\|x\|,\|y\|\}$ . It is easy to see that $(E\times E, \|\cdot\|_1)$ is a Banach space. We define a cone $P$ of $E \times E$ by

$$P = \{(x,y) \in E\times E : x(t) \geqslant kt^{\gamma_2}\|(x,y)\|_1, \ y(t) \geqslant kt^{\gamma_2}\|(x,y)\|_1, \ t\in[0,1]\},$$

where $k = \frac{k_1}{k_2} \in (0,1)$, in which $k_1$ and $k_2$ are defined by Lemma 2.7. For any $r>0$, let $P_r = \{(x,y)\in P : \|(x,y)\|_1 < r\}$, $\partial P_r = \{(x,y) \in P : \|(x,y)\|_1 = r\}$.

Define an operator $T : P \setminus \{\theta\} \to E \times E$ by

$$T(x,y) = (T_1(x,y), T_2(x,y)),$$

where the operators $T_1, T_2 : P\setminus\{\theta\} \to Q$ are defined by

$$\begin{aligned} T_1(x,y)(t) = &\int_0^1 G_1(t,s)\left[p_1(s)f_1(s,x(s),y(s)) + q_1(s)g_1(s,x(s),y(s))\right] ds \\ &+ \int_0^1 H_1(t,s)\left[p_2(s)f_2(s,x(s),y(s)) + q_2(s)g_2(s,x(s),y(s))\right] ds, \\ T_2(x,y)(t) = &\int_0^1 G_2(t,s)\left[p_2(s)f_2(s,x(s),y(s)) + q_2(s)g_2(s,x(s),y(s))\right] ds \\ &+ \int_0^1 H_2(t,s)\left[p_1(s)f_1(s,x(s),y(s)) + q_1(s)g_1(s,x(s),y(s))\right] ds. \end{aligned}$$

**Lemma 2.10.** *Assume that* $(H_1)$ *and* $(H_2)$ *hold. Then, for any* $0 < r < R < +\infty$, $T : (\overline{P_R} \setminus P_r) \to P$ *is a completely continuous operator.*

*Proof.* Firstly, we claim that $T(x, y)$ is well-defined for $(x, y) \in P \setminus \{\theta\}$. In fact, since $(x, y) \in P \setminus \{\theta\}$, we can see that

$$x(t) \geqslant kt^{\gamma_2}\|(x, y)\|_1 > 0, \quad y(t) \geqslant kt^{\gamma_2}\|(x, y)\|_1 > 0, \quad t \in (0, 1].$$

Let $c$ be a positive number such that $c > 1$ and $\|(x, y)\|_1/c < 1$. From $(H_1)$, $(H_2)$ and Remark 2.8, we have

$$\begin{aligned} f_i(t, x(t), y(t)) &\leqslant f_i(t, c, kt^{\gamma_2}\|(x, y)\|_1) \\ &\leqslant c^{\lambda_i} f_i(t, 1, \frac{kt^{\gamma_2}\|(x, y)\|_1}{c}) \\ &\leqslant c^{\lambda_i}\left(\frac{k\|(x, y)\|_1}{c}\right)^{-\mu_i} f(t, 1, t^{\gamma_2}) \\ &= c^{\lambda_i+\mu_i}(k\|(x, y)\|_1)^{-\mu_i} f_i(t, 1, t^{\gamma_2}), \quad i = 1, 2, \end{aligned} \tag{2.12}$$

$$g_i(t, x(t), y(t)) \leqslant c^{\xi_i+\eta_i}(k\|(x, y)\|_1)^{-\eta_i} g_i(t, t^{\gamma_2}, 1), \quad i = 1, 2. \tag{2.13}$$

Hence, for any $t \in [0, 1]$, we get

$$\begin{aligned} T_1(x, y)(t) \leqslant & k_1 \int_0^1 p_1(s)f_1(s, x(s), y(s)) + q_1(s)g_1(s, x(s), y(s))ds \\ & + k_1 \int_0^1 p_2(s)f_2(s, x(s), y(s)) + q_2(s)g_2(s, x(s), y(s))ds \\ \leqslant & k_1 c^{\lambda_1+\mu_1}(k\|(x, y)\|_1)^{-\mu_1} \int_0^1 p_1(s)f_1(s, 1, s^{\gamma_2})ds \\ & + k_1 c^{\xi_1+\eta_1}(k\|(x, y)\|_1)^{-\eta_1} \int_0^1 q_1(s)g_1(s, s^{\gamma_2}, 1)ds \\ & + k_1 c^{\lambda_2+\mu_2}(k\|(x, y)\|_1)^{-\mu_2} \int_0^1 p_2(s)f_2(s, 1, s^{\gamma_2})ds \\ & + k_1 c^{\xi_2+\eta_2}(k\|(x, y)\|_1)^{-\eta_2} \int_0^1 q_2(s)g_2(s, s^{\gamma_2}, 1)ds \\ < & \infty. \end{aligned}$$

Similarly, we can prove $T_2(x, y)(t) < \infty$. Thus we can say that $T$ is well-defined on $P \setminus \{\theta\}$.

