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Controllability result of nonlinear higher order fractional damped dynamical system

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Abstract

In this paper, we investigate the controllability of nonlinear fractional damped dynamical system, which involved fractional Caputo derivatives of any different orders. In the process of proof, we mainly use the Schaefer's fixed-point theorem and Mittag-Leffler matrix function. At last, we give an example to illustrate our main result.

Keywords: Controllability, Mittag-Leffler matrix, Schaefer's fixed-point theorem, Gramian matrix.
2010 MSC: 47H10, 54H25.

1. Introduction

In this article, we study the controllability result for following system:

$$\begin{cases} {}^{C}D^{\alpha}x(t) = A\,{}^{C}D^{\beta}x(t) + Bu(t) + f\big(t, x(t), u(t), {}^{C}D^{\alpha}x(t), {}^{C}D^{\beta}x(t)\big), \\ x(0) = x_0, \quad x'(0) = x_1, \cdots, x^{(p)}(0) = x_p, \end{cases} \tag{1.1}$$

where $p-1 < \alpha \leqslant p$, $q-1 < \beta \leqslant q$, $q \leqslant p-1$, A is an $n \times n$ matrix and B is an $n \times m$ matrix, $x \in R^n$, $u \in L^{\infty}(J, R^m)$, $t \in [0, T]$ and the nonlinear function f being continuous. In order to solve the problem, we will use Mittag-Leffler matrix function, Gramian matrix and the theorem of Schaefer's fixed-point.

Fractional differential equation has increasingly attracted the attention of many researchers during the last three decades, see [2–4, 6, 7, 14, 18, 20–24, 26, 27, 30, 32]. The various types of fractional differential equation, playing significant roles and tools, are used for solving some mathematical issues of general physical phenomena in physics and engineering. Especially, the field of control theory sparked the interest of many scholars which can be seen from the literatures [10, 19, 25]. In recent years, several authors [8, 11–13, 15, 17] have made a detailed research about controllability results of linear and proposed many new ideas about the low-order fractional equation.

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Yonggang and Xiu'e [31] introduced a fractional oscillator satisfied the following differential equation:

$$\frac{d^2x(t)}{dt^2} + \omega_0^{2-\varepsilon}\,{}_0D_t^{\varepsilon}[x(t)] = A\delta(t),$$

where the elastic restoring force $\omega_0^{2-\varepsilon}\,{}_0D_t^{\varepsilon}[x(t)]$ of arbitrary order ε with $0 < \varepsilon < 1$. $x(t)$ is an unknown function that in the equilibrium position, t is a time variable, ω_0 is the proper frequency, A is the strength of pulses, $\delta(t)$ is the Dirac δ function.

Achar et al. [1] studied the response of several specific forcing functions and analyzed resonance characteristic of the fractional oscillator model. Tofighi [29] have defined and obtained the expression of the fractional oscillator system. Al-rabth et al. [5] took advantage of the differential transform technique and efficient algorithm to solve a fractional oscillator equation.

Balachandran et al. [9] studied the controllability result for the type of fractional damped system represented by the following equation:

$$\begin{cases} {}^{C}D^{\alpha}x(t) = A\,{}^{C}D^{\beta}x(t) + Bu(t) + f\big(t, x(t)\big), \\ x(0) = x_0, \quad x'(0) = x_0', \end{cases}$$

where $0 < \beta \leqslant 1 < \alpha \leqslant 2$, A and B are $n \times n$ and $n \times m$ matrices respectively, and $f : J \times R^n \to R^n$ is a continuous function.

Motivated by the work mentioned above, in this article we study the nonlinear fractional damped dynamical system of (1.1). To the best of our knowledge, the controllability of nonlinear fractional damped dynamical system of order $p-1 < \alpha \leqslant p$ $(p \in N)$ have not been discussed.

This paper is arranged as follows. In Section 1, we illustrate the background and motivation of writing this article. In Section 2, we make preparation of basic knowledge for the main result, and controllability of linear system is proved. In Section 3, the main result of this article is obtained. Finally, an example is provided to illustrate the main result in Section 4.

2. Preliminaries

2.1. *Definitions and preliminary facts*

In this section, we introduce some necessary definitions which are used throughout this paper.

Definition 2.1 ([18]). The Caputo fractional derivative of order $\alpha \in R$ with $n-1 < \alpha \leqslant n$, $n \in N$, for a suitable function f is defined as

$$\left({}^{C}D_{0+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $f^{(n)}(s)=\frac{d^n f}{ds^n}$. In particular, if $1 < \alpha \leqslant 2$ then

$$\left({}^{C}D_{0+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f''(s)ds.$$

For the brevity, the Caputo fractional derivative ${}^{C}D_{0+}^{\alpha}$ is taken as ${}^{C}D^{\alpha}$.

