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The distributional Henstock-Kurzweil integral and applications II

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Abstract

In this paper, we study a special Banach lattice D_{HK}, which is induced by the distributional Henstock-Kurzweil integral, and discuss its lattice properties. We show that D_{HK} is an AM-space with the Archimedean property and the Dunford-Pettis property but it is not Dedekind complete. We also present two fixed point theorems in D_{HK}. Meanwhile, two examples are worked out to demonstrate the results.

Keywords: Distributional Henstock-Kurzweil integral, Banach lattice, AM-space, Archimedean property, Dunford-Pettis property, order continuity.
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1. Introduction

This is a continuation of the preceding paper [22], where the distributional Henstock-Kurzweil integral and its properties were studied. The distributional Henstock-Kurzweil integral defined by using Schwartz distributional derivative is a very wide integral form. It includes the Henstock-Kurzweil integral and the Lebesgue integral, see details in [12–15, 19, 21, 22]. The space of Henstock-Kurzweil integrable distributions, denoted by D_{HK}, is a completion of the space of Henstock-Kurzweil integrable functions.

The outline of the present paper is as follows. Section 2 is devoted to the basic notations of the distributional Henstock-Kurzweil integral. In Section 3, an inner product is introduced in the space D_{HK} and so D_{HK} is an inner product space. Section 4 proves that the space D_{HK} is a Banach lattice with a norm cone. Besides, D_{HK} is also an AM-space with the Archimedean property and the Dunford-Pettis property, the details are carried out in Section 5. In Section 6, we show that the norm on D_{HK} is σ-order continuous. However, D_{HK} is not Dedekind complete. Finally, we end this paper with applications, where two fixed point theorems are presented in D_{HK} and two examples are given to demonstrate the results.

2. Basic definitions and preliminaries

For convenience, we use the same notations as in [22] and list some basic ones as follows.

Let (a, b) be an open interval in $\mathbb{R}$, we define

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$$\mathcal{D}((a,b)) = \{\varphi : (a,b) \to \mathbb{R} \mid \varphi \in C_c^\infty \text{ and } \varphi \text{ has a compact support in } (a,b)\}.$$

The distributions on (a,b) are defined to be the continuous linear functionals on $\mathcal{D}((a,b))$. The dual space of $\mathcal{D}((a,b))$ is denoted by $\mathcal{D}'((a,b))$.

For all $f \in \mathcal{D}'((a,b))$, we define the distributional derivative f' of f to be a distribution satisfying $\langle f', \varphi\rangle = -\langle f, \varphi'\rangle$, where $\varphi \in \mathcal{D}((a,b))$ is a test function. Further, we write distributional derivative as f' and its pointwise derivative as $f'(t)$ where $t \in \mathbb{R}$. From now on, all derivatives in this paper will be distributional derivatives unless stated otherwise.

Denote the space of continuous functions on $[a,b]$ by $C([a,b])$. Let

$$C_0 = \{F \in C([a,b]) : F(a) = 0\}.$$

Then C_0 is a Banach space under the norm

$$\|F\|_\infty = \sup_{t\in[a,b]} |F(t)| = \max_{t\in[a,b]} |F(t)|.$$

Definition 2.1 ([22, Definition 1])**.** A distribution $f \in \mathcal{D}'((a,b))$ is said to be Henstock-Kurzweil integrable (shortly D_{HK}) on an interval $[a,b]$, if there exists a continuous function $F \in C_0$ such that $F' = f$, i.e., the distributional derivative of F is f. The distributional Henstock-Kurzweil integral of f on $[a,b]$ is denoted by $\int_a^b f(t)dt = F(b) - F(a)$. The function F is called the primitive of f. For short, $\int_a^b f = F(b) - F(a)$.

For every $f \in D_{HK}$, $\varphi \in \mathcal{D}((a,b))$, we write

$$\langle f, \varphi\rangle = \int_a^b f(t)\varphi(t)dt = -\int_a^b F(t)\varphi'(t)dt.$$

The distributional Henstock-Kurzweil integral is very wide and it includes the integrals of Riemann, Lebesgue, Henstock-Kurzweil, restricted and wide Denjoy (see [14, 21, 22]).

For $f \in D_{HK}$, define the Alexiewicz norm in D_{HK} as

$$\|f\| = \|F\|_\infty = \sup_{t\in[a,b]} |F(t)| = \max_{t\in[a,b]} |F(t)|.$$

Under the Alexiewicz norm, D_{HK} is a Banach space, see [21, Theorem 2]. In [5], the author first proved that the completion, under the Alexiewicz norm, of the family of all Henstock-Kurzweil integrable functions in $[a,b]$, is the space D_{HK}.

