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Viscosity approximation methods for the implicit midpoint rule of nonexpansive mappings in CAT(0) Spaces

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Abstract

The purpose of this paper is to introduce the implicit midpoint rule of nonexpansive mappings in CAT(0) spaces. The strong convergence of this method is proved under certain assumptions imposed on the sequence of parameters. Moreover, it is shown that the limit of the sequence generated by the implicit midpoint rule solves an additional variational inequality. Applications to nonlinear Volterra integral equations and nonlinear variational inclusion problem are included. The results presented in the paper extend and improve some recent results announced in the current literature.

Keywords: Viscosity, implicit midpoint rule, nonexpansive mapping, projection, variational inequality, CAT(0) space.
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1. Introduction

The implicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, please refer to [2–5, 15, 17, 18, 21].

For the ordinary differential equation

$$x'(t) = f(t), \quad x(0) = x_0, \tag{1.1}$$

the implicit midpoint rule generates a sequence $\{x_n\}$ by the recursion procedure

$$x_{n+1} = x_n + hf\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geqslant 0, \tag{1.2}$$

where $h > 0$ is a stepsize. It is known that if $f : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{x_n\}$ generated by (1.2) converges to the exact solution of (1.1) as $h \to 0$ uniformly over $t \in [0, T]$ for any fixed $T > 0$.

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Based on the above fact, Alghamdi et al. [2] presented the following semi-implicit midpoint rule for nonexpansive mappings in the setting of Hilbert space H:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(\frac{x_n + x_{n+1}}{2}), \quad n \geqslant 0, \tag{1.3}$$

where $\alpha_n \in (0,1)$ and $T : H \to H$ is a nonexpansive mapping. They proved the weak convergence of (1.3) under some additional conditions on $\{\alpha_n\}$.

Recently, Xu et al. [22] and Yao et al. [23] in a Hilbert spaces presented the following viscosity implicit midpoint rule for nonexpansive mappings:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad \forall n \geqslant 0,$$

where $\alpha_n \in (0,1)$ and f is a contraction. Under suitable conditions and by using a very complicated method, the authors proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T, which is also the unique solution of the following variational inequality

$$\langle (I-f)q, x-q\rangle \geqslant 0, \quad \forall x \in \mathrm{Fix}(T).$$

Very recently, Zhao et al. [25] presented the following viscosity implicit midpoint rule for an asymptotically nonexpansive mapping T in a Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n(\frac{x_n + x_{n+1}}{2}), \quad \forall n \geqslant 0,$$

and under suitable conditions some strong convergence theorems to a fixed point of T are proved.

On the other hand, the theory and applications of CAT(0) space have been studied extensively by many authors.

Recall that a metric space (X, d) is called a CAT(0) space, if it is geodetically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples of CAT(0) spaces include pre-Hilbert spaces, R-trees [8, 16], Euclidean buildings [9], and many others. A complete CAT(0) space is often called a Hadamard space. A subset K of a CAT(0) space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$, where $[x, y]$ is the uniquely geodesic joining x and y. For a thorough discussion of CAT(0) spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [8].

Motivated and inspired by the research going on in this direction, it is natural to put forward the following.

Open Question: Can we establish the viscosity implicit midpoint rule for nonexpansive mapping in CAT(0) and generalize the main results in [22, 23] to CAT(0) spaces?

The purpose of this paper is to give an affirmative answer to the above open question. In our paper we introduce and consider the following semi-implicit algorithm which is called the viscosity implicit midpoint rule in CAT(0):

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad n \geqslant 0. \tag{1.4}$$

Under suitable conditions, some strong converge theorems to a fixed point of the nonexpansive mapping in CAT(0) space are proved. Moreover, it is shown that the limit of the sequence $\{x_n\}$ generated by (1.4) solves an additional variational inequality. As applications, we shall utilize the results presented in the paper to study the existence problems of solutions of nonlinear variational inclusion problem, and nonlinear Volterra integral equations. The results presented in the paper also extend and improve the main results in Xu [22], Yao et al. [23], and others.