Definition 2.2 ([27]). The Mittag-leffler matrix function for an arbitrary square matrix A is

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,$$
$$E_{\alpha,1}(A) = E_{\alpha}(A), \quad \text{with } \beta = 1.$$

Lemma 2.3 ([16]). *The Mittag-leffler matrix function derivative of order p $(p \in N)$ is defined as*

$$\Big(\frac{d}{dt}\Big)^{(p)}\big(t^{\alpha-1}E_{\alpha-\beta,\alpha}(At^{\alpha-\beta})\big) = t^{\alpha-p-1}E_{\alpha-\beta,\alpha-p}(At^{\alpha-\beta}), \ (p \in N).$$

In order to find the solution of system (1.1), take Laplace transform and inverse Laplace transform on both sides of the following formula, we get the solution of the system (1.1)

$$\begin{aligned} x(t) = &\int_0^t (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})Bu(s)ds \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})f(\cdot)ds \\ &- \sum_{k=0}^{q-1} x^k(0)t^{\alpha-\beta+k}E_{\alpha-\beta,\alpha-\beta+1+k}(At^{\alpha-\beta}) \\ &+ \sum_{k=0}^{p-1} x^k(0)t^k E_{\alpha-\beta,1+k}(At^{\alpha-\beta}). \end{aligned}$$

2.2. *Linear system*

Consider the liner fractional dynamical system represented by fractional differential equation of the form

$$\begin{cases} {}^C D^\alpha x(t) = A\,{}^C D^\beta x(t) + Bu(t), & t \in [0, T], \\ x(0) = x_0, \quad x'(0) = x_1, \quad \cdots, & x^{(p)}(0) = x_p, \end{cases} \tag{2.1}$$

where $p-1 < \alpha \leqslant p$, $q-1 < \beta \leqslant q$ and $q \leqslant p-1$, A is an $n \times n$ matrix and B is an $n \times m$ matrix, $x \in R^n$, $u \in L^2(J, R^m)$.

Definition 2.4. The system (2.1) is said to be controllable on J, if for every $x_0, x_1, \dots, x_p, y \in R^n$, if there exists a control $u(t)$ such that the solution $x(t)$ of such system satisfies the conditions $x(0) = x_0$, $x'(0) = x_1$, $\cdots$, $x^{(p)}(0) = x_p$, $x(T) = y$.

Define the controllability Grammian matrix W as

$$\begin{aligned} W = &\int_0^T (T-s)^{2(\alpha-1)} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})BB^* \\ &\times E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})ds. \end{aligned}$$

Theorem 2.5. *The linear system* (2.1) *is controllable on* J*, iff the* $n \times n$ *Grammian matrix*

$$\begin{aligned} W = &\int_0^T (T-s)^{2(\alpha-1)} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})BB^* \\ &\times E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})ds, \end{aligned}$$

is invertible.

Proof. Suppose W is invertible, then given $x_0, x_1, \dots, x_p$ and y can choose the input function $u(t)$ as

$$\begin{aligned} u(t) = &(T-t)^{\alpha-1}B^* E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})W^{-1} \\ &(y + \sum_{k=0}^{q-1} \Phi(T)_1^k - \sum_{k=0}^{p-1} \Phi(T)_2^k), \end{aligned}$$

where

$$\begin{aligned} \sum_{k=0}^{q-1} \Phi(t)_1^k &= \sum_{k=0}^{q-1} x^k(0)t^{\alpha-\beta+k}E_{(\alpha-\beta),(\alpha-\beta+1+k)}(At^{\alpha-\beta}), \\ \sum_{k=0}^{p-1} \Phi(t)_2^k &= \sum_{k=0}^{p-1} x^k(0)t^k E_{(\alpha-\beta),(1+k)}(At^{\alpha-\beta}). \end{aligned}$$

The corresponding solution of the system (2.1) at $t = T$ can be written as

$$\begin{aligned} x(T) = &\int_0^T (T-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})B(T-s)^{\alpha-1}B^* \\ &\times E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})W^{-1}\Big[y + \sum_{k=0}^{q-1}\Phi(T)_1^k \\ &- \sum_{k=0}^{p-1}\Phi(T)_2^k\Big]ds - \sum_{k=0}^{q-1}\Phi(T)_1^k + \sum_{k=0}^{p-1}\Phi(T)_2^k \\ = &\, y, \end{aligned}$$

so that the system (2.1) is controllable on J.