Secondly, we show that $T(\overline{P_R} \setminus P_r) \subset P$. By Lemma 2.7, for all $\tau, t, s \in [0, 1]$, we obtain

$$\begin{aligned} &G_1(t, s) \geqslant kt^{\alpha-1}G_1(\tau, s), && G_2(t, s) \geqslant kt^{\beta-1}G_2(\tau, s), \\ &H_1(t, s) \geqslant kt^{\alpha-1}H_1(\tau, s), && H_2(t, s) \geqslant kt^{\beta-1}H_2(\tau, s), \\ &H_1(t, s) \geqslant kt^{\alpha-1}G_2(\tau, s), && G_1(t, s) \geqslant kt^{\alpha-1}H_2(\tau, s), \\ &H_2(t, s) \geqslant kt^{\beta-1}G_1(\tau, s), && G_2(t, s) \geqslant kt^{\beta-1}H_1(\tau, s). \end{aligned}$$

Hence, for $(x, y) \in (\overline{P_R} \setminus P_r)$, $t \in [0, 1]$, we have

$$\begin{aligned} T_1(x, y)(t) \geqslant & kt^{\alpha-1} \int_0^1 G_1(\tau, s)\left[p_1(s)f_1(s, x(s), y(s)) + q_1(s)g_1(s, x(s), y(s))\right] ds \\ & + kt^{\alpha-1} \int_0^1 H_1(\tau, s)\left[p_2(s)f_2(s, x(s), y(s)) + q_2(s)g_2(s, x(s), y(s))\right] ds \\ \geqslant & kt^{\gamma_2}T_1(x, y)(\tau), \quad \forall \tau \in [0, 1], \end{aligned}$$

$$
\begin{aligned}
T_1(x,y)(t) \geqslant & kt^{\alpha-1}\int_0^1 H_2(\tau,s)\,[p_1(s)f_1(s,x(s),y(s))+q_1(s)g_1(s,x(s),y(s))]\,ds \\
& + kt^{\alpha-1}\int_0^1 G_2(\tau,s)\,[p_2(s)f_2(s,x(s),y(s))+q_2(s)g_2(s,x(s),y(s))]\,ds \\
\geqslant & kt^{\gamma_2}T_2(x,y)(\tau), \quad \forall \tau\in[0,1].
\end{aligned}
$$

Then, $T_1(x,y)(t)\geqslant kt^{\gamma_2}\|T_1(x,y)\|$ and $T_1(x,y)(t)\geqslant kt^{\gamma_2}\|T_2(x,y)\|$, that is,

$$
T_1(x,y)(t)\geqslant kt^{\gamma_2}\|(T_1(x,y),T_2(x,y))\|_1.
$$

In the same way, we can prove that

$$
T_2(x,y)(t)\geqslant kt^{\gamma_2}\|(T_1(x,y),T_2(x,y))\|_1.
$$

Therefore, $T(\overline{P_R}\setminus P_r)\subset P$.

Next, we prove that $T$ is a compact operator. Suppose $V\subset\overline{P_R}\setminus P_r$ is an arbitrary bounded set in $E\times E$. Then from the above proof, we know that $T(V)$ is uniformly bounded. In the following, we shall show that $T(V)$ is equicontinuous on $[0,1]$. For all $(x,y)\in V,\ t\in[0,1]$, by using Lemma 2.5, we have

$$
\begin{aligned}
T_1(x,y)(t) = & \int_0^1 G_1(t,s)\big[p_1(s)f_1(s,x(s),y(s))+q_1(s)g_1(s,x(s),y(s))\big]ds \\
& + \int_0^1 H_1(t,s)\big[p_2(s)f_2(s,x(s),y(s))+q_2(s)g_2(s,x(s),y(s))\big]ds \\
= & \frac{\Delta_1}{\Delta}t^{\alpha-1}\int_0^1\left(\int_0^1 g_1(\tau,s)b(\tau)dB(\tau)\right)\big[p_1(s)f_1(s,x(s),y(s)) \\
& + q_1(s)g_1(s,x(s),y(s))\big]ds \\
& + \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big[p_1(s)f_1(s,x(s),y(s)) \\
& + q_1(s)g_1(s,x(s),y(s))\big]ds \\
& + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\big[p_1(s)f_1(s,x(s),y(s)) \\
& + q_1(s)g_1(s,x(s),y(s))\big]ds \\
& + \frac{t^{\alpha-1}}{\Delta}\int_0^1\left(\int_0^1 g_2(\tau,s)a(\tau)dA(\tau)\right)\big[p_2(s)f_2(s,x(s),y(s)) \\
& + q_2(s)g_2(s,x(s),y(s))\big]ds.
\end{aligned}
$$

Differentiating the above formula with respect to $t$ and combining $(H_1)$ and $(H_2)$, we obtain