2. Preliminaries

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = td(x,y), \quad \text{and} \quad d(z,y) = (1-t)d(x,y).$$

The following lemmas play an important role in our paper.

Lemma 2.1 ([13]). *Let X be a CAT(0) space, $x, y, z \in X$ and $t \in [0,1]$. Then*

(i) $d(tx \oplus (1-t)y,\ z) \leqslant td(x,z) + (1-t)d(y,z)$;
(ii) $d^2(tx \oplus (1-t)y,\ z) \leqslant td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$.

Lemma 2.2 ([6]). *Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0,1]$. Then*

$$d(\lambda p \oplus (1-\lambda)q,\ \lambda r \oplus (1-\lambda)s) \leqslant \lambda d(p,r) + (1-\lambda)d(q,s).$$

Berg and Nikolaev [7] introduced the concept of quasilinearization as follows. Let us denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, quasilinearization is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \qquad (a,b,c,d \in X).$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwartz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leqslant d(a,b)d(c,d)$$

for all $a, b, c, d \in X$. It is well-known [7] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.

Let C be a nonempty closed convex subset of a complete CAT(0) space X. The metric projection $P_C : X \to C$ is defined by

$$u = P_C(x) \iff d(u,x) = \inf\{d(y,x) : y \in C\}, \quad \forall x \in X.$$

Lemma 2.3 ([11]). *Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C(x)$ if and only if u is a solution of the following variational inequality*

$$\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geqslant 0, \quad \forall y \in C,$$

i.e., u satisfies the following inequality:

$$d^2(x,y) - d^2(y,u) - d^2(u,x) \geqslant 0,\ \forall y \in C.$$

Lemma 2.4 ([14]). *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.5 ([1]). *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leqslant 0$ for all $y \in X$.*

Lemma 2.6 ([12]). *Let X be a complete CAT(0) space. Then for all $u, x, y \in X$, the following inequality holds*

$$d^2(x,u) \leqslant d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Lemma 2.7 ([19]). *Let X be a complete CAT(0) space. For any $t \in [0,1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leqslant t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leqslant t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leqslant t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Lemma 2.8 ([20]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \delta_n, \ \forall n \geqslant 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let* C *be a closed convex subset of a complete* CAT(0) *space* X, *and* $T : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Let* f *be a contraction on* C *with coefficient* $k \in [0,1)$, *and for the arbitrary initial point* $x_0 \in C$, *let* $\{x_n\}$ *be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad n \geqslant 0,$$

where $\{\alpha_n\} \in (0,1)$ *satisfies the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Then the sequence $\{x_n\}$ *converges strongly to* $\tilde{x} = P_{\mathrm{Fix}(T)} f(\tilde{x})$, *which is a fixed point of* T *and it is also a solution of the following variational inequality:*

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geqslant 0, \quad \forall x \in \mathrm{Fix}(T),$$

i.e., $\tilde{x}$ *satisfies the following inequality equation:*

$$d^2(f(\tilde{x}), x) - d^2(\tilde{x}, x) - d^2(f(\tilde{x}), \tilde{x}) \geqslant 0, \ \forall x \in \mathrm{Fix}(T).$$

Proof. We divide the proof into five steps.

Step 1. We prove that $\{x_n\}$ is bounded. To see this, we take $p \in \mathrm{Fix}(T)$ to deduce that

$$\begin{aligned} d(x_{n+1}, p) &= d\left(\alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\ &\leqslant \alpha_n d(f(x_n), p) + (1-\alpha_n) d\left(T\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\ &\leqslant \alpha_n (d(f(x_n), f(p)) + d(f(p), p)) + (1-\alpha_n) d\left(T\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right) \\ &\leqslant \alpha_n k d(x_n, p) + \alpha_n d(f(p), p) + (1-\alpha_n) d\left(\frac{x_n \oplus x_{n+1}}{2}, p\right) \\ &\leqslant \alpha_n k d(x_n, p) + \alpha_n d(f(p), p) + \frac{1-\alpha_n}{2}(d(x_n, p) + d(x_{n+1}, p)). \end{aligned}$$