On the other hand, suppose that the the system (2.1) is controllable, but for the sake of a contradiction, assume that the matrix W is not invertible. If W is not invertible, then there exists a vector $z \neq 0$ such that

$$\begin{aligned} z^*Wz = &\int_0^T z^*(T-s)^{2(\alpha-1)} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})BB^* \\ &\times E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})zds \\ = &\, 0, \end{aligned}$$

hence

$$z^*(T-s)^{2(\alpha-1)} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})B = 0, \quad t \in J.$$

Consider the initial points $x_0 = x_1 = \cdots = x_p = 0$ and the final point $y = z$, so the system (2.1) is controllable and there exists a control $u(t)$ on J that steers the response from 0 to $y = z$ at $t = T$,

$$y = z = \int_0^T (T-s)^{2(\alpha-1)} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})Bu(s)ds,$$

then

$$\begin{aligned} z^*z = &\int_0^T z^*(T-s)^{2(\alpha-1)} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})Bu(s)ds \\ \neq &\, 0, \end{aligned}$$

which is a contradiction. Thus the matrix W is invertible. □

Lemma 2.6 (Schaefer's Theorem [28]). *Let S be a normed space, T a continuous mapping of S into S, which is compact on each bounded subset of X. Then, either*

(i) *the equation $x = \lambda T(x)$ has a solution for $\lambda = 1$, or*

(ii) *the set of all such solutions x for $0 < \lambda < 1$ is unbounded.*

We assume the following hypotheses.

(E_1) For each $t \in J$, the function $f(t,\cdot,\cdot,\cdot,\cdot) : J \times R^n \times R^m \times R^n \times R^n \to R^n$ is continuous, the function $f(\cdot, x(\cdot), u(\cdot), y(\cdot), z(\cdot)) : J \to R^n$ is strongly measurable for each $x, y, z \in R^n$, $u \in R^m$.

(E_2)
$$\|f(t, x(t), u(t), {}^C D^\alpha x(t), {}^C D^\beta x(t))\| \leqslant M,$$
where $t \in J$, $x \in R^n$, $u \in R^m$, $M \in R$.

(E_3) Let

$$\begin{aligned}
&L=\sum_{k=0}^{q-1}\Phi(t)_1^k-\sum_{k=0}^{p-1}\Phi(t)_2^k; \qquad \triangle=y+L;\\
&n_1=\sup|(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})|;\\
&n_2=\sup|(t-s)^{\alpha-p-1}E_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta})|;\\
&n_3=\sup|\Big[\sum_{k=0}^{q-1}\Phi(t)_1^k+\sum_{k=0}^{p-1}\Phi(t)_2^k\Big]^{(p)}|;\\
&n_4=\sup|(t-s)^{p-\alpha-1}|.
\end{aligned}$$

3. Nonlinear system

Consider the nonlinear fractional dynamical system represented by the fractional differential equation of the form

$$\begin{cases} {}^CD^{\alpha}x(t)=A\,{}^CD^{\beta}x(t)+Bu(t)+f\big(t,x(t),u(t),{}^CD^{\alpha}x(t),{}^CD^{\beta}x(t)\big),\\ x(0)=x_0,\quad x'(0)=x_1,\cdots,x^{(p)}(0)=x_p,\end{cases} \tag{3.1}$$

where $p-1<\alpha\leqslant p$, $q-1<\beta\leqslant q$, $q\leqslant p-1$, A is an $n\times n$ matrix and B is an $n\times m$ matrix, $x\in R^n$, $u\in L^{\infty}(J,R^m)$, $t\in[0,T]$ and the nonlinear function f being continuous, the solution of (3.1) is given by

$$\begin{aligned}
x(t)=&\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})Bu(s)ds\\
&+\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\\
&\times f\big(s,x(s),u(s),{}^CD^{\alpha}x(s),{}^CD^{\beta}x(s)\big)ds\\
&-\sum_{k=0}^{q-1}\Phi(t)_1^k+\sum_{k=0}^{p-1}\Phi(t)_2^k.
\end{aligned}$$

Theorem 3.1. *Assume that hypotheses* (E_1)-(E_3) *hold and the linear system* (2.1) *is controllable on* J*, then the nonlinear system* (3.1) *is controllable on* J.