$$
\begin{aligned}
|T_1(x,y)'(t)| = & \frac{(\alpha-1)\Delta_1}{\Delta}t^{\alpha-2}\int_0^1\left(\int_0^1 g_1(\tau,s)b(\tau)dB(\tau)\right)\big[p_1(s)f_1(s,x(s),y(s)) \\
& + q_1(s)g_1(s,x(s),y(s))\big]ds \\
& + \int_0^t \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}-(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}\big[p_1(s)f_1(s,x(s),y(s)) \\
& + q_1(s)g_1(s,x(s),y(s))\big]ds \\
& + \int_t^1 \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\big[p_1(s)f_1(s,x(s),y(s)) \\
& + q_1(s)g_1(s,x(s),y(s))\big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\alpha-1)t^{\alpha-2}}{\Delta}\int_0^1\left(\int_0^1 g_2(\tau,s)a(\tau)dA(\tau)\right)\big[p_2(s)f_2(s,x(s),y(s))\\
&+q_2(s)g_2(s,x(s),y(s))\big]ds\\
\leqslant&\frac{(\alpha-1)\Delta_1}{\Delta}\int_0^1\left(\int_0^1 g_1(\tau,s)b(\tau)dB(\tau)\right)[p_1(s)f_1(s,x(s),y(s))\\
&+q_1(s)g_1(s,x(s),y(s))]ds\\
&+\int_0^t\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}-(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}\big[p_1(s)f_1(s,x(s),y(s))\\
&+q_1(s)g_1(s,x(s),y(s))\big]ds\\
&+\int_t^1\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\big[p_1(s)f_1(s,x(s),y(s))\\
&+q_1(s)g_1(s,x(s),y(s))\big]ds\\
&+\frac{(\alpha-1)t^{\alpha-2}}{\Delta}\int_0^1\left(\int_0^1 g_2(\tau,s)a(\tau)dA(\tau)\right)\big[p_2(s)f_2(s,x(s),y(s))\\
&+q_2(s)g_2(s,x(s),y(s))\big]ds\\
\leqslant&(\alpha-1)k_1c^{\lambda_1+\mu_1}(k\|(x,y)\|_1)^{-\mu_1}\int_0^1 p_1(s)f_1(s,1,s^{\gamma_2})ds\\
&+(\alpha-1)k_1c^{\xi_1+\eta_1}(k\|(x,y)\|_1)^{-\eta_1}\int_0^1 q_1(s)g_1(s,s^{\gamma_2},1)ds\\
&+k_1c^{\lambda_1+\mu_1}(k\|(x,y)\|_1)^{-\mu_1}\int_0^t\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}-(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}\\
&p_1(s)f_1(s,1,s^{\gamma_2})ds\\
&+k_1c^{\xi_1+\eta_1}(k\|(x,y)\|_1)^{-\eta_1}\int_0^t\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}-(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}\\
&q_1(s)g_1(s,s^{\gamma_2},1)ds\\
&+k_1c^{\lambda_1+\mu_1}(k\|(x,y)\|_1)^{-\mu_1}\int_t^1\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)f_1(s,1,s^{\gamma_2})ds\\
&+k_1c^{\xi_1+\eta_1}(k\|(x,y)\|_1)^{-\eta_1}\int_t^1\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}}{\Gamma(\alpha)}q_1(s)g_1(s,s^{\gamma_2},1)ds\\
&+(\alpha-1)k_1c^{\lambda_2+\mu_2}(k\|(x,y)\|_1)^{-\mu_2}\int_0^1 p_2(s)f_2(s,1,s^{\gamma_2})ds\\
&+(\alpha-1)k_1c^{\xi_2+\eta_2}(k\|(x,y)\|_1)^{-\eta_2}\int_0^1 q_2(s)g_2(s,s^{\gamma_2},1)ds\\
\leqslant&c^{\lambda_1+\mu_1}(kr)^{-\mu_1}\left[(\alpha-1)k_1\int_0^1 p_1(s)f_1(s,1,s^{\gamma_2})ds\right.\\
&+\int_0^t\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}-(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}p_1(s)f_1(s,1,s^{\gamma_2})ds\\
&\left.+\int_t^1\frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}}{\Gamma(\alpha)}p_1(s)f_1(s,1,s^{\gamma_2})ds\right]\\
&+c^{\xi_1+\eta_1}(kr)^{-\eta_1}\left[(\alpha-1)k_1\int_0^1 q_1(s)g_1(s,s^{\gamma_2},1)ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}-(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)} q_1(s)g_1(s,s^{\gamma_2},1)ds\\
&+\left.\int_t^1 \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}}{\Gamma(\alpha)} q_1(s)g_1(s,s^{\gamma_2},1)ds\right]\\
&+(\alpha-1)k_1c^{\lambda_2+\mu_2}(kr)^{-\mu_2}\int_0^1 p_2(s)f_2(s,1,s^{\gamma_2})ds\\
&+(\alpha-1)k_1c^{\xi_2+\eta_2}(kr)^{-\eta_2}\int_0^1 q_2(s)g_2(s,s^{\gamma_2},1)ds =: K(t).
\end{aligned}
$$

Exchanging the integration order, we have

$$
\begin{aligned}
\int_0^1 K(t)dt =&c^{\lambda_1+\mu_1}(kr)^{-\mu_1}\left[(\alpha-1)k_1\int_0^1 p_1(s)f_1(s,1,s^{\gamma_2})ds\right]\\
&+c^{\xi_1+\eta_1}(kr)^{-\eta_1}\left[(\alpha-1)k_1\int_0^1 q_1(s)g_1(s,s^{\gamma_2},1)ds\right]\\
&+(\alpha-1)k_1c^{\lambda_2+\mu_2}(kr)^{-\mu_2}\int_0^1 p_2(s)f_2(s,1,s^{\gamma_2})ds\\
&+(\alpha-1)k_1c^{\xi_2+\eta_2}(kr)^{-\eta_2}\int_0^1 q_2(s)g_2(s,s^{\gamma_2},1)ds\\
&<+\infty.
\end{aligned}
$$

From the absolute continuity of the integral, we know that $T_1(V)$ is equicontinuous on [0,1]. Thus, according to the Ascoli-Arzela theorem, $T_1(V)$ is a relatively compact set. In the same way, we can prove that $T_2(V)$ is a relatively compact set. Therefore, $T(V)$ is relatively compact.

