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Some transcendence properties of integrals of Bessel functions

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Abstract

We prove that some integrals of Bessel functions are transcendence over ring of Bessel functions with coefficients from the field of rational fractions of one variable.

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1. Introduction

In 1990 Lawrence Markus formulated the following problem:

Problem 1.1. Consider the differential field $F < J_0, J_1, J_2, \cdots >$, where F is the differential field of all elementary functions over $\mathbb{C}(x)$. Is $\int J_1^3(x)dx$ in this field $F < J_0, J_1, J_2, \cdots >$?

In the paper [11] the partial answer was obtained to this question. More exactly, Sibuya proved that

$$\int J_n(x)dx \begin{cases} \in \mathbb{C}(x)[J_0(x), J_1(x)], & \text{if } n \equiv 1(2), \\ \notin \mathbb{C}(x)[J_0(x), J_1(x)], & \text{if } n \equiv 0(2), \end{cases}$$

and also $\int J_1^3(x)dx \notin \mathbb{C}(x)[J_0(x), J_1(x)]$.

In this paper we develop a technique from the paper [11] and prove that

1. $\int J_0^m(x)dx \notin \mathbb{C}(x)[J_0(x), J_1(x)]$, if $m \geqslant 2$.
2. $\int J_1^m(x)dx \notin \mathbb{C}(x)[J_0(x), J_1(x)]$, if $m \geqslant 2$.
3. $\int J_n^2(x)dx \notin \mathbb{C}(x)[J_0(x), J_1(x)]$, if $n \geqslant 0$.

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In connection with the obtained statements, the following hypothesis naturally arises:
Hypothesis. If $n \geqslant 0$ and $m \geqslant 2$ then

$$\int J_n^m(x)dx \notin \mathbb{C}(x)[J_0(x), J_1(x)].$$

Problems of this type are directly connected with the questions of differential algebra [3, 4, 7, 8]. At first, it is a statement as Liouville's theorem that integrals of some differential equations are expressed by elementary functions. Classical result of this kind is the theorem of Holder [2] that the Gamma function $\Gamma(x)$ of Euler is not a solution any algebraical differential equation over field $\mathbb{C}(x)$ of rational functions of a complex variable x. In our paper we use Siegel's theorem [12] about algebraical independence functions x, $J_0(x)$, $J_1(x)$ over the field $\mathbb{C}$. Also these questions are connected with the problems of analytical theory of numbers and problems of transcendence. For example, Shidlovskii proved usual theorems on the transcendence and algebraical independence of values in algebraical points, a sufficiently wide class of entire functions which are solutions linear differential equations with polynomial coefficients (see [1]). Some consequences of this direction can be found in [5, 9, 10].

2. Main results

Let $\mathbb{C}$ be the field of complex numbers, x the complex variable and let $\mathbb{C}(x)$ be the field of rational functions with variable x. For every nonnegative integer n, let us denote by $J_n(x)$ the Bessel function of the first kind of order n, i.e.,

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!(n+l)!} \left(\frac{x}{2}\right)^{2l}.$$

We consider the ring $\mathbb{C}(x)[J_0(x), J_1(x)]$ of polynomials in $J_0(x)$ and $J_1(x)$ with coefficients in $\mathbb{C}(x)$.

We remind some well-known facts about the Bessel functions [6, 13].

1. If $\delta = x\dfrac{d}{dx}$, then functions $J_n(x)$, $n \geqslant 0$, are solutions of the equation

$$\delta^2 y + (x^2 - n^2)y = 0.$$

In particular, $\delta^2 J_0(x) = -x^2 J_0(x)$.

2. The following recurrence relations hold:

$$\begin{aligned}
J_{n+1}(x) &= J_{n-1}(x) - 2J_n(x)', \quad n \geqslant 1,\\
xJ_{n+1}(x) &= nJ_n(x) - xJ_n(x)', \quad n \geqslant 0,\\
J_{n+1}(x) &= \frac{2n}{x}J_n(x) - J_{n-1}(x), \quad n \geqslant 1,\\
J_1(x) &= -J_0(x)'.
\end{aligned}$$

Here $\dfrac{d}{dx}$ denotes $'$. From the third relation it can be obtained that

$$\mathbb{C}(x)[J_0(x), J_1(x)] = \mathbb{C}(x)[J_n(x), J_{n+1}(x)], \quad n \geqslant 0.$$

From the second relation

$$\mathbb{C}(x)[J_n(x), J_{n+1}(x)] = \mathbb{C}(x)[J_n(x), \delta J_n(x)], \quad n \geqslant 0.$$

From these relations we obtain that the field $\mathbb{C}(x)(J_n(x), \delta J_n(x))$ is a differential algebra and it is equal to $\mathbb{C}(x)(J_0(x), \delta J_0(x)) = \mathbb{C}(x)(J_0(x), J_1(x))$.

