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Tripled random coincidence point and common fixed point results of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras

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Abstract

In this paper, based on the concept of cone b-metric space over Banach algebra, which was introduced by Huang and Radenovic [H.-P. Huang, S. Radenović, J. Nonlinear Sci. Appl., **8** (2015), 787–799], we obtain some tripled common random fixed point and tripled random fixed point theorems with several generalized Lipschitz constants in such spaces. We consider the obtained assertions without the assumption of normality of cones. The presented results generalize some coupled common fixed point theorems in the existing literature. ©2017 all rights reserved.

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1. Introduction

Fixed point theory plays a basic role in applications of many branches of mathematics. Finding the fixed point of contractive mapping becomes the center of strong research activity [3, 9, 10, 24, 32, 37]. Recently, Huang and Zhang [18] and Bakhtin [8] introduced cone metric space and b-metric space, respectively, as some generalizations of usual metric spaces. They greatly expanded the famous Banach contraction principle in such setting. Since then, a lot of papers have appeared on cone metric spaces and b-metric spaces (see [1, 5, 11, 15, 21, 28, 33, 35]). Hussain and Shah [20] introduced cone b-metric space and generalized both cone metric space and b-metric space. Aydi et al. [7] and Fadail and Ahmad [14] introduced tripled fixed point of w-compatible mappings in abstract metric spaces and coupled coincidence point and common coupled fixed point results in cone b-metric spaces, respectively. However, latterly, some authors made a conclusion that fixed point results in cone metric spaces and cone b-metric spaces are just equivalent to those in metric spaces and b-metric spaces, respectively (see [6, 13, 23]). But fortunately, very recently, Liu and Xu [27] introduced the concept of cone metric space over Banach

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algebra and proved the non-equivalence of fixed point results in these new spaces and usual metric spaces. Further, Huang and Radenović [17] introduced the notion of cone b-metric space over Banach algebra and showed some fixed point results in such spaces are not some consequences of the corresponding b-metric spaces. As a result, it is essentially necessary to investigate fixed point results in cone metric spaces over Banach algebras. Random coincidence point theorems are stochastic generalizations of classical coincidence point theorems, and play an important role in the theory of random differential and integral equations. Random fixed point theorems for contractive mapping on complete separable metric space have been proved by several authors (see [4, 12, 19, 25, 26, 36]). Ćirić [12] proved some coupled random fixed point and coupled random coincidence results in partially ordered metric spaces. Afterwards, many coupled random coincidence results in partially ordered metric spaces were considered (see [4, 19, 36]). In this paper, by the concept of cone b-metric space over Banach algebra introduced by [17], we obtain tripled common random fixed point and tripled random fixed point theorems with several generalized Lipschitz constants in cone b-metric spaces over Banach algebras by omitting the normality of cones. The presented results improve the main results of [7, 14] in a large extent.

2. Preliminaries

Let $\mathcal{A}$ be a Banach algebra with a unit e, and θ the zero element of $\mathcal{A}$. A nonempty closed convex subset P of $\mathcal{A}$ is called a cone if

(i) $\{\theta, e\} \subset P$;
(ii) $P^2 = PP \subset P$, $P \bigcap (-P) = \{\theta\}$;
(iii) $\lambda P + \mu P \subset P$ for all $\lambda, \mu \geqslant 0$.

On this basis, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will indicate that $y - x \in \text{int}P$, where int$P$ stands for the interior of P. Write $\|\cdot\|$ as the norm on $\mathcal{A}$. A cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leqslant M\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

In the following we always suppose that $\mathcal{A}$ is a Banach algebra with a unit e. P is a cone in $\mathcal{A}$ with int$P \neq \emptyset$, and $\preceq$ is a partial ordering with respect to P.

Definition 2.1. Let X be a nonempty set and $\mathcal{A}$ a Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq s(d(x, z) + d(z, y))$ for all $x, y, z \in X$,

where $s \geqslant 1$ is a constant. Then d is called a cone b-metric on X, and (X, d) is called a cone b-metric space over Banach algebra $\mathcal{A}$.

Example 2.2. Let $\mathcal{A} = \mathbb{R}^2$ and $P = \{(x_1, x_2) \in \mathcal{A} : x_1, x_2 \geqslant 0\}$ be a cone in $\mathcal{A}$. Let $\|x\| = |x_1| + |x_2|$ for $x = (x_1, x_2) \in \mathcal{A}$. Take $y = (y_1, y_2)$. Define multiplication in $\mathcal{A}$, $xy = (x_1y_1, x_1y_2 + x_2y_1)$. Then $\mathcal{A}$ is a Banach algebra with a unit $e = (1, 0)$. Define $X = [0, 1] \times [0, 1]$, $d : X \times X \to \mathcal{A}$ by $d(x, y) = (|x_1 - x_2|^\alpha, |y_1 - y_2|^\alpha)$ $(\alpha \geqslant 1)$. Since $a^p + b^p \leqslant (a + b)^p$ for all $a, b \geqslant 0, p \geqslant 1$, we have $d(x, y) \preceq s[d(x, z) + d(z, y)]$, for any $x, y, z \in X$, where $s = 2^{\alpha-1}$. Then it is not hard to verify that (X, d) is a complete cone b-metric space over Banach algebra $\mathcal{A}$.

Definition 2.3 ([20]). Let (X, d) be a cone b-metric space over Banach algebra, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever, for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geqslant N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$.

(iii) (X, d) is complete if every Cauchy sequence is convergent.

The following lemmas are often used (in particular when dealing with cone b-metric spaces in which the cones need not to be normal).

Lemma 2.4 ([20]). *Let (X, d) be a cone