Finally, we prove that $T : (\overline{P_R}\setminus P_r) \to Q$ is continuous. We need to prove only $T_1, T_2 : (\overline{P_R}\setminus P_r) \to Q$ are continuous. Suppose that $(x_n, y_n), (x_0, y_0) \in \overline{P_R}\setminus P_r$ and $\|(x_n, y_n) - (x_0, y_0)\|_1 \to 0$ $(n \to \infty)$. Let $S = \sup\{\|(x_n, y_n)\|_1 | n = 0, 1, 2, \cdots\}$. We choose a positive constant $M$ such that $S/M < 1$ and $M > 1$. From (2.12) and (2.13), for any $t \in (0, 1)$, we know

$$
\begin{aligned}
&f_i(t,x_n(t),y_n(t)) \leqslant M^{\lambda_i+\mu_i}(kr)^{-\mu_i}f_i(t,1,t^{\gamma_2}),\quad n=0,1,2,\cdots,\quad i=1,2,\\
&g_i(t,x_n(t),y_n(t)) \leqslant M^{\xi_i+\eta_i}(kr)^{-\eta_i}g_i(t,t^{\gamma_2},1),\quad n=0,1,2,\cdots,\quad i=1,2.
\end{aligned}
\tag{2.14}
$$

Then by Lemma 2.7, for any $t \in [0, 1]$, we get

$$
\begin{aligned}
|T_1(x_n,y_n)(t)-T_1(x_0,y_0)(t)| \leqslant& k_1\int_0^1 \big[|p_1(s)||f_1(s,x_n(s),y_n(s))-f_1(s,x_0(s),y_0(s))|\\
&+|q_1(s)||g_1(s,x_n(s),y_n(s))-g_1(s,x_0(s),y_0(s))|\big]ds\\
&+k_1\int_0^1 \big[|p_2(s)||f_2(s,x_n(s),y_n(s))-f_2(s,x_0(s),y_0(s))|\\
&+|q_2(s)||g_2(s,x_n(s),y_n(s))-g_2(s,x_0(s),y_0(s))|\big]ds.
\end{aligned}
\tag{2.15}
$$

For any $\epsilon > 0$, by $(H_3)$, there exists a positive number $\delta \in (0, \frac{1}{2})$ such that

$$
\begin{aligned}
&\int_{H_{(\delta)}} k_1M^{\lambda_i+\mu_i}(kr)^{-\mu_i}p_i(s)f_i(t,1,t^{\gamma_2})ds < \frac{\epsilon}{4},\\
&\int_{H_{(\delta)}} k_1M^{\xi_i+\eta_i}(kr)^{-\eta_i}q_i(s)g_i(t,t^{\gamma_2},1)ds < \frac{\epsilon}{4},
\end{aligned}
\tag{2.16}
$$

where $H_{(\delta)} = [0,\delta] \cup [1-\delta, 1]$. On the other hand, for $(x,y) \in \overline{P_R} \setminus P_r$ and $t \in [\delta, 1-\delta]$, we have

$$0 < rk\delta \leqslant x(t), \quad y(t) \leqslant R. \tag{2.17}$$

Since $f_i(t,x,y)$ and $g_i(t,x,y)$ $(i = 1,2)$ are uniformly continuous in $[\delta, 1-\delta] \times [rk\delta, b] \times [rk\delta, b]$, we have

$$\begin{aligned}\lim_{n\to+\infty} &|f_i(s,x_n(s),y_n(s)) - f_i(s,x_0(s),y_0(s))|\\ &= \lim_{n\to+\infty} |g_i(s,x_n(s),y_n(s)) - g_i(s,x_0(s),y_0(s))|\\ &= 0,\end{aligned} \tag{2.18}$$

holds uniformly on $[\delta, 1-\delta]$ for $s$. Then the Lebesgue dominated convergence theorem yields that

$$\begin{aligned}&\int_\delta^{1-\delta} |p_i(s)|\,|f_i(s,x_n(s),y_n(s)) - f_i(s,x_0(s),y_0(s))|\,ds \to 0,\\ &\int_\delta^{1-\delta} |q_i(s)|\,|g_i(s,x_n(s),y_n(s)) - g_i(s,x_0(s),y_0(s))|\,ds \to 0, \quad n \to \infty.\end{aligned} \tag{2.19}$$

Thus, for above $\epsilon > 0$, there exists a natural number $N$ such that for $n > N$ we have

$$\begin{aligned}k_1 \int_\delta^{1-\delta} \Big[&|p_1(s)||f_1(s,x_n(s),y_n(s)) - f_1(s,x_0(s),y_0(s))|\\ &+ |q_1(s)||g_1(s,x_n(s),y_n(s)) - g_1(s,x_0(s),y_0(s))|\Big]\,ds\\ &+ k_1 \int_\delta^{1-\delta} \Big[|p_2(s)||f_2(s,x_n(s),y_n(s)) - f_2(s,x_0(s),y_0(s))|\\ &+ |q_2(s)||g_2(s,x_n(s),y_n(s)) - g_2(s,x_0(s),y_0(s))|\Big]\,ds < \frac{\epsilon}{2}.\end{aligned} \tag{2.20}$$

It follows from (2.14), (2.15), (2.16), (2.17), (2.18), (2.19), (2.20) that when $n > N$