In addition, we will use the following fact.

3. Functions x, $J_0(x)$, $J_1(x)$ are algebraically independence over the field $\mathbb{C}$ [12].

Now we proceed to formulate and prove main results of this paper.

Theorem 2.1. *For every* $m \geqslant 2$, *we have*

$$\int J_0(x)^m dx \notin \mathbb{C}(x)[J_0(x), J_1(x)].$$

Proof. Assume to the contrary that we have

$$\int J_0(x)^m dx \in \mathbb{C}(x)[J_0(x), J_1(x)].$$

Since $xJ_1(x) = -\delta J_0(x)$, it follows that

$$\int J_0(x)^m dx \in \mathbb{C}(x)[J_0(x), \delta J_0(x)].$$

Hence there is a polynomial $F(x, \xi, \eta) \in \mathbb{C}(x)[\xi, \eta]$ such that

$$\begin{aligned} J_0(x)^m = {} & \frac{\partial}{\partial x} F(x, J_0(x), \delta J_0(x)) + (J_0(x))' \frac{\partial}{\partial \xi} F(x, J_0(x), \delta J_0(x)) \\ & + (\delta J_0(x))' \frac{\partial}{\partial \eta} F(x, J_0(x), \delta J_0(x)), \end{aligned}$$

or

$$\begin{aligned} xJ_0(x)^m = {} & x\frac{\partial}{\partial x} F(x, J_0(x), \delta J_0(x)) + \delta J_0(x) \frac{\partial}{\partial \xi} F(x, J_0(x), \delta J_0(x)) \\ & + \delta^2 J_0(x) \frac{\partial}{\partial \eta} F(x, J_0(x), \delta J_0(x)). \end{aligned}$$

Since $\delta^2 J_0(x) = -x^2 J_0(x)$ and function x, $J_0(x)$, $J_1(x)$ are algebraically independence over field $\mathbb{C}$, we have the identity

$$x\xi^m = x\frac{\partial}{\partial x} F(x, \xi, \eta) + \eta \frac{\partial}{\partial \xi} F(x, \xi, \eta) - x^2 \xi \frac{\partial}{\partial \eta} F(x, \xi, \eta). \tag{2.1}$$

Write F as a sum

$$F(x, \xi, \eta) = F_0(x) + (F_{10}(x)\eta + F_{11}(x)\xi) + \cdots + \sum_{i=0}^{m} F_{mi} \eta^{m-i} \xi^i + \cdots,$$

where $\sum\limits_{i=0}^{m} F_{mi} \eta^{m-i} \xi^i$ is the homogeneous component of F of order m with variables ξ, η. Substitute this presentation of F into (2.1) and consider its homogeneous component of order m:

$$x\xi^m = x \sum_{i=0}^{m} F_{mi}(x)' \eta^{m-i} \xi^i + \sum_{i=0}^{m} iF_{mi}(x) \eta^{m-i+1} \xi^{i-1} - \sum_{i=0}^{m} x^2 F_{mi}(x)(m-i) \eta^{m-i-1} \xi^{i+1}. \tag{2.2}$$

Splitting (2.2) on monomials of ξ, η, we obtain the system of identities:

$$\begin{aligned} x &= \delta F_{mm}(x) - x^2 F_{m,m-1}(x), \\ 0 &= \delta F_{m,m-1}(x) + mF_{mm} - 2x^2 F_{m,m-2}(x), \\ 0 &= \delta F_{m,m-2}(x) + (m-1)F_{m,m-1} - 3x^2 F_{m,m-3}(x), \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &= \delta F_{m1}(x) + 2F_{m2} - mx^2 F_{m0}(x), \\ 0 &= \delta F_{m0}(x) + F_{m1}. \end{aligned}$$