Let $g : [a,b] \to \mathbb{R}$, its variation is $V(g) = \sup \sum_n |g(s_n) - g(t_n)|$ where the supremum is taken over every sequence $\{(t_n, s_n)\}$ of disjoint intervals in $[a,b]$. A function g is of bounded variation on $[a,b]$, if $V(g)$ is finite. Denote the space of functions of bounded variation by $\mathcal{BV}$. The space $\mathcal{BV}$ is a Banach space with norm $\|g\|_{\mathcal{BV}} = |g(a)| + V(g)$.

The dual of D_{HK} is $\mathcal{BV}$ (see cf. [21]) and we have

Lemma 2.2 ([21, Theorem 7])**.** (Hölder inequality) *Let* $f \in D_{HK}$. *If* $g \in \mathcal{BV}$, *then*

$$\left|\int_a^b fg\right| \leqslant 2\|f\|\|g\|_{\mathcal{BV}}.$$

3. An inner product in D_{HK}

In this section we introduce an inner product in D_{HK} so that it is an inner product space.

Let $f,\ g \in D_{HK}$ with the primitives $F,\ G \in C_0$, respectively. We say that $f = g$ if $F(t) = G(t)$ everywhere.

Define

$$\langle f, g\rangle = \langle F, G\rangle = \int_a^b F(t)G(t)dt. \tag{3.1}$$

Now, we prove that (3.1) is an inner product in D_{HK}.

(i) For any $f \in D_{HK}$,

$$\langle f, f\rangle = \langle F, F\rangle = \int_a^b F^2(t)dt \geqslant 0,$$

and $\langle f, f\rangle = 0$ if and only if $F(t) = 0$ almost everywhere, i.e., $f = 0$.

(ii) For any $f, g \in D_{HK}$,

$$\langle f, g\rangle = \int_a^b F(t)G(t)dt = \int_a^b G(t)F(t)dt = \langle g, f\rangle.$$

(iii) For any $f, g, h \in D_{HK}$,

$$\begin{aligned}\langle f, g+h\rangle &= \int_a^b F(t)(G(t)+H(t))dt \\ &= \int_a^b F(t)G(t)dt + \int_a^b F(t)H(t)dt = \langle f, g\rangle + \langle f, h\rangle.\end{aligned}$$

By (i), (ii) and (iii), we obtain:

Theorem 3.1. *The space D_{HK} is an inner product space with the inner product given in (3.1).*

The inner product (3.1) induces a norm

$$\|f\|_{\langle,\rangle} = \left(\int_a^b F^2(t)dt\right)^{\frac{1}{2}}.$$

It is easy to obtain

$$\|f\|_{\langle,\rangle} \leqslant (b-a)^{\frac{1}{2}}\|f\|.$$

This means that the norm $\|\cdot\|$ is stronger than $\|\cdot\|_{\langle,\rangle}$. However, the two norms $\|\cdot\|_{\langle,\rangle}$ and $\|\cdot\|$ in D_{HK} are not equivalent, because D_{HK} is complete under the norm $\|\cdot\|$ but not complete under the norm $\|\cdot\|_{\langle,\rangle}$.

Remark 3.2. The norm $\|\cdot\|$ on D_{HK} does not induce an inner product, since $\|\cdot\|$ does not satisfy the parallelogram law.

Remark 3.3. Although D_{HK} is an inner product space, it is not complete under the norm $\|\cdot\|_{\langle,\rangle}$. That is, D_{HK} is not a Hilbert space under the norm $\|\cdot\|_{\langle,\rangle}$. We know that the Hilbert space is self-conjugate. Since the dual of D_{HK} is $\mathcal{BV}$, D_{HK} is not self-conjugate and therefore D_{HK} is not a Hilbert space.

4. The ordering in D_{HK} and Banach lattice

We shall first present some basic properties of order Banach space.