Proof. We give the space $X=\{x:x^{(p)},{}^CD^{\alpha}x,{}^CD^{\beta}x\in C(J,R^n)$ and $u(t)\in L^{\infty}(J,R^m)\}$ be a Banach space endowed with the norm $\|x\|_X=\max_{t\in J}\{\|x\|,\|{}^CD^{\alpha}x\|,\|{}^CD^{\beta}x\|,\|u\|\}$. Using the hypothesis, for an arbitrary $x(\cdot)$, the control

$$u(t)=(T-t)^{\alpha-1}B^*E_{\alpha-\beta,\alpha}(A^*(T-t)^{\alpha-\beta})W^{-1}\triangledown,$$

where

$$\begin{aligned}
\triangledown=&y+\sum_{k=0}^{q-1}\Phi(T)_1^k-\sum_{k=0}^{p-1}\Phi(T)_2^k\\
&-\int_0^T(T-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})\\
&\times f\big(s,x(s),u(s),{}^CD^{\alpha}x(s),{}^CD^{\beta}x(s)\big)ds.
\end{aligned}$$

Now we shall show that the nonlinear operator $F:X\to X$

$$(Fx)(t)=\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})Bu(s)ds$$

$$\begin{aligned}&+\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\\ &\times f\big(s,x(s),u(s),{}^{C}D^{\alpha}x(s),{}^{C}D^{\beta}x(s)\big)ds\\ &-\sum_{k=0}^{q-1}\Phi(t)_1^k+\sum_{k=0}^{p-1}\Phi(t)_2^k,\end{aligned}$$

has a fixed point. This fixed point is a solution of (3.1). Substituting the control $u(t)$ in the above, we can get

$$\begin{aligned}(Fx)(t) =&\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})(T-s)^{\alpha-1}\\ &\times BB^*E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})W^{-1}\bigtriangledown ds\\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\\ &\times f\big(s,x(s),u(s),{}^{C}D^{\alpha}x(s),{}^{C}D^{\beta}x(s)\big)ds\\ &-\sum_{k=0}^{q-1}\Phi(t)_1^k+\sum_{k=0}^{p-1}\Phi(t)_2^k.\end{aligned}$$

Clearly, $(Fx)(T)=y$, this means that the system (3.1) was steered from the x_0 to y by the control u, if we can obtain a fixed point of the nonlinear operator F. The first step is to obtain a priori bound of the set

$$\zeta(F)=\{x\in X : x=\lambda Fx, \lambda\in[0,1]\}.$$

Let $x\in\zeta(F)$, then $x=\lambda Fx$ for some $0<\lambda<1$. So for each $t\in J$, we have

$$\begin{aligned}x(t) =&\lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})(T-s)^{\alpha-1}\\ &\times BB^*E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})W^{-1}\bigtriangledown ds\\ &+\lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\\ &\times f\big(s,x(s),u(s),{}^{C}D^{\alpha}x(s),{}^{C}D^{\beta}x(s)\big)ds\\ &-\lambda\sum_{k=0}^{q-1}\Phi(t)_1^k+\lambda\sum_{k=0}^{p-1}\Phi(t)_2^k,\end{aligned}$$

then

$$\begin{aligned}\|x(t)\| \leqslant& \int_0^t n_1BB^*n_1W^{-1}[\bigtriangledown+\int_0^T n_1Md\xi]ds\\ &+\int_0^t n_1Mds+L\\ \leqslant& n_1^2\|B\|\|B^*\|\|W^{-1}\|\|T\bigtriangledown\|\\ &+n_1T^2M+n_1TM+L,\end{aligned}$$

and

$$\|u(t)\| \leqslant \|(T-t)^{\alpha-1}B^*E_{\alpha-\beta,\alpha}(A^*(T-t)^{\alpha-\beta})$$

$$\begin{aligned}
&\times W^{-1}\Big[y+\sum_{k=0}^{q-1}\Phi(T)_1^k-\sum_{k=0}^{p-1}\Phi(T)_2^k\\
&-\int_0^T (T-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})\\
&\times f\big(s,x(s),u(s),{}^{C}D^{\alpha}x(s),{}^{C}D^{\beta}x(s)\big)ds\Big]\|\\
\leqslant\; & n_1\|B^*\|\|W^{-1}\|\Big[\|\triangle\|+Tn_1M\Big].
\end{aligned}$$

Due to Lemma 2.3, we have

$$\begin{aligned}
x^{(p)}(t)\leqslant &\int_0^t (t-s)^{\alpha-p-1}E_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta})BB^*\\
&\times E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})W^{-1}\bigtriangledown ds\\
&+\int_0^t (t-s)^{\alpha-p-1}E_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta})\\
&\times f\big(s,x(s),u(s),{}^{C}D^{\alpha}x(s),{}^{C}D^{\beta}x(s)\big)ds\\
&-\Big[\sum_{k=0}^{q-1}\Phi(t)_1^k+\sum_{k=0}^{p-1}\Phi(t)_2^k\Big]^{(p)}.
\end{aligned}$$