Let $H_i = F_{mi}$, $i = 0, \cdots, m$, $H = (H_0, \cdots, H_m)^t$ is the vector-column, "t" - transpose, and

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \dots\dots & 0 & 0 \\ 0 & 0 & 2 & \dots\dots & 0 & 0 \\ 0 & 0 & 0 & \dots\dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots\dots & 0 & m \\ 0 & 0 & 0 & \dots\dots & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & \dots\dots & 0 & 0 \\ m & 0 & 0 & \dots\dots & 0 & 0 \\ 0 & m-1 & 0 & \dots\dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots\dots & 0 & 0 \\ 0 & 0 & 0 & \dots\dots & 1 & 0 \end{pmatrix},$$

are the matrices of order $(m+1) \times (m+1)$. Then we can write this system as

$$\delta H + (A_0 - x^2 A_1) H = xe, \tag{2.3}$$

where $e = (0, 0, \cdots, 0, 1)^t$. Vector-column $H(x)$ is a rational solution of this equation. The operator δ increases the degree of simplest fraction on unit

$$\delta \frac{1}{(x-x_0)^k} = \frac{kx_0}{(x-x_0)^{k+1}} - \frac{k}{(x-x_0)^k}, \quad k \geqslant 1.$$

Therefore, if we present $H(x)$ as the sum of simplest fractions and substitute it in this equation, then we obtain

$$H(x) = \frac{C_k}{x^k} + \frac{C_{k-1}}{x^{k-1}} + \cdots + \frac{C_1}{x} + B_0 + B_1 x + \cdots + B_n x^n,$$

where $C_k, \cdots, B_n$ are vector-columns with elements from $\mathbb{C}$. It means that the decomposition $H(x)$ as the sum of simplest fractions has only fractions of the form $1/x^k$.

Substituting H into (2.3) and equating coefficients under identical degrees x, we obtain the system of equations

$$\begin{aligned} -kC_k + A_0 C_k &= 0, \\ -(k-1)C_{k-1} + A_0 C_{k-1} &= 0, \\ -(k-2)C_{k-2} + A_0 C_{k-2} - A_1 C_k &= 0, \\ \dots\dots\dots\dots\dots\dots& \\ -C_1 + A_0 C_1 - A_1 C_3 &= 0, \\ A_0 B_0 - A_1 C_2 &= 0, \\ B_1 + A_0 B_1 - A_1 C_1 &= e, \\ 2B_2 + A_0 B_2 - A_1 B_0 &= 0, \\ \dots\dots\dots\dots\dots\dots& \\ nB_n + A_0 B_n - A_1 B_{n-2} &= 0, \\ A_1 B_{n-1} &= 0, \\ A_1 B_n &= 0. \end{aligned} \tag{2.4}$$

Since the eigenvalues of matrix A_0 are equal to zero, it follows from first k equations that we find

$$C_k = C_{k-1} = \cdots = C_1 = 0.$$

From the remaining equations, we consider subsystem with odd indices

$$\begin{aligned} (E + A_0)B_1 &= e, \\ (3E + A_0)B_3 &= A_1 B_1, \\ \dots\dots\dots\dots\dots\dots& \\ ((2s+1)E + A_0)B_{2s+1} &= A_1 B_{2s-1}, \\ A_1 B_{2s+1} &= 0. \end{aligned}$$

For arbitrary matrix $A = (a_{ij})$ of order $p \times q$ over the field $\mathbb{R}$, we denote by $\sigma(A)$ the matrix of order $p \times q$ which consists of 0 and ± 1, and $\sigma(A)_{ij} = \text{sign} a_{ij}$.

Now calculate sequentially the vectors $\sigma(B_1)$, $\sigma(B_3), \cdots$.