A closed subset X_+ of a Banach space X is called an order cone, if $X_+ + X_+ \subseteq X_+$, $X_+ \cap (-X_+) = \{0\}$ and $cX_+ \subseteq X_+$ for each $c \geqslant 0$. It is easy to see that the order relation $\preceq$ defined by

$$x \preceq y, \text{ if and only if } y - x \in X_+,$$

is a partial ordering in X, and that $X_+ = \{y \in X \mid 0 \preceq y\}$ is an order cone in X. The space X, equipped with this partial ordering, is called an ordered Banach space. For any $r > 0$, $B_r = \{x \in X : \|x\| \leqslant r\}$ is called a closed ball in X. The order interval $[y, z] = \{x \in X \mid y \preceq x \preceq z\}$ is a closed subset of X for all $y, z \in X$. A sequence (subset) of X is called order bounded, if it is contained in an order interval $[y, z]$ of X. We say that an order cone X_+ of a Banach space is normal, if there exists a constant $\gamma \geqslant 1$ such that

$$0 \preceq x \preceq y \text{ in } X \text{ implies } \|x\| \leqslant \gamma\|y\|.$$

X_+ is called regular, if all increasing and order bounded sequences of X_+ converge. If all norm-bounded and increasing sequences of X_+ converge, we say that X_+ is fully regular. As for the proof of the following result, see, e.g., [11, Theorem 2.2.2].

Lemma 4.1. *Let X_+ be an order cone of a Banach space X. If X_+ is fully regular, it is also regular, and if X_+ is regular, it is also normal. Converse holds if X is reflexive.*

Assume that X is an order linear space. If for every $x, y \in X$, there exists $z \in X$ such that $x \preceq z$, $y \preceq z$, and if $x \preceq u$, $y \preceq u$ then $z \preceq u$, then X is called a Riesz space (or lattice) and we denote $z = x \vee y$.

A vector subspace M of a Riesz space X is said to be a Riesz subspace (or a vector sublattice), whenever M is closed under the lattice operations of X, i.e., whenever for each pair $x, y \in M$ the vector $x \vee y$ (taken in X) belongs to M.

For a vector x in a lattice X, define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$, then we call them the positive part, the negative part and the absolute value (or modulus) respectively. Moreover, $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Note that $|x| = 0$, if and only if $x = 0$.

Definition 4.2. Assume that X is a Banach space, if X is a lattice and

$$|x| \preceq |y| \ \text{ in } X \text{ implies } \ \|x\| \leqslant \|y\|, \tag{4.1}$$

then X is called a Banach lattice and the norm $\|\cdot\|$ satisfying (4.1) is called a lattice norm.

Recall that $C([a, b])$ is a Banach lattice with the uniform norm and so is $C_0([a, b])$. For $F \in C_0([a, b])$, the positive part $F^+ = F \vee 0 = \max_{t\in[a,b]}\{F(t), 0\}$, the negative part $F^- = (-F) \vee 0 = \max_{t\in[a,b]}\{-F(t), 0\}$, and hence $F = F^+ - F^-$ and the absolute value $|F| = F^+ + F^-$. Moreover, F^+, F^-, $|F|$ all belong to $C_0([a, b])$.

Let $f \in D_{HK}$ with the primitive $F \in C_0([a, b])$, define

$$f^+ = (F^+)', \quad f^- = (F^-)', \quad |f| = |F|'.$$

Then,

$$f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

See details in [21].

In $C_0([a, b])$ there exists a pointwise order: for $F, G \in C_0([a, b])$, $F \leqslant G$, if and only if $F(t) \leqslant G(t)$, for all $t \in [a, b]$. For $f, g \in D_{HK}$ with primitives $F, G \in C_0([a, b])$, respectively, let

$$f \preceq g \ (\text{or } g \succeq f), \ \text{ if and only if } \ F \leqslant G. \tag{4.2}$$

Theorem 4.3 ([21, Theorem 23]). *D_{HK} is a Banach lattice.*

In the Banach lattice D_{HK}, define

$$D_{HK+} = \{f \in D_{HK} : f \succeq 0\}. \tag{4.3}$$

Then D_{HK+} is an order cone. Moreover, one has

$$\begin{aligned} 0 \preceq f \preceq g \Rightarrow 0 \leqslant F \leqslant G \Rightarrow 0 \leqslant F(t) \leqslant G(t) \Rightarrow \|F\|_\infty \leqslant \|G\|_\infty \\ \Rightarrow \|f\| \leqslant \|g\|. \end{aligned}$$

Therefore, the following statement holds.

Theorem 4.4. *D_{HK+} is a normal cone in D_{HK}.*

Remark 4.5. In [22], another ordering was introduced in D_{HK} and the cone D_{HK+} there is proved to be regular. However, in Section 6, we will prove that the cone D_{HK+} in (4.3) is not regular, still less full regular.

5. AM-space

This section shows that D_{HK} is an AM-space. Moreover, we prove that D_{HK} possesses the Archimedean property (Theorem 5.7) and the Dunford-Pettis property (Theorem 5.8).