So

$$\begin{aligned}
\|x^{(p)}(t)\|\leqslant\; & n_2\|B\|\|B^*\|n_1\|W^{-1}\|(\|\triangle\|+Tn_1M)T\\
&+n_2TM+n_3\\
=\; & M_0.
\end{aligned}$$

We can get

$$\|{}^{C}D^{\alpha}x(t)\|\leqslant\|\frac{1}{\Gamma(p-\alpha)}\|\|\int_0^t n_4M_0ds\|.$$

Hence, ${}^{C}D^{\alpha}x(t)$ is bounded. In the same way, ${}^{C}D^{\beta}x(t)$ is bounded too. And because

$$\|x\|_X=\max_{t\in J}\{\|x\|,\|{}^{C}D^{\alpha}x\|,\|{}^{C}D^{\beta}x\|,\|u\|\},$$

so $\zeta(F)$ is bounded. Next we prove that F is completely continuous operator.

Let $B_\rho=\{x\in X;\|x\|\leqslant\rho\}$, then we first show that F maps B_ρ into equicontinuous family. Let $x\in B_\rho,\;\; t_1,t_2\in J,\;\; 0<t_1<t_2<T,$

$$\begin{aligned}
\|(Fx)(t_2)-(Fx)(t_1)\|\leqslant\;&\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_2-s)^{\alpha-\beta})(T-s)^{\alpha-1}\\
&\times BB^*E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})W^{-1}\bigtriangledown ds\|\\
&+\|\int_0^{t_1}[(t_2-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_2-s)^{\alpha-\beta})\\
&-(t_1-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_1-s)^{\alpha-\beta})]\\
&\times(T-s)^{\alpha-1}BB^*E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})W^{-1}\bigtriangledown ds\|\\
&+\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_2-s)^{\alpha-\beta})\\
&\times f\big(s,x(s),u(s),{}^{C}D^{\alpha}x(s),{}^{C}D^{\beta}x(s)\big)ds\|
\end{aligned}$$

$$
\begin{aligned}
&+\|\int_0^{t_1}[(t_2-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_2-s)^{\alpha-\beta})\\
&-(t_1-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_1-s)^{\alpha-\beta})]\\
&\times f\big(s,x(s),u(s),{}^CD^{\alpha}x(s),{}^CD^{\beta}x(s)\big)ds\|\\
&+\|\sum_{k=0}^{q-1}\Phi(t_2)_1^k-\sum_{k=0}^{q-1}\Phi(t_1)_1^k\|\\
&+\|\sum_{k=0}^{p-1}\Phi(t_2)_2^k-\sum_{k=0}^{p-1}\Phi(t_1)_2^k\|.
\end{aligned}
$$

So,

$$
\begin{aligned}
\|(Fu)(t_2)-(Fu)(t_1)\| \leqslant \|[&(T-t_2)^{\alpha-1}B^*E_{\alpha-\beta,\alpha}(A^*(T-t_2)^{\alpha-\beta})\\
&-(T-t_1)^{\alpha-1}B^*E_{\alpha-\beta,\alpha}(A^*(T-t_1)^{\alpha-\beta})]W^{-1}\bigtriangledown\|.
\end{aligned}
$$

and

$$
\begin{aligned}
\|{}^CD^{\alpha}(Fx)(t_2)-{}^CD^{\alpha}(Fx)(t_1)\| &= \|\frac{1}{\Gamma(1-\alpha)}\int_0^{t_2}(t_2-s)^{-\alpha}(Fx)^{(p)}(s)ds\\
&\quad-\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_2-s)^{-\alpha}(Fx)^{(p)}(s)ds\|\\
&\leqslant \frac{1}{\Gamma(1-\alpha)}\Big(\|\int_{t_1}^{t_2}(t_2-s)^{-\alpha}(Fx)^{(p)}(s)ds\|\\
&\quad+\|\int_0^{t_1}[(t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}](Fx)^{(p)}(s)ds\|\Big).
\end{aligned}
$$

Clearly when $t_2 \to t_1$,

$$
\begin{aligned}
\|(Fx)(t_2)-(Fx)(t_1)\| &\to 0,\\
\|{}^CD^{\alpha}(Fx)(t_2)-{}^CD^{\alpha}(Fx)(t_1)\| &\to 0,\\
\|{}^CD^{\beta}(Fx)(t_2)-{}^CD^{\beta}(Fx)(t_1)\| &\to 0,\\
\|(Fu)(t_2)-(Fu)(t_1)\| &\to 0.
\end{aligned}
$$

So F maps B_ρ into an equicontinuous family of functions. Then the family FB_ρ is uniformly bounded.