Note that if $u > 0$ then

$$\sigma((uE + A_0)^{-1}) = \begin{pmatrix} 1 & -1 & 1 & \dots\dots & (-1)^m \\ 0 & 1 & -1 & \dots\dots & (-1)^{m+1} \\ 0 & 0 & 1 & \dots\dots & (-1)^{m+2} \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots\dots & -1 \\ 0 & 0 & 0 & \dots\dots & 1 \end{pmatrix}.$$

Consequently

$$\begin{aligned} &\sigma(B_1) = \sigma((E + A_0)^{-1}e) = \pm(1, -1, 1, \cdots)^t, \\ &\sigma(B_3) = \sigma((3E + A_0)^{-1}B_1) = \pm(1, -1, 1, \cdots)^t, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\sigma(B_{2s+1}) = \pm(1, -1, 1, \cdots)^t. \end{aligned}$$

But from the equation $A_1 B_{2s+1} = 0$, it follows that $B_{2s+1} = (0, 0, \cdots, \star)^t$, a contradiction, and the proof is complete. □

Theorem 2.2. *For every $m \geqslant 2$, we have*

$$\int J_1(x)^m dx \notin \mathbb{C}(x)[J_0(x), J_1(x)].$$

Proof. Proceed by contradiction: suppose that

$$\int J_1(x)^m dx \in \mathbb{C}(x)[J_0(x), J_1(x)].$$

Reasoning as in Theorem 2.1, we obtain the relation

$$\begin{aligned} x^m J_1(x)^m = x^m \frac{\partial}{\partial x} F(x, J_0(x), \delta J_0(x)) + x^{m-1} \delta J_0(x) \frac{\partial}{\partial \xi} F(x, J_0(x), \delta J_0(x)) \\ + x^{m-1} \delta^2 J_0(x) \frac{\partial}{\partial \eta} F(x, J_0(x), \delta J_0(x)). \end{aligned}$$

Using algebraical independence of functions x, $J_0(x)$, $J_1(x)$ and using relation $\delta^2 J_0(x) = -x^2 J_0(x)$ we obtain the identity

$$(-1)^m \eta^m = x^m \frac{\partial}{\partial x} F(x, \xi, \eta) + x^{m-1} \eta \frac{\partial}{\partial \xi} F(x, \xi, \eta) - x^{m+1} \xi \frac{\partial}{\partial \eta} F(x, \xi, \eta).$$

Considering homogeneous component of degree m on variables ξ, η we obtain the relation

$$\delta H + (A_0 - x^2 A_1) H = x^{1-m} \nu, \tag{2.5}$$

where $\nu = (1, 0, \cdots, 0,)^t$. Vector function $H(x)$ is a rational solution of this equation. Writing $H(x)$ as a sum of simplest fractions and substituting in the equation (2.5) and reasoning as in Theorem 2.1, we obtain

$$H(x) = \frac{C_k}{x^k} + \frac{C_{k-1}}{x^{k-1}} + \cdots + \frac{C_1}{x} + B_0 + B_1 x + \cdots + B_n x^n,$$

where $C_k, \cdots, B_n$ - vector-columns are in $\mathbb{C}^{m+1}$.

Substituting H in (2.5) and equating coefficients under same degrees x, we obtain the system of equations

$$\begin{aligned}
-kC_k + A_0C_k &= 0,\\
-(k-1)C_{k-1} + A_0C_{k-1} &= 0,\\
-(k-2)C_{k-2} + A_0C_{k-2} - A_1C_k &= 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
-mC_m + A_0C_m - A_1C_{m+2} &= 0,\\
-(m-1)C_{m-1} + A_0C_{m-1} - A_1C_{m+1} &= \nu,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
-C_1 + A_0C_1 - A_1C_3 &= 0,\\
A_0B_0 - A_1C_2 &= 0,\\
B_1 + A_0B_1 - A_1C_1 &= 0,\\
2B_2 + A_0B_2 - A_1B_0 &= 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
nB_n + A_0B_n - A_1B_{n-2} &= 0,\\
A_1B_{n-1} &= 0,\\
A_1B_n &= 0.
\end{aligned}$$

Since the eigenvalues of matrix A_0 are equal to zero, we have

$$C_k = C_{k-1} = \cdots = C_m = 0.$$

Let m be an even number. Then subsystem with odd indices has the form

$$\begin{aligned}
(A_0 - (m-1)E)B_{m-1} &= \nu,\\
(A_0 - (m-3)E)B_{m-3} &= A_1C_{m-1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
(A_0 - E)C_1 &= A_1C_3,\\
(A_0 + E)B_1 &= A_1C_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
(A_0 + (2s+1)E)B_{2s+1} &= A_1B_{2s-1},\\
A_1B_{2s+1} &= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sigma(C_{m-1}) = (-1, 0, 0, \cdots)^t,\\
&\sigma(C_{m-3}) = \pm(1, -1, 0, \cdots)^t,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sigma(B_{2s+1}) = \pm(1, -1, 1, \cdots)^t.
\end{aligned}$$

But from the equation $A_1B_{2s+1} = 0$, it follows that $B_{2s+1} = (0, 0, \cdots, \star)^t$, a contradiction.