It then follows that

$$\frac{1+\alpha_n}{2} d(x_{n+1}, p) \leqslant \frac{1+(2k-1)\alpha_n}{2} d(x_n, p) + \alpha_n d(f(p), p),$$

and, moreover

$$\begin{aligned}
d(x_{n+1},p) &\leqslant \frac{1+(2k-1)\alpha_n}{1+\alpha_n} d(x_n,p) + \frac{2\alpha_n}{1+\alpha_n} d(f(p),p) \\
&= \left(1-\frac{2(1-k)\alpha_n}{1+\alpha_n}\right) d(x_n,p) + \frac{2(1-k)\alpha_n}{1+\alpha_n}\,\frac{1}{1-k} d(f(p),p) \\
&\leqslant \max\left\{ d(x_n,p), \frac{1}{1-k} d(f(p),p)\right\}.
\end{aligned}$$

By induction we readily obtain

$$d(x_n,p) \leqslant \max\{d(x_0,p), \frac{1}{1-k} d(f(p),p)\},$$

for all $n \geqslant 0$. Hence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$ and $\{T(\frac{x_n \oplus x_{n+1}}{2})\}$.

Step 2. We show that $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$. Observe that

$$\begin{aligned}
d(x_{n+1},x_n) &= d\left(\alpha_n f(x_n) \oplus (1-\alpha_n) T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1}) T\left(\frac{x_{n-1}\oplus x_n}{2}\right)\right) \\
&\leqslant d\left(\alpha_n f(x_n) \oplus (1-\alpha_n) T(\frac{x_n \oplus x_{n+1}}{2}), \alpha_n f(x_n) \oplus (1-\alpha_n) T(\frac{x_{n-1}\oplus x_n}{2})\right) \\
&\quad + d\left(\alpha_n f(x_n) \oplus (1-\alpha_n) T(\frac{x_{n-1}\oplus x_n}{2}), \alpha_n f(x_{n-1}) \oplus (1-\alpha_n) T(\frac{x_{n-1}\oplus x_n}{2})\right) \\
&\quad + d\left(\alpha_n f(x_{n-1}) \oplus (1-\alpha_n) T(\frac{x_{n-1}\oplus x_n}{2}), \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1}) T(\frac{x_{n-1}\oplus x_n}{2})\right) \\
&\leqslant (1-\alpha_n) d\left(T(\frac{x_n \oplus x_{n+1}}{2}), T(\frac{x_{n-1}\oplus x_n}{2})\right) + \alpha_n d(f(x_n), f(x_{n-1})) \\
&\quad + |\alpha_n - \alpha_{n-1}| d\left(f(x_{n-1}), T(\frac{x_{n-1}\oplus x_n}{2})\right) \\
&\leqslant (1-\alpha_n) d\left(\frac{x_n \oplus x_{n+1}}{2}, \frac{x_{n-1}\oplus x_n}{2}\right) + \alpha_n k d(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}| M \\
&\leqslant \frac{(1-\alpha_n)}{2}\left[d(x_{n+1},x_n) + d(x_n,x_{n-1})\right] + \alpha_n k d(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}| M.
\end{aligned}$$

Here $M > 0$ is a constant such that

$$M \geqslant \sup\left\{ d\left(f(x_{n-1}), T(\frac{x_{n-1}\oplus x_n}{2})\right), n \geqslant 0\right\}.$$

It turns out that

$$\frac{1+\alpha_n}{2} d(x_{n+1},x_n) \leqslant \frac{1+(2k-1)\alpha_n}{2} d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}|.$$

Consequently, we arrive at

$$\begin{aligned}
d(x_{n+1},x_n) &\leqslant \frac{1+2k\alpha_n - \alpha_n}{1+\alpha_n} d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}| \\
&= \frac{1+\alpha_n + 2k\alpha_n - 2\alpha_n}{1+\alpha_n} d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}| \\
&= \left(1-\frac{2(1-k)\alpha_n}{1+\alpha_n}\right) d(x_n,x_{n-1}) + M|\alpha_n - \alpha_{n-1}|.
\end{aligned}$$