Next we show that F is a compact operator. Obviously, the closure of FB_ρ is compact. Let $0 \leqslant t \leqslant T$, t be fixed and τ a real number satisfying $0 < \tau < t$. For $x \in B_\rho$, we define

$$
\begin{aligned}
(F_\tau x)(t) = &\int_0^{t-\tau}(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})\\
&\times(T-s)^{\alpha-1}BB^*E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})\\
&\times W^{-1}\Big[y+\sum_{k=0}^{q-1}\Phi(T)_1^k-\sum_{k=0}^{p-1}\Phi(T)_2^k\\
&-\int_0^T(T-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})\\
&\times f\big(\xi,x(\xi),u(\xi),{}^CD^{\alpha}x(\xi),{}^CD^{\beta}x(\xi)\big)d\xi\Big]ds\\
&-\sum_{k=0}^{q-1}\Phi(t)_1^k+\sum_{k=0}^{p-1}\Phi(t)_2^k+\int_0^{t-\tau}(t-s)^{\alpha-1}
\end{aligned}
$$

$$\times E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})$$
$$\times f\big(s, x(s), u(s), {}^{C}D^{\alpha}x(s), {}^{C}D^{\beta}x(s)\big)ds.$$

By using the previous method, we can get the bounded and equicontinuous property of F_τ. Hence,

$$S_\tau(t) = \{(F_\tau x)(t),\ x \in B_\rho\},$$

is relatively compact in X for every $0 < \tau < t$ and for every $x \in B_\rho$,

$$\begin{aligned}\|(Fx)(t) - (F_\tau x)(t)\| \leqslant\ & \| \int_{t-\tau}^{t} (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})(T-s)^{\alpha-1} \\ & \times BB^{*}E_{\alpha-\beta,\alpha}(A^{*}(T-s)^{\alpha-\beta})W^{-1} \bigtriangledown ds\| \\ & + \| \int_{t-\tau}^{t} (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta}) \\ & \times f\big(s, x(s), u(s), {}^{C}D^{\alpha}x(s), {}^{C}D^{\beta}x(s)\big)ds\| \\ \leqslant\ & n_1^2 \tau \|B\|\|B^{*}\|\|W^{-1}\|\| \bigtriangledown \| + n_1 \tau M,\end{aligned}$$

then

$$\begin{aligned}\|(Fx)^{(p)}(t) - (F_\tau x)^{(p)}(t)\| \leqslant\ & \int_{t-\tau}^{t} (t-s)^{\alpha-p-1} E_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta}) \\ & \times BB^{*}E_{\alpha-\beta,\alpha}(A^{*}(T-s)^{\alpha-\beta})W^{-1} \\ & \times \Big[\triangle - \int_0^T (T-\xi)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(T-\xi)^{\alpha-\beta}) \\ & \times f\big(\xi, x(\xi), u(\xi), {}^{C}D^{\alpha}x(\xi), {}^{C}D^{\beta}x(\xi)\big)d\xi \Big] ds \\ & + \int_{t-\tau}^{t} (t-s)^{\alpha-p-1} E_{\alpha-\beta,\alpha-p}(A(t-s)^{\alpha-\beta}) \\ & \times f\big(s, x(s), u(s), {}^{C}D^{\alpha}x(s), {}^{C}D^{\beta}x(s)\big)ds \\ \leqslant\ & n_2^2 \tau \|B\|\|B^{*}\|\|W^{-1}\|\| \bigtriangledown \| + n_2 \tau M.\end{aligned}$$

So $\|(Fx)(t) - (F_\tau x)(t)\| \to 0$, $\|(Fx)^{(p)}(t) - (F_\tau x)^{(p)}(t)\| \to 0$, as $\tau \to 0$.
Hence,

$$\begin{aligned}\|{}^{C}D^{\alpha}(Fx)(t) - {}^{C}D^{\alpha}(F_\tau x)(t)\| &= \| \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}[(Fx)^{(p)}(s) - (F_\tau x)^{(p)}]ds\| \\ &\leqslant \| \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}[(Fx)^{(p)}(s) - (F_\tau x)^{(p)}]ds\| \to 0,\ \tau \to 0.\end{aligned}$$

In the same way, $\|{}^{C}D^{\beta}(Fx)(t) - {}^{C}D^{\beta}(F_\tau x)(t)\| \to 0$, as $\tau \to 0$.