$$\begin{aligned}\|T_1&(x_n,y_n) - T_1(x_0,y_0)\|\\ &\leqslant k_1 \int_0^1 \big[|p_1(s)||f_1(s,x_n(s),y_n(s)) - f_1(s,x_0(s),y_0(s))|\\ &\quad + |q_1(s)||g_1(s,x_n(s),y_n(s)) - g_1(s,x_0(s),y_0(s))|\big]\,ds\\ &\quad + k_1 \int_0^1 \big[|p_2(s)||f_2(s,x_n(s),y_n(s)) - f_2(s,x_0(s),y_0(s))|\\ &\quad + |q_2(s)||g_2(s,x_n(s),y_n(s)) - g_2(s,x_0(s),y_0(s))|\big]\,ds\\ &\leqslant k_1 \int_{H(\delta)} M^{\lambda_1+\mu_1}(kr)^{-\mu_1} p_1(s) f_1(s,1,s^{\gamma_2}) + c^{\xi_1+\eta_1}(kr)^{-\eta_1} q_1(s) g_1(s,s^{\gamma_2},1)\,ds\\ &\quad + k_1 \int_{H(\delta)} M^{\lambda_2+\mu_2}(kr)^{-\mu_2} p_2(s) f_2(s,1,s^{\gamma_2}) + M^{\xi_2+\eta_2}(kr)^{-\eta_2} q_2(s) g_2(s,s^{\gamma_2},1)\,ds\\ &\quad + k_1 \int_\delta^{1-\delta} \big[|p_1(s)||f_1(s,x_n(s),y_n(s)) - f_1(s,x_0(s),y_0(s))|\\ &\quad + |q_1(s)||g_1(s,x_n(s),y_n(s)) - g_1(s,x_0(s),y_0(s))|\big]\,ds\\ &\quad + k_1 \int_\delta^{1-\delta} \big[|p_2(s)||f_2(s,x_n(s),y_n(s)) - f_2(s,x_0(s),y_0(s))|\\ &\quad + |q_2(s)||g_2(s,x_n(s),y_n(s)) - g_2(s,x_0(s),y_0(s))|\big]\,ds < \epsilon.\end{aligned}$$

This implies that $T_1 : (\overline{P_R} \setminus P_r) \to Q$ is continuous. Similarly, we can prove that $T_2 : (\overline{P_R} \setminus P_r) \to Q$ is continuous. So, $T : (\overline{P_R} \setminus P_r) \to Q$ is continuous. By summing up, we get that $T : (\overline{P_R} \setminus P_r) \to P$ is completely continuous. □

To prove the main results, we need the following well-known fixed point theorem.

**Lemma 2.11** ([5]). *Let $P$ be a positive cone in a Banach space $E$, $\Omega_1$ and $\Omega_2$ be two bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$, $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator, where $\theta$ denotes the zero element of $E$ and $P$ is a cone of $E$. Suppose that one of the following two conditions holds:*

(i) $\|Au\| \leqslant \|u\|, \ \forall u \in P \cap \partial\Omega_1; \quad \|Au\| \geqslant \|u\|, \ \forall u \in P \cap \partial\Omega_2;$

(ii) $\|Au\| \geqslant \|u\|, \ \forall u \in P \cap \partial\Omega_1; \quad \|Au\| \leqslant \|u\|, \ \forall u \in P \cap \partial\Omega_2.$

*Then $A$ has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

## 3. Main results

In this section, we shall give sufficient conditions for the existence and uniqueness of a positive solution for the BVP (1.1).

**Theorem 3.1.** *Assume that conditions $(H_0)$-$(H_3)$ hold. Then the BVP (1.1) has at least one positive solution $(x^*, y^*)$ and there exists a real number $0 < m < 1$ such that*

$$mt^{\gamma_1} \leqslant x^*(t) \leqslant \frac{1}{m} t^{\gamma_1}, \quad mt^{\gamma_1} \leqslant y^*(t) \leqslant \frac{1}{m} t^{\gamma_1}, \quad t \in [0,1], \tag{3.1}$$

*where $\gamma_1 = \min\{\alpha - 1, \beta - 1\}$.*

*Proof.* We first prove that the differential system (1.1) has at least one positive solution $(x^*, y^*)$. Choose $d$ and $D$ such that

$$0 < d \leqslant \min_{i=1,2} \left\{ \left( \left(\frac{1}{4}\right)^{\gamma_2} k_2 k^{\max\{\lambda_i,\mu_i\}} \int_0^1 s(1-s)^{\gamma_2} (p_i(s) f_i(s, s^{\gamma_2}, 1) \right.\right.$$
$$\left.\left. + q_i(s) g_i(s, 1, s^{\gamma_2}))ds \right)^{\frac{1}{1-\max\{\lambda_i,\mu_i\}}}, \frac{1}{2} \right\},$$
$$D \geqslant \max \left\{ \left[ k_1 \left( \int_0^1 (p_1(s) f_1(s, 1, s^{\gamma_2}) + q_1(s) g_1(s, s^{\gamma_2}, 1))ds + \int_0^1 (p_2(s) f_2(s, 1, s^{\gamma_2}) \right.\right.\right.$$
$$\left.\left.\left. + q_2(s) g_2(s, s^{\gamma_2}, 1))ds \right) \right]^{\frac{1}{1-\max\{\lambda_1,\lambda_2,\xi_1,\xi_2\}}}, \frac{1}{k}, 2 \right\}.$$

Clearly $0 < d < 1 < D$. By Lemma 2.10, $T : \overline{P_D} \setminus P_d \to P$ is completely continuous. Extend $T$ (denote $T$ yet) to $T : \overline{P_D} \to P$ which is completely continuous. Then, for $(x, y) \in \partial P_d$, we have

$$dkt^{\gamma_2} \leqslant x(t), \ \ y(t) \leqslant d, \quad t \in [0,1].$$

By Remark 2.8 and $(H_1)$-$(H_3)$, we get

$$T_i(x,y)(t) \geqslant \left(\frac{1}{4}\right)^{\gamma_2} k_2 \int_0^1 s(1-s)^{\gamma_2} (p_i(s) f_i(s, dks^{\gamma_2}, d) + q_i(s) g_i(s, d, dks^{\gamma_2}))ds$$

$$
\begin{aligned}
&\geqslant \left(\frac{1}{4}\right)^{\gamma_2} k_2 \int_0^1 s(1-s)^{\gamma_2}(p_i(s)f_i(s, dks^{\gamma_2}, 1) + q_i(s)g_i(s, 1, dks^{\gamma_2}))ds \\
&\geqslant \left(\frac{1}{4}\right)^{\gamma_2} k_2 \int_0^1 s(1-s)^{\gamma_2}(d^{\lambda_i}k^{\lambda_i}p_i(s)f_i(s, s^{\gamma_2}, 1) + d^{\xi_i}k^{\eta_i}q_i(s)g_i(s, 1, s^{\gamma_2}))ds \\
&\geqslant \left(\frac{1}{4}\right)^{\gamma_2} k_2 d^{\max\{\lambda_i,\xi_i\}}k^{\max\{\lambda_i,\xi_i\}} \int_0^1 s(1-s)^{\gamma_2}(p_i(s)f_i(s, s^{\gamma_2}, 1) + q_i(s)g_i(s, 1, s^{\gamma_2}))ds \\
&\geqslant d = \|(x,y)\|_1 \quad i = 1,2,\ t \in \left[\frac{1}{4}, \frac{3}{4}\right].
\end{aligned}
$$