Let m be an odd number. Then subsystem with even indices has the form

$$\begin{aligned}
(A_0 - (m-1)E)B_{m-1} &= \nu,\\
(A_0 - (m-3)E)B_{m-3} &= A_1C_{m-1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
(A_0 - 2E)C_2 &= A_1C_4,\\
A_0B_0 &= A_1C_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
(A_0 + 2sE)B_{2s} &= A_1B_{2s-2},\\
A_1B_{2s} &= 0.
\end{aligned} \tag{2.6}$$

It can be supposed that $B_{2s} \not\equiv 0$. From the last equation, we find $B_{2s} = \alpha(0, \cdots, 1)^t$, $\alpha \not\equiv 0$.

Beginning from the first equation of system (2.6) we consistently find

$$\sigma(C_{m-1}) = \pm(1, 0, 0, \cdots)^t,$$

$$\sigma(C_{m-3}) = \pm(1, -1, 0, \cdots)^t,$$

and so on

$$\sigma(C_2) = \pm(1, -1, 1, \cdots, 0, 0, \cdots)^t.$$

where the first $(m-1)/2$ elements of vector C_2 are zero.

From the equation $A_0B_0 = A_1C_2$, we have $\sigma(B_0) = (\star, 0, 1, \cdots)^t$.

Moving from the last to the first equation of system (2.6) we consistently find

$$\begin{aligned} B_{2(s-1)} &= \alpha(0, \cdots, 0, m/2, 2s, \star)^t, \\ B_{2(s-2)} &= \alpha(0, \cdots, 0, m(m-3)/8, (m(s-1) + 2s(m-1))/3, \star, \star, \star)^t, \end{aligned}$$

and so on. Since $\sigma(B_0) = (\star, 0, 1, \cdots)^t$, we have $s \geqslant (m-1)/2$. Hence when we move from the vector B_{2k} to the vector $B_{2(k-1)}$ we obtain two nonzero elements on the places where there were two last zeros of the vector B_{2k}. Therefore, there is such a number k that

$$B_{2k} = \alpha(0, x, y, \cdots)^t.$$

But from the equation

$$(A_0 + 2kE)B_{2k} = A_1B_{2k-2},$$

we obtain

$$\begin{aligned} (A_0 + 2kE)B_{2k} &= \alpha(2kx, \star, \cdots)^t, \\ A_1B_{2k-2} &= \alpha(0, \star, \cdots)^t, \end{aligned}$$

a contradiction. Thus, the theorem is proved. □

Next we need eigenvalues of matrix $A_0 + n^2A_1$. The following lemma holds.

Lemma 2.3. *The eigenvalues of the matrix $A_0 + n^2A_1$ are equal to $(m-2k)n$, $k = 0, 1, \cdots, m$, and the coordinates eigenvector v_k, corresponding to the eigenvalue $(m-2k)n$, are equal to the coefficients of the polynomial $f_k(x) = (1-n^2x^2)^k(1+nx)^{m-2k}$, arranged in the order of ascending powers of variable x.*

Proof. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in \mathbb{C}[x]$ and $f'(x)$ its derivative on variable x. Consider the vector $v = (a_0, a_1, \cdots, a_m)^T$. We have

$$(A_0 + n^2A_1)v = (a_1, 2a_2, \cdots, ma_m, 0)^T + n^2(0, ma_0, (m-1)a_1, \cdots, a_{m-1})^T.$$