Definition 5.1. A lattice norm on a Riesz space is:

1. an M-norm, if $x, y \succeq 0$ implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$;
2. an L-norm, if $x, y \succeq 0$ implies $\|x + y\| = \|x\| + \|y\|$.

A normed Riesz space equipped with an M-norm (resp. an L-norm) is called an M-space. A norm complete M-space (resp. L-space) is an AM-space (resp. AL-space).

Theorem 5.2. D_{HK} *is an AM-space.*

Proof. Let $f, g \in D_{HK}$ and $f, g \succeq 0$ with the primitives F and G. Then $F, G \in C_0$ and $F(t) \geqslant 0$, $G(t) \geqslant 0$ for every $t \in [a, b]$. Therefore,

$$\begin{aligned}\|f \vee g\| = \|F \vee G\|_\infty = \max_t\{F(t), G(t)\} = \max\{\|F\|_\infty, \|G\|_\infty\} \\ = \max\{\|f\|, \|g\|\}.\end{aligned}$$

This means that Alexiewicz norm $\|\cdot\|$ in D_{HK} is M-norm. Note that D_{HK} is complete, hence D_{HK} is an AM-space. □

Lemma 5.3 ([1, Theorem 9.27])**.** *A Banach lattice is an AL-space (resp. an AM-space), if and only if its dual is an AM-space (resp. an AL-space).*

Theorem 5.4. $\mathcal{BV}$ *is an AL-space.*

Proof. By Theorem 5.2, D_{HK} is AM-space. It follows from Lemma 5.3 that $\mathcal{BV}$ is an AL-space, because $\mathcal{BV}$ is the dual of D_{HK}. □

A vector $e > 0$ in a Riesz space X is an order unit, or simply a unit, if for each $x \in X$ there exists some $\lambda > 0$ such that $|x| \preceq \lambda e$. It is easy to see that D_{HK} is a Banach lattice with unit.

Two Riesz spaces X and Y are lattice isomorphic, (or Riesz isomorphic or simply isomorphic), if there exists a one-to-one, onto, lattice preserving linear operator $T : X \to Y$. That is, besides being linear, one-to-one, and surjective, T also satisfies the identities

$$T(x \vee y) = T(x) \vee T(y) \quad \text{and} \quad T(x \wedge y) = T(x) \wedge T(y),$$

for all $x, y \in X$.

The Kakutani-Bohnenblust-M.Krein-S.Krein theorem ([1, Theorem 9.32]) shows that a Banach lattice is an AM-space with unit, if and only if it is lattice isometric to $C(K)$ for some compact Hausdorff space K. The space K is unique (up to homeomorphism). So, we have the following result.

Theorem 5.5. *Banach lattice* D_{HK} *is lattice isometric to* $C([a, b])$*.*

Proof. The proof follows from Theorem 5.2 and the Kakutani-Bohnenblust-M.Krein-S.Krein theorem. □

Now, we consider the Archimedean property and the Dunford-Pettis Property of D_{HK}.

Recall that a net $\{x_\alpha\}$ in a Riesz space is decreasing, written $x_\alpha \downarrow$, if $\alpha \geqslant \beta$ implies $x_\alpha \preceq x_\beta$. The symbol $x_\alpha \uparrow$ indicates an increasing net, while $x_\alpha \uparrow \preceq x$ (resp. $x_\alpha \downarrow \succeq x$) denotes an increasing (resp. decreasing) net that is order bounded from above (resp. below) by x. The notation $x_\alpha \downarrow x$ means that $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$. The meaning of $x_\alpha \uparrow x$ is similar.

Definition 5.6. A Riesz space X is Archimedean, whenever $\frac{1}{n}x \downarrow 0$ holds in X for each $x \in X^+$.

Theorem 5.7. D_{HK} *has the Archimedean property.*

Proof. Suppose that $f \in D_{HK}$ with the primitive $F \in C_0([a, b])$ and $0 \preceq f$ on $[a, b]$. Then $0 \leqslant F(t)$ for each $t \in [a, b]$. So, $\frac{1}{n}F(t) \downarrow 0$ in $\mathbb{R}$ for each t. Hence, by the Dini theorem, $\frac{1}{n}F \downarrow 0$ uniformly. It follows that $\frac{1}{n}f \downarrow 0$ in D_{HK}. By Definition 5.6, D_{HK} has the Archimedean property and the proof is complete. □

A Banach space X has the Dunford-Pettis Property, if $x_n \xrightarrow{w} x$ in X and $x'_n \xrightarrow{w} x'$ in X' imply $\langle x'_n, x_n\rangle \to \langle x', x\rangle$, where "$\xrightarrow{w}$" stands for the weak convergence, see more details in [2, 3].