This guarantees that

$$\|T(x,y)\|_1 \geqslant \|(x,y)\|_1, \ \ \forall (x,y) \in \partial P_d. \tag{3.2}$$

On the other hand, for any $(x,y) \in \partial P_D$, we have

$$Dkt^{\gamma_2} \leqslant x(t), \quad y(t) \leqslant D, \quad t \in [0,1].$$

Therefore, by Lemma 2.7, for any $(x,y) \in \partial P_D$ and $t \in [0,1]$, we have

$$
\begin{aligned}
T_i(x,y)(t) \leqslant & k_1 \int_0^1 p_1(s)f_1(s, D, Dks^{\gamma_2}) + q_1(s)g_1(s, Dks^{\gamma_2}, D)ds \\
& \quad + k_1 \int_0^1 p_2(s)f_2(s, D, Dks^{\gamma_2}) + q_2(s)g_2(s, Dks^{\gamma_2}, D)ds \\
\leqslant & k_1 \int_0^1 p_1(s)f_1(s, D, s^{\gamma_2}) + q_1(s)g_1(s, s^{\gamma_2}, D)ds \\
& \quad + k_1 \int_0^1 p_2(s)f_2(s, D, s^{\gamma_2}) + q_2(s)g_2(s, s^{\gamma_2}, D)ds \\
\leqslant & k_1 \int_0^1 D^{\lambda_1}p_1(s)f_1(s, 1, s^{\gamma_2}) + D^{\xi_1}q_1(s)g_1(s, s^{\gamma_2}, 1)ds \\
& \quad + k_1 \int_0^1 D^{\lambda_2}p_2(s)f_2(s, 1, s^{\gamma_2}) + D^{\xi_2}q_2(s)g_2(s, s^{\gamma_2}, 1)ds \\
\leqslant & k_1 D^{\max\{\lambda_1,\lambda_2,\xi_1,\xi_2\}} \left( \int_0^1 p_1(s)f_1(s, 1, s^{\gamma_2}) + q_1(s)g_1(s, s^{\gamma_2}, 1)ds \right. \\
& \quad \left. + \int_0^1 p_2(s)f_2(s, 1, s^{\gamma_2}) + q_2(s)g_2(s, s^{\gamma_2}, 1)ds \right) \\
\leqslant & D = \|(x,y)\|_1.
\end{aligned}
$$

This guarantees that

$$\|T(x,y)\|_1 \leqslant \|(x,y)\|_1, \ \ \forall (x,y) \in \partial P_D. \tag{3.3}$$

By the complete continuity of $T$, (3.2) and (3.3), and Lemma 2.11, we obtain that $T$ has a fixed point $(x^*, y^*)$ in $\overline{P_D} \setminus P_d$. Consequently, BVP (1.1) has a positive solution $(x^*, y^*)$ in $\overline{P_D} \setminus P_d$.

Next we prove that there exists a real number $0 < m < 1$ satisfying (3.1). Firstly, we show that for any $\theta \in (0, \frac{1}{2})$ we have

$$mt^{\gamma_1} \leqslant x^*(t) \leqslant \frac{1}{m}t^{\gamma_1}, \quad mt^{\gamma_1} \leqslant y^*(t) \leqslant \frac{1}{m}t^{\gamma_1}, \quad t \in [\theta, 1]. \tag{3.4}$$

From Lemma 2.10, we know that $(x^*, y^*) \in P \setminus \{\theta\}$. So we obtain that

$$0 < k\|(x^*, y^*)\|_1 t^{\gamma_2} \leqslant x^*(t), \quad y^*(t) \leqslant \|(x^*, y^*)\|_1.$$

Let $h$ be a constant such that $\frac{\|(x^*,y^*)\|_1}{h} < 1$ and $h > \frac{1}{d} > 1$. By Lemma 2.5 we get

$$\begin{aligned}
x^*(t) \leqslant & k_1 t^{\gamma_1}\left[\int_0^1 p_1(s) f_1\left(s, h, \frac{k\|(x^*, y^*)\|_1}{h} s^{\gamma_2}\right)\right. \\
& + q_1(s) g_1\left(s, \frac{k\|(x^*, y^*)\|_1}{h} s^{\gamma_2}, h\right) ds \\
& + \int_0^1 p_2(s) f_2\left(s, h, \frac{k\|(x^*, y^*)\|_1}{h} s^{\gamma_2}\right) \\
& \left. + q_2(s) g_2\left(s, \frac{k\|(x^*, y^*)\|_1}{h} s^{\gamma_2}, h\right) ds\right] \\
\leqslant & k_1 t^{\gamma_1}\left[\int_0^1 h^{\lambda_1+\mu_1}(k\|(x^*, y^*)\|_1)^{-\mu_1} p_1(s) f_1(s, 1, s^{\gamma_2})\right. \\
& + h^{\xi_1+\eta_1}(k\|(x^*, y^*)\|_1)^{-\eta_1} q_1(s) g_1(s, s^{\gamma_2}, 1) ds \\
& + \int_0^1 h^{\lambda_2+\mu_2}(k\|(x^*, y^*)\|_1)^{-\mu_2} p_2(s) f_2(s, 1, s^{\gamma_2}) \\
& \left. + h^{\xi_2+\eta_2}(k\|(x^*, y^*)\|_1)^{-\eta_2} q_2(s) g_2(s, s^{\gamma_2}, 1) ds\right] \\
=: & C t^{\gamma_1}, \quad t \in [\theta, 1].
\end{aligned}$$