Since

$$\begin{aligned} a_1 + 2a_2x + \cdots + ma_mx^{m-1} &= f(x)', \\ n^2(ma_0x + (m-1)a_1x^2 + \cdots + a_{m-1}x^m) &= -n^2x^{m+2}\left(\frac{f(x)}{x^m}\right)', \end{aligned}$$

it follows that it is enough to prove the equality

$$f_k(x)' - n^2x^{m+2}\left(\frac{f(x)}{x^m}\right)' = (m-2k)nf_k(x).$$

Let $f = f_k(x)$. Then

$$f' = -2kn^2x\frac{f}{1-n^2x^2} + (m-2k)n\frac{f}{1+nx}.$$

From this, we obtain

$$\begin{aligned} f' - n^2x^{m+2}\frac{x^m f' - mx^{m-1}f}{x^{2m}} - (m-2k)nf \\ &= -\frac{2kn^2x}{1-n^2x^2}f + \frac{(m-2k)n}{1+nx}f - n^2x^2f' + mn^2xf - (m-2k)nf \\ &= f\left(-\frac{2kn^2x}{1-n^2x^2} + \frac{(m-2k)n}{1+nx} + \frac{2kn^4x^3}{1-n^2x^2} - \right. \\ &\quad \left. - \frac{(m-2k)n^3x^2}{1+nx} + mn^2x + (m-2k)n\right). \end{aligned}$$

Simple computation shows that the last expression is equal to zero, hence the proof is complete. □

Theorem 2.4. *For every $l \geqslant 2$, we have*

$$\int J_l(x)^2 dx \notin \mathbb{C}(x)[J_0(x), J_1(x)].$$

Proof. Suppose to the contrary that we have

$$\int J_l(x)^2 dx \in \mathbb{C}(x)[J_0(x), J_1(x)] = \mathbb{C}(x)[J_l(x), \delta J_l(x)].$$

Then there is a polynomial $F(x, \xi, \eta) \in \mathbb{C}(x)[\xi, \eta]$ such that

$$x\xi^2 = x\frac{\partial}{\partial x}F(x,\xi,\eta) + \eta\frac{\partial}{\partial \xi}F(x,\xi,\eta) + (n^2 - x^2)\xi\frac{\partial}{\partial \eta}F(x,\xi,\eta).$$

Consider homogeneous of degree two of this relation. We have

$$\delta H + (A_0 + (l^2 - x^2)A_1)H = xe,$$

where $e = (0, 0, 1)^t$. The vector function $H(x)$ is a rational solution of this equation. Writing $H(x)$ as a sum of simplest fractions and substituting into this equation we obtain

$$H(x) = \frac{C_k}{x^k} + \frac{C_{k-1}}{x^{k-1}} + \cdots + \frac{C_1}{x} + B_0 + B_1x + \cdots + B_nx^n,$$

where $C_k, \cdots, B_n$ are vector columns with elements from $\mathbb{C}$.

We obtain a system of equations for $C_k, \cdots, B_n$ which is analogous to system (2.4). Since the eigenvalues of matrix $A_0 + l^2A_1$ are equal to $-2l$, 0, $2l$, there is inequality $k \leqslant 2l$ in the system (2.4).

Consider the subsystem with odd indices. From the first equations we obtain

$$C_1 = C_3 = \cdots = 0.$$

Then the remaining equations are

$$\begin{aligned} (E + A_0 + l^2A_1)B_1 &= e, \\ (3E + A_0 + l^2A_1)B_3 &= A_1B_1, \\ &\cdots\cdots \\ ((2s+1)E + A_0 + l^2A_1)B_{2s+1} &= A_1B_{2s-1}, \\ A_1B_{2s+1} &= 0. \end{aligned}$$

A direct computation shows that

$$T_p = \sigma(pE + A_0 + l^2 A_1)^{-1} = \begin{pmatrix} \star & -1 & 1 \\ \star & 1 & -1 \\ \star & \star & \star \end{pmatrix}, \; p = 1, 3, \cdots .$$

Therefore,

$$\sigma(B_1) = \pm(1, -1, \star)^t, \;\; \sigma(A_1 B_1) = \pm(0, 1, -1)^t,$$
$$\sigma(B_3) = \pm(1, -1, \star)^t, \;\; \sigma(A_1 B_3) = \pm(0, 1, -1)^t,$$

and so on

$$\sigma(B_{2s+1}) = \pm(1, -1, \star)^t.$$

But from the equation $A_1 B_{2s+1} = 0$ follows that $B_{2s+1} = (0, 0, \star)^t$, a contradiction. The proof is complete. □

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