Since $\{\alpha_n\} \in (0,1)$, then $1+\alpha_n < 2$, $\frac{1}{1+\alpha_n} > \frac{1}{2}$, $(1-\frac{2(1-k)\alpha_n}{1+\alpha_n}) < (1-(1-k)\alpha_n)$. We have

$$d(x_{n+1}, x_n) \leqslant (1-(1-k)\alpha_n)d(x_n, x_{n-1}) + M|\alpha_n - \alpha_{n-1}|. \tag{3.1}$$

By virtue of the conditions (ii) and (iii), we can apply Lemma 2.8 to (3.1) to obtain $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Step 3. We show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. In fact, we have

$$\begin{aligned}
d(x_n, Tx_n) &\leqslant d(x_n, x_{n+1}) + d\left(x_{n+1}, T(\frac{x_n \oplus x_{n+1}}{2})\right) + d\left(T(\frac{x_n \oplus x_{n+1}}{2}), Tx_n\right)\\
&\leqslant d(x_n, x_{n+1}) + \alpha_n d\left(f(x_n), T(\frac{x_n \oplus x_{n+1}}{2})\right) + d\left(\frac{x_n \oplus x_{n+1}}{2}, x_n\right)\\
&\leqslant d(x_n, x_{n+1}) + \alpha_n d\left(f(x_n), T(\frac{x_n \oplus x_{n+1}}{2})\right) + \frac{1}{2}d(x_n, x_{n+1})\\
&\leqslant \frac{3}{2}d(x_n, x_{n+1}) + \alpha_n M \to 0 \ \ (\text{as } n \to \infty).
\end{aligned}$$

Step 4. Now we prove

$$\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle \leqslant 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that Δ-converges to $\tilde{x}$ and

$$\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle = \limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_j}\tilde{x}}\rangle. \tag{3.2}$$

Since $\{x_{n_j}\}$ Δ-converges to $\tilde{x}$, by Lemma 2.5, we have

$$\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_j}\tilde{x}}\rangle \leqslant 0.$$

This together with (3.2) shows that

$$\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle = \limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n_j}\tilde{x}}\rangle \leqslant 0. \tag{3.3}$$

Step 5. Finally, we prove that $x_n \to \tilde{x} \in Fix(T)$ as $n \to \infty$. For any $n \in \mathbb{N}$, we set $z_n = \alpha_n\tilde{x} \oplus (1-\alpha_n)T(\frac{x_n \oplus x_{n+1}}{2})$. It follows from Lemma 2.6 and Lemma 2.7 that

$$\begin{aligned}
d^2(x_{n+1}, \tilde{x}) &\leqslant d^2(z_n, \tilde{x}) + 2\langle \overrightarrow{x_{n+1}z_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\leqslant (1-\alpha_n)^2 d^2\left(T(\frac{x_n \oplus x_{n+1}}{2}), \tilde{x}\right) + 2\Bigg[\alpha_n\langle \overrightarrow{f(x_n)z_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\quad + (1-\alpha_n)\left\langle \overrightarrow{T(\frac{x_n \oplus x_{n+1}}{2})\, z_n}, \overrightarrow{x_{n+1}\tilde{x}}\right\rangle\Bigg]\\
&\leqslant (1-\alpha_n)^2 d^2\left(\frac{x_n \oplus x_{n+1}}{2}, \tilde{x}\right) + 2\Bigg[\alpha_n\alpha_n\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\quad + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{f(x_n)T(\frac{x_n \oplus x_{n+1}}{2})}, \overrightarrow{x_{n+1}\tilde{x}}\right\rangle + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{T(\frac{x_n \oplus x_{n+1}}{2})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\right\rangle\\
&\quad + (1-\alpha_n)^2\left\langle \overrightarrow{T(\frac{x_n \oplus x_{n+1}}{2})\, T(\frac{x_n \oplus x_{n+1}}{2})}, \overrightarrow{x_{n+1}\tilde{x}}\right\rangle\Bigg]\\
&\leqslant (1-\alpha_n)^2 d^2\left(\frac{x_n \oplus x_{n+1}}{2}, \tilde{x}\right) + 2\Bigg[\alpha_n^2\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\quad + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{f(x_n)T(\frac{x_n \oplus x_{n+1}}{2})}, \overrightarrow{x_{n+1}\tilde{x}}\right\rangle + \alpha_n(1-\alpha_n)\left\langle \overrightarrow{T(\frac{x_n \oplus x_{n+1}}{2})\,\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\right\rangle\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (1-\alpha_n)^2 d^2\left(\frac{x_n \oplus x_{n+1}}{2}, \tilde{x}\right) + 2[\alpha_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leqslant (1-\alpha_n)^2 d^2\left(\frac{x_n \oplus x_{n+1}}{2}, \tilde{x}\right) + 2\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leqslant (1-\alpha_n)^2 \left[\frac{1}{2} d^2(x_n, \tilde{x}) + \frac{1}{2} d^2(x_{n+1}, \tilde{x}) - \frac{1}{4} d^2(x_n, x_{n+1})\right] \\
&\quad + 2\alpha_n \langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}} + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leqslant \frac{(1-\alpha_n)^2}{2}[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] + 2\alpha_n k d(x_n, \tilde{x}) d(x_{n+1}, \tilde{x}) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leqslant \frac{(1-\alpha_n)^2}{2}[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] + \alpha_n k[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leqslant \left(\frac{(1-\alpha_n)^2}{2} + \alpha_n k\right)[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leqslant \frac{1-2(1-k)\alpha_n}{2}[d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})] + \alpha_n^2 M_1 + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle.
\end{aligned}
$$