Theorem 5.8. *D_{HK} and $\mathcal{BV}$ possesses the Dunford-Pettis Property.*

Proof. The Grothendieck theorem ([1, Theorem 9.37]) shows that an AM-space (or AL-space) possesses the Dunford-Pettis Property. Since D_{HK} is an AM-space and $\mathcal{BV}$ is an AL-space, the assertion follows immediately. □

6. The σ-order continuity

In this section, we show that the norm on D_{HK} is σ-order continuous but D_{HK} is not Dedekind complete.

Definition 6.1 ([1])**.** A lattice norm $\|\cdot\|$ on a Riesz space is
(a) order continuous, if $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$.
(b) σ-order continuous, if $x_n \downarrow 0$ implies $\|x_n\| \downarrow 0$.

Obviously, order continuity implies σ-order continuity. The converse is false, even for Banach lattices.

Theorem 6.2. *The norm $\|\cdot\|$ on D_{HK} defined as in (4.2) is σ-order continuous.*

Proof. Suppose that $f_n \in D_{HK}$ with the primitive F_n, $n = 1, 2, \cdots$, and $f_n \downarrow 0$. Then $F_n(t) \downarrow 0$ for each $t \in [a, b]$. By the Dini Theorem, $\{F_n\}$ uniformly converges to 0. It implies $\|F_n\|_\infty \downarrow 0$ in $C([a, b])$ and therefore $\|f_n\| \downarrow 0$ in D_{HK}. So, the norm $\|\cdot\|$ on the D_{HK} is σ-order continuous and the proof is complete. □

A Riesz space X is order complete, or Dedekind complete, if every nonempty subset that is order bounded from above has a supremum. (Equivalently, if every nonempty subset that is bounded from below has an infimum).

Assume that X is a Banach lattice, if for any upper bounded sequence $\{x_n\}$ has supremum $\vee_n x_n$, then X is called σ-complete (or K_σ-space). If for any upper bounded set has supremum, then X is called K-space. Obviously, K-space implies K_σ-space. The converse is false.

It is a pity that in D_{HK} the monotone convergence theorem is not true and so D_{HK} is not an K_σ-space, although D_{HK} is a Banach lattice.

In fact, in $C_0([a, b])$ there exists $\{F_n\}$ such that $-1 \leqslant F_n \uparrow\leqslant \mathbf{0}$ in $C_0[a, b]$, where $\mathbf{0}$ is the zero function, but $\{F_n\}$ does not have a supremum in $C_0([a, b])$. For example,

$$F_n(t) = \begin{cases} 0, & \text{if } 0 \leqslant t \leqslant \frac{1}{2} - \frac{1}{n}, \\ -n(t - \frac{1}{2}) - 1, & \text{if } \frac{1}{2} - \frac{1}{n} < t < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{6.1}$$

The limit function of F_n is

$$F(t) = \begin{cases} 0, & \text{if } 0 \leqslant t < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \tag{6.2}$$

which is not in $C_0([0, 1])$.

By (6.1) and (6.2), it is easy to verify that D_{HK} is not order complete, that is,

Theorem 6.3. *D_{HK} is not K_σ-space, and also not K-space.*

According to Theorem 6.3, we obtain the following consequence.

Corollary 6.4. *The cone D_{HK+} is not regular and so is not full regular.*

However, D_{HK} can have Dedekind completions as $\hat{D}_{HK}$, since, by [1, Theorem 8.8], every Archimedean Riesz space has a unique (up to lattice isomorphism) Dedekind completion. That is, the Dedekind completion of D_{HK} is an order complete Riesz space $\hat{D}_{HK}$ having a Riesz subspace M that is lattice isomorphic to D_{HK} (hence M can be identified with D_{HK}) satisfying

$$\hat{f} = \sup\{f \in M : f \preceq \hat{f}\} = \inf\{g \in M : \hat{f} \preceq g\},$$

for each $\hat{f} \in \hat{D}_{HK}$.

7. Fixed point theorems and applications

In this section, we apply the conclusions in Section 5 to establish fixed point theorems in D_{HK}. The obtained results are used to prove the existence of solutions of an operator equation and a Volterra integral equation.