Hence, relatively compact sets $S_\tau(t) = \{(F_\tau x)(t),\ x \in B_\rho\}$ are arbitrary close to the set $\{(Fx)(t),\ x \in B_\rho\}$, so $\{(Fx)(t),\ x \in B_\rho\}$ is compact in X by the Arzela-Ascoli theorem.

Next we show that F is continuous. Let $\{x_n\}$ be a sequence in X, $\|x_n - x\| \to 0$, as $n \to \infty$. Then for all n and $t \in J$, there is an integer k_0 such that $\|x_n\| \leqslant k_0$, $\|u_n\| \leqslant k_0$, $\|{}^{C}D^{\alpha}x_n\| \leqslant k_0$, $\|{}^{C}D^{\beta}x_n\| \leqslant k_0$. Hence $\|x(t)\| \leqslant k_0$, $\|u(t)\| \leqslant k_0$, $\|{}^{C}D^{\alpha}x(t)\| \leqslant k_0$, $\|{}^{C}D^{\beta}x(t)\| \leqslant k_0$. $x, u, {}^{C}D^{\alpha}x, {}^{C}D^{\beta}x \in X$. By (E_1)

$$f\big(t, x_n(t), u_n(t), {}^{C}D^{\alpha}x_n(t), {}^{C}D^{\beta}x_n(t)\big) \to f\big(t, x(t), u(t), {}^{C}D^{\alpha}x(t), {}^{C}D^{\beta}x(t)\big),$$

for each $t \in J$ and since

$$\|f\big(t, x_n(t), u_n(t), {}^{C}D^{\alpha}x_n(t), {}^{C}D^{\beta}x_n(t)\big) - f\big(t, x(t), u(t), {}^{C}D^{\alpha}x(t), {}^{C}D^{\beta}x(t)\big)\| \leqslant 2M.$$

By the Fatou-Lebesgue theorem we have

$$\begin{aligned}\|(Fx_n)(t)-(Fx)(t)\| \leqslant \| \int_0^t n_1^3 BB^* W^{-1} \int_0^s & \big[f\big(\xi, x_n(\xi), u_n(\xi), {}^C D^{\alpha} x_n(\xi), {}^C D^{\beta} x_n(\xi)\big) \\ & - f\big(\xi, x(\xi), u(\xi), {}^C D^{\alpha} x(\xi), {}^C D^{\beta} x(\xi)\big) d\xi\big] ds \\ & + \int_0^t n_1 [f\big(s, x_n(s), u_n(s), {}^C D^{\alpha} x_n(s), {}^C D^{\beta} x_n(s)\big) \\ & - f\big(s, x(s), u(s), {}^C D^{\alpha} x(s), {}^C D^{\beta} x(s)\big)\big] ds\|.\end{aligned}$$

Clearly $\|(Fx_n)(t)-(Fx)(t)\| \to 0$, as $n \to \infty$.

By x and corresponding relationship of u, we get $\|(Fu_n)(t)-(Fu)(t)\| \to 0$, as $n \to \infty$, then

$$\begin{aligned}\|(Fx_n)^{(p)}(t)-(Fx)^{(p)}(t)\| \leqslant \| \int_0^T n_2^3 BB^* W^{-1} & \big[f\big(\xi, x_n(\xi), u_n(\xi), {}^C D^{\alpha} x_n(\xi), {}^C D^{\beta} x_n(\xi)\big) \\ & - f\big(\xi, x(\xi), u(\xi), {}^C D^{\alpha} x(\xi), {}^C D^{\beta} x(\xi)\big) d\xi\big] ds \\ & + \int_0^T n_2 \big[f\big(s, x_n(s), u_n(s), {}^C D^{\alpha} x_n(s), {}^C D^{\beta} x_n(s)\big) \\ & - f\big(s, x(s), u(s), {}^C D^{\alpha} x(s), {}^C D^{\beta} x(s)\big)\big] ds\|.\end{aligned}$$

So $\|(Fx_n)^{(p)}(t)-(Fx)^{(p)}(t)\| \to 0$, as $n \to \infty$.