On the other hand, it is obvious to see that $\gamma_2 - \gamma_1 \geqslant 0$, where $\gamma_1 = \min\{\alpha-1, \beta-1\}$, $\gamma_2 = \max\{\alpha-1, \beta-1\}$. So we get

$$\begin{aligned}
x^*(t) \geqslant & k\|(x^*, y^*)\|_1 t^{\gamma_2} \\
= & k\|(x^*, y^*)\|_1 t^{\gamma_2-\gamma_1} t^{\gamma_1} \\
\geqslant & k\|(x^*, y^*)\|_1 \theta^{\gamma_2-\gamma_1} t^{\gamma_1}, \quad t \in [\theta, 1].
\end{aligned}$$

In the same way, we can prove that $y^*(t) \leqslant Ct^{\gamma_1}$ and $y^*(t) \geqslant k\|(x^*, y^*)\|_1 \theta^{\gamma_2-\gamma_1} t^{\gamma_1}, t \in [\theta, 1]$. Then, we pick out $m$ such that

$$m = \min\left\{k\theta^{(\gamma_2-\gamma_1)}\|(x^*, y^*)\|_1, \frac{1}{C}, \frac{1}{2}\right\},$$

which implies that (3.4) holds. Moreover, from the arbitrariness of $\theta$, we get that for any $t \in (0, 1]$, (3.4) is satisfied. Specially, when $t = 0$, by the boundary value conditions of (1.1), we have $x^*(0) = y^*(0) = 0$. So that we get that for any $t \in [0, 1]$, (3.1) holds. This completes the proof of Theorem 3.1. □

**Theorem 3.2.** *Assume that conditions* $(H_0)$-$(H_3)$ *hold. If* $\lambda_i + \mu_i < 1$ *and* $\xi_i + \eta_i < 1$ $(i = 1, 2)$, *then the BVP* (1.1) *has a unique positive solution* $(x^*, y^*)$ *and it satisfies* (3.1).

*Proof.* By assuming the contrary, we find that the BVP (1.1) has a positive solution $(x_*, y_*)$ different from $(x^*, y^*)$. By (3.1) there exist $\rho_1, \rho_2 > 0$ such that

$$\begin{aligned}
\rho_1 t^{\gamma_1} \leqslant x^*(t), \quad y^*(t) \leqslant \frac{1}{\rho_1} t^{\gamma_1}, \quad \forall t \in [0, 1], \\
\rho_2 t^{\gamma_1} \leqslant x_*(t), \quad y_*(t) \leqslant \frac{1}{\rho_2} t^{\gamma_1}, \quad \forall t \in [0, 1].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \rho_1\rho_2 x_*(t) \leqslant x^*(t) \leqslant \frac{1}{\rho_1\rho_2} x_*(t), \\
& \rho_1\rho_2 y_*(t) \leqslant y^*(t) \leqslant \frac{1}{\rho_1\rho_2} y_*(t), \quad \forall t \in [0, 1].
\end{aligned}$$

Clearly, $\rho_1\rho_2 \neq 1$. Put

$$\rho^* = \sup\left\{\rho > 0 \mid \rho x_*(t) \leqslant x^*(t) \leqslant \frac{1}{\rho} x_*(t),\ \rho y_*(t) \leqslant y^*(t) \leqslant \frac{1}{\rho} y_*(t),\ \forall t \in [0, 1]\right\}.$$

It is easy to see that $1 > \rho^* \geqslant \rho_1\rho_2 > 0$ and

$$\rho^* x_*(t) \leqslant x^*(t) \leqslant \frac{1}{\rho^*} x_*(t), \quad \rho^* y_*(t) \leqslant y^*(t) \leqslant \frac{1}{\rho^*} y_*(t), \quad \forall t \in [0,1].$$

By $(H_1)$ and $(H_2)$, we have

$$\begin{aligned} f_i(t, x^*(t), y^*(t)) \geqslant & f_i(t, \rho^* x_*(t), \frac{1}{\rho^*} y_*(t)) \\ \geqslant & (\rho^*)^{\lambda_i+\mu_i} f_i(t, x_*(t), y_*(t)) \\ \geqslant & (\rho^*)^{\sigma} f_i(t, x_*(t), y_*(t)), \end{aligned}$$

$$\begin{aligned} g_i(t, x^*(t), y^*(t)) \geqslant & g_i(t, \rho^* x_*(t), \frac{1}{\rho^*} y_*(t)) \\ \geqslant & (\rho^*)^{\xi_i+\eta_i} g_i(t, x_*(t), y_*(t)) \\ \geqslant & (\rho^*)^{\sigma} g_i(t, x_*(t), y_*(t)), \qquad i = 1, 2, \end{aligned}$$

where $\sigma = \max\{\lambda_i + \mu_i, \xi_i + \eta_i, i = 1, 2\}$ such that $\sigma < 1$. Therefore, we have