Here $M_1 > 0$ is a constant such that

$$
M_1 \geqslant \sup\{d^2(x_n, \tilde{x}),\ n \geqslant 0\}.
$$

It follows that

$$
\begin{aligned}
d^2(x_{n+1}, \tilde{x}) &\leqslant \frac{1-2(1-k)\alpha_n}{1+2(1-k)\alpha_n} d^2(x_n, \tilde{x}) + \frac{2\alpha_n^2}{1+2(1-k)\alpha_n} M_1 + \frac{4\alpha_n}{1+2(1-k)\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leqslant \left(1 - \frac{2(1-k)\alpha_n}{1+(1-k)\alpha_n}\right) d^2(x_n, \tilde{x}) + \frac{2\alpha_n^2}{1+(1-k)\alpha_n} M_1 + \frac{4\alpha_n}{1+(1-k)\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle.
\end{aligned}
$$

Since $1 + (1-k)\alpha_n < 2 - k$, $\frac{1}{1+(1-k)\alpha_n} > \frac{1}{2-k}$, we have

$$
\begin{aligned}
d^2(x_{n+1}, \tilde{x}) &\leqslant \left(1 - \frac{2(1-k)\alpha_n}{2-k}\right) d^2(x_n, \tilde{x}) + \frac{2\alpha_n^2}{1+(1-k)\alpha_n} M_1 + \frac{4\alpha_n}{1+(1-k)\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leqslant \left(1 - \frac{2(1-k)\alpha_n}{2-k}\right) d^2(x_n, \tilde{x}) + 2\alpha_n^2 M_1 + \frac{4\alpha_n}{1+(1-k)\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle.
\end{aligned}
$$

Take $\gamma_n = \frac{2(1-k)\alpha_n}{2-k}$, $\delta_n = 2\alpha_n^2 M_1 + \frac{4\alpha_n}{1+(1-k)\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle$. It follows from conditions (i), (ii), and (3.3) that $\{\gamma_n\} \subset (0,1)$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and

$$
\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty} \frac{2-k}{1-k}\left(\alpha_n M_1 + \frac{2}{1+(1-k)\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\right) \leqslant 0.
$$

From Lemma 2.8 we have that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Remark 3.2. Since every Hilbert space is a complete CAT(0) space, Theorem 3.1 is an improvement and generalization of the main results in Xu et al. [22] and Yao et al. [23].

The following result can be obtained from Theorem 3.1 immediately.