Let B be a subset of an order Banach space X. An operator $T : B \to B$ is a nonexpansive operator, if $\|T(x) - T(y)\| \leqslant \|x - y\|$, $\forall x, y \subset B$.

Lemma 7.1 ([20, Corollary 1])**.** *Suppose X is an AM-space. If $B_r \subset X$ is a closed ball and $T : B_r \to B_r$ is a nonexpansive operator, then T has a fixed point in B_r.*

Lemma 7.2 ([20, Corollary 2])**.** *Suppose X is an AM-space. If $I \subset X$ is a closed order interval and $T : I \to I$ is a nonexpansive operator, then T has a fixed point in I.*

According to Theorem 5.2 and Lemmas 5.3, 7.1 and 7.2, it is easy to see the following results.

Theorem 7.3. *If $T : B_r \to B_r$ is a nonexpansive operator, where*

$$B_r = \{x \in D_{HK} : \|x\| \leqslant r\}. \tag{7.1}$$

Then the operator T has a fixed point in B_r.

Theorem 7.4. *If $T : I \to I$ is a nonexpansive operator, where*

$$I = [y, z] = \{x \in D_{HK} : y \preceq x \preceq z\}.$$

Then the operator T has a fixed point in I.

Example 7.5. Consider an operator equation

$$Tx = f(t, x), \quad t \in [0, 1],$$

where $x \in D_{HK}$, $f : [0, 1] \times D_{HK} \to D_{HK}$. If there exist $y, z \in D_{HK}$ such that

$$y \preceq f(., x) \preceq z, \quad \forall x \in [y, z],$$

and

$$\|f(., x_1) - f(., x_2)\| \leqslant \|x_1 - x_2\|, \quad \forall x_1, x_2 \in [y, z].$$

Then T has a fixed point in $[y, z]$.

Proof. For each $x \in [y, z]$, one has

$$y \preceq Tx = f(t, x) \preceq z,$$

i.e., $T([y, z]) \subset [y, z]$. Moreover, for any $x_1, x_2 \in [y, z]$, it is easy to see that

$$\|Tx_1 - Tx_2\| = \|f(t, x_1) - f(t, x_2)\| \leqslant \|x_1 - x_2\|,$$

which implies that T is a nonexpansive operator. In view of Theorem 7.4, the assertion follows immediately. □

Example 7.6. Consider a Volterra integral equation of the type

$$x(t) = g(t) + \int_0^t K(t,s)f(s,x(s))ds, \quad t \in [0,1], \tag{7.2}$$

where $x, g \in D_{HK}$, $f : [0,1] \times D_{HK} \to D_{HK}$, $K : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function with bounded variation. If there exist positive constants r, L such that

$$\|g\| \leqslant \frac{r}{2}, \quad \|K\| \leqslant \frac{1}{2L}, \tag{7.3}$$

and

$$\|f(.,x) - f(.,y)\| \leqslant L\|x-y\|, \quad \|f(.,x)\| \leqslant \frac{L}{2}\|x\|, \quad \forall x, y \in B_r, \tag{7.4}$$

where B_r is defined as in (7.1). Then, the Volterra integral equation (7.2) has a solution.

Proof. Define an operator $T : B_r \to D_{HK}$,

$$Tx(t) := g(t) + \int_0^t K(t,s)f(s,x(s))ds, \quad t \in [0,1]. \tag{7.5}$$

From (7.3)-(7.5) and Lemma 2.2, it follows that

$$\|Tx\| \leqslant \|g\| + 2\|K\|\|f\| \leqslant \frac{r}{2} + \frac{r}{2} = r,$$

and

$$\|Tx - Ty\| \leqslant 2\|K\|\|f(.,x) - f(.,y)\| \leqslant \|x - y\|.$$

Therefore, $T : B_r \to B_r$ is a nonexpansive operator. By virtue of Theorem 7.3, T has a fixed point in B_r, i.e., the Volterra integral equation (7.2) has a solution. □

Remark 7.7. In Examples 7.5 and 7.6, we deal with equations involving distributions, e.g., let

$$f(t,x) = h(x) + \left(\sum_{n=1}^{\infty} \frac{\sin n^2 \pi t}{n^2} \right)',$$

where $h(x)$ is continuous with respect to $x \in C([0,1])$ and $(.)'$ denotes the distributional derivative. According to [22, Remark 1], $f(t,x)$ is neither Henstock-Kurzweil integrable nor Lebesgue integrable on $[0,1]$, so approaches in the literatures [4, 6–10, 16–18] are no longer effective. This implies that our results are more general.

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