$$\begin{aligned} x^*(t) = & T_1(x^*, y^*)(t) = \int_0^1 G_1(t,s)\,[p_1(s) f_1(s, x^*(s), y^*(s)) + q_1(s) g_1(s, x^*(s), y^*(s))]\,ds \\ & + \int_0^1 H_1(t,s)\,[p_2(s) f_2(s, x^*(s), y^*(s)) + q_2(s) g_2(s, x^*(s), y^*(s))]\,ds \\ \geqslant & (\rho^*)^{\sigma} \left[ \int_0^1 G_1(t,s)\,[p_1(s) f_1(s, x^*(s), y^*(s)) + q_1(s) g_1(s, x^*(s), y^*(s))]\,ds \right. \\ & \left. + \int_0^1 H_1(t,s)\,[p_2(s) f_2(s, x^*(s), y^*(s)) + q_2(s) g_2(s, x^*(s), y^*(s))]\,ds \right] \\ = & (\rho^*)^{\sigma} T_1(x_*, y_*)(t) = (\rho^*)^{\sigma} x_*(t). \end{aligned}$$

Similarly, we can get

$$y^*(t) \geqslant (\rho^*)^{\sigma} y_*(t), \quad x_*(t) \geqslant (\rho^*)^{\sigma} x^*(t), \quad y_*(t) \geqslant (\rho^*)^{\sigma} y^*(t).$$

Noticing that $(\rho^*)^{\sigma} > \rho^*$ $(0 < \rho^*, \sigma < 1)$, we get to a contradiction with the maximality of $\rho^*$. Thus, the BVP (1.1) has a unique positive solution $(x^*, y^*)$. This completes the proof of Theorem 3.2. □

*Remark* 3.3. Compared with the result in [3, 21], we can see that for any $\alpha \in (n-1, n], \beta \in (m-1, m], n, m \in \mathbb{N}$, we can get the uniqueness of positive solutions of the BVP (1.1). That is, we do not need the condition of $\alpha = \beta$. So our result is better than that in [3, 21].

## 4. An example

We give an explicit example to illustrate our main result in Section 3. Let us consider the singular differential system with coupled boundary conditions

$$\begin{cases} D_{0^+}^{\frac{5}{2}} x(t) + \dfrac{\sqrt{x}}{\sqrt[3]{y(1-t)t}} + \dfrac{\sqrt[3]{y}}{\sqrt{x}} = 0, & t \in (0,1), \\ D_{0^+}^{\frac{7}{2}} y(t) + \dfrac{\sqrt[3]{x}}{\sqrt{y}} + \dfrac{\sqrt{y}}{\sqrt[3]{x(1-t)t}} = 0, & t \in (0,1), \\ x(0) = x'(0) = 0, \ \ x(1) = y\left(\dfrac{1}{3}\right) + y\left(\dfrac{1}{2}\right), \\ y(0) = y'(0) = y''(0) = 0, \ \ y(1) = \displaystyle\int_0^1 x(s)\,ds^2. \end{cases} \tag{4.1}$$

Let $\alpha = \frac{5}{2}, \beta = \frac{7}{2}$,

$$f_1(t,x,y) = \frac{\sqrt{x}}{\sqrt[3]{y(1-t)t}}, \quad g_1(t,x,y) = \frac{\sqrt[3]{y}}{\sqrt{x}},$$
$$f_2(t,x,y) = \frac{\sqrt[3]{x}}{\sqrt{y}}, \quad g_2(t,x,y) = \frac{\sqrt{y}}{\sqrt[3]{x(1-t)t}},$$
$$a(t) = b(t) = 1,$$

$$A(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{3}\right), \\ 1, & t \in \left[\frac{1}{3}, \frac{1}{2}\right), \\ 2, & t \in \left[\frac{1}{2}, 1\right], \end{cases}$$
$$B(t) = t^2.$$
$$\lambda_1 = \mu_2 = \frac{1}{2}, \quad \lambda_2 = \mu_1 = \frac{1}{3}, \quad \xi_1 = \eta_2 = \frac{1}{2}, \quad \xi_2 = \eta_1 = \frac{1}{3},$$

then

$$\int_0^1 p_1 f_1(s,1,1-s)ds = B\left(\frac{2}{3}, \frac{1}{6}\right), \quad \int_0^1 q_1 g_1(s,1-s,1)ds = B\left(1, \frac{1}{2}\right),$$
$$\int_0^1 p_2 f_2(s,1,1-s)ds = B\left(1, \frac{1}{2}\right), \quad \int_0^1 q_2 g_2(s,1-s,1)ds = B\left(\frac{2}{3}, \frac{1}{3}\right).$$

So all conditions of Theorems 3.1 and 3.2 are satisfied for (4.1) and our conclusion follows from Theorems 3.1 and 3.2, namely the BVP (4.1) has a unique positive solution $(x^*, y^*)$ and there exists a real number $0 < m < 1$ such that

$$mt^{\gamma_1} \leqslant x^*(t) \leqslant \frac{1}{m}t^{\gamma_1}, \quad mt^{\gamma_1} \leqslant y^*(t) \leqslant \frac{1}{m}t^{\gamma_1}, \quad t \in [0,1],$$

where $\gamma_1 = \min\{\alpha - 1, \beta - 1\} = \frac{3}{2}$.

## 5. Conclusions

In this paper, by using the mixed monotone operators and the Guo-Krasnosel'skii fixed point theorem, we have established the existence and uniqueness of positive solutions for a class of singular fractional differential systems with coupled integral boundary conditions for any real number $\alpha, \beta \in (0, +\infty)$. It is worth noting that in this paper we divide the functions into the former of $f_i + g_i$ $(i = 1, 2)$ and add different conditions to $f_i$ and $g_i$. From this point, our result is more general than that in [3, 21].

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