Theorem 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Let f be a contraction on C with coefficient $k \in [0,1)$, and for the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$
x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geqslant 0, \tag{3.4}
$$

where $\{\alpha_n\} \in (0,1)$ *satisfies the conditions* (i), (ii), *and* (iii) *in Theorem* 3.1. *Then the sequence* $\{x_n\}$ *defined by* (3.4) *converges strongly to* $\tilde{x}$ *such that* $\tilde{x} = P_{Fix(T)}f(\tilde{x})$ *which is equivalent to the following variational inequality:*

$$\langle \tilde{x} - f(\tilde{x}),\ x - \tilde{x}\rangle \geqslant 0, \quad \forall\, x \in Fix(T).$$

4. Applications

4.1. Application to nonlinear variational inclusion problem

Let H be a real Hilbert space, $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then, the *resolvent mapping* $J_\lambda^M : H \to H$ *associated with* M, is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x),\ \forall x \in H,$$

for some $\lambda > 0$, where I stands for the identity operator on H.

We note that for all $\lambda > 0$ the resolvent operator J_λ^M is a single-valued nonexpansive mapping.

The "so-called" *monotone variational inclusion problem (in short, MVIP)* [10] is to find $x^* \in H$ such that

$$0 \in M(x^*). \tag{4.1}$$

From the definition of resolvent mapping J_λ^M, it is easy to know that (MVIP) (4.1) is equivalent to find $x^* \in H$ such that

$$x^* \in Fix(J_\lambda^M) \ \text{ for some } \lambda > 0.$$

For any given function $x_0 \in H$, define a sequence by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_\lambda^M\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geqslant 0. \tag{4.2}$$

From Theorem 3.3 we have the following.

Theorem 4.1. *Let* M *and* J_λ^M *be the same as above. Let* $f : H \to H$ *be a contraction. Let* $\{x_n\}$ *be the sequence defined by* (4.2). *If the sequence* $\{\alpha_n\} \in (0,1)$ *satisfies the conditions* (i), (ii), *and* (iii) *in Theorem* 3.1 *and* $Fix(J_\lambda^M) \neq \emptyset$, *then* $\{x_n\}$ *converges strongly to the solution of monotone variational inclusion* (4.1), *which is also a solution of the following variational inequality:*

$$\langle \tilde{x} - f(\tilde{x}), x - \tilde{x}\rangle \geqslant 0, \quad \forall\, x \in Fix(J_\lambda^M).$$

4.2. Application to nonlinear Volterra integral equations

Let us consider the following nonlinear Volterra integral equation

$$x(t) = g(t) + \int_0^t F(t,s,x(s))ds, \quad t \in [0,1], \tag{4.3}$$

where g is a continuous function on $[0,1]$ and $F : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following condition.

$$|F(t,s,x) - F(t,s,y)| \leqslant |x-y|, \quad t,s \in [0,1], \quad x,y \in \mathbb{R}.$$

Then equation (4.3) has at least one solution in $L^2[0,1]$ (see, for example, [24]).

Define a mapping $T : L^2[0,1] \to L^2[0,1]$ by

$$(Tx)(t) = g(t) + \int_0^t F(t,s,x(s))ds, \ \ t \in [0,1].$$

It is easy to see that T is a nonexpansive mapping. This means that finding the solution of integral equation (4.3) is reduced to find a fixed point of the nonexpansive mapping T in $L^2[0,1]$.

For any given function $x_0 \in L^2[0,1]$, define a sequence of functions $\{x_n\}$ in $L^2[0,1]$ by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geqslant 0. \tag{4.4}$$

From Theorem 3.3 we have the following.

Theorem 4.2. *Let* F, g, T, $L^2[0,1]$ *be the same as above. Let* f *be a contraction on* $L^2[0,1]$ *with coefficient* $k \in [0,1)$. *Let* $\{x_n\}$ *be the sequence defined by* (4.4). *If the sequence* $\{\alpha_n\} \in (0,1)$ *satisfies the conditions* (i), (ii), *and* (iii) *in Theorem* 3.1, *then* $\{x_n\}$ *converges strongly in* $L^2[0,1]$ *to the solution of integral equation* (4.3) *which is also a solution of the following variational inequality:*

$$\langle \tilde{x} - f(\tilde{x}), x - \tilde{x} \rangle \geqslant 0, \quad \forall\, x \in \mathrm{Fix}(T).$$

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