This implies that

$$\begin{aligned}& \|{}^C D^{\alpha}(Fx_n)(t) - {}^C D^{\alpha}(Fx)(t)\| \\ & \qquad \leqslant \frac{1}{\Gamma(p-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \|(Fx_n)^p(s)-(Fx)^p(s)\| ds \to 0, \ n \to \infty.\end{aligned}$$

Hence F is continuous. Finally, the set $\zeta(F)=\{x \in X; x=\lambda Fx, \lambda \in [0,1]\}$ is bounded as shown in front part. By Schaefer's theorem, the operator F has a fixed point in X. This fixed point is the solution of (3.1). In summary, the system (3.1) is controllable on $[0,T]$. □

4. An example

Consider the problem of nonlinear fractional dynamical system

$$\begin{cases} {}^C D^{\alpha} x(t) = A\, {}^C D^{\beta} x(t) + Bu(t) + f\big(t, x(t), u(t), {}^C D^{\alpha} x(t), {}^C D^{\beta} x(t)\big), \\ x(0)=x_0, \quad x'(0)=x_1, \cdots, x^{(p)}(0)=x_p, \end{cases} \tag{4.1}$$

where $p-1<\alpha \leqslant p$, $q-1<\beta \leqslant q$, $p, q \in \mathbb{N}$, $t \in J$, and

$$A=\begin{pmatrix} 4 & 3 & 7 & 0 \\ 3 & 6 & 2 & 5 \\ 7 & 2 & -11 & 9 \\ 0 & 5 & 9 & 10 \end{pmatrix}, \quad B=\begin{pmatrix} \frac{1}{e} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix},$$

and the non-linear function is defined by

$$f\big(t, x(t), u(t), {}^C D^{\alpha} x(t), {}^C D^{\beta} x(t)\big) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin({}^C D^{\alpha} x_1(t)) + \cos({}^C D^{\beta} x_2(t)) + \frac{\sqrt{x_3(t)+u^2(t)}}{\ln(x_4(t))} \end{pmatrix}.$$

First, let us consider the Mittag-Leffler matrix function for the given system,

$$E_{\alpha-\beta,\alpha}(At^{\alpha-\beta}) = \sum_{k=0}^{\infty} \frac{A^k t^{k(\alpha-\beta)}}{\Gamma((\alpha-\beta)k+\alpha)}.$$

By doing some matrix calculations, we can get the following matrices:

$$E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})B = \frac{1}{e}\begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix},$$

$$B^* E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta}) = \frac{1}{e}\begin{pmatrix} l_1 & l_2 & l_3 & l_4 \end{pmatrix},$$

where

$$\begin{aligned}
l_1(t) &= \frac{1}{\Gamma(\alpha)} + \frac{4t^{\alpha-\beta}}{\Gamma(2\alpha-\beta)} + \frac{74t^{2(\alpha-\beta)}}{\Gamma(3\alpha-2\beta)} + \cdots, \\
l_2(t) &= \frac{3t^{\alpha-\beta}}{\Gamma(2\alpha-\beta)} + \frac{44t^{2(\alpha-\beta)}}{\Gamma(3\alpha-2\beta)} + \frac{790t^{3(\alpha-\beta)}}{\Gamma(4\alpha-3\beta)} + \cdots, \\
l_3(t) &= \frac{7t^{\alpha-\beta}}{\Gamma(2\alpha-\beta)} - \frac{43t^{2(\alpha-\beta)}}{\Gamma(3\alpha-2\beta)} + \frac{1781t^{3(\alpha-\beta)}}{\Gamma(4\alpha-3\beta)} + \cdots, \\
l_4(t) &= \frac{78t^{2(\alpha-\beta)}}{\Gamma(3\alpha-2\beta)} + \frac{613t^{3(\alpha-\beta)}}{\Gamma(4\alpha-3\beta)} + \frac{26109t^{4(\alpha-\beta)}}{\Gamma(5\alpha-4\beta)} + \cdots, \\
l_n &= l_n(T-s).
\end{aligned}$$

So,

$$\begin{aligned}
E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta})BB^*E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta} &= \frac{1}{e^2}\begin{pmatrix} l_1^2 & l_1l_2 & l_1l_3 & l_1l_4 \\ l_1l_2 & l_2^2 & l_2l_3 & l_2l_4 \\ l_1l_3 & l_2l_3 & l_3^2 & l_3l_4 \\ l_1l_4 & l_2l_4 & l_3l_4 & l_4^2 \end{pmatrix} \\
&= l^*.
\end{aligned}$$

Hence, the controllability matrix W for the system is found by

$$\begin{aligned}
W &= \int_0^T (T-s)^{2(\alpha-1)} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta}) \\
&\quad BB^* E_{\alpha-\beta,\alpha}(A^*(T-s)^{\alpha-\beta})ds \\
&= \int_0^T (T-s)^{2(\alpha-1)} l^* ds \\
&> 0, \quad \text{for } T > 0.
\end{aligned}$$

From the above, we have shown that W is an invertible matrix. Moreover, the nonlinear function f satisfies the hypothesis. Hence the system (4.1) is controllable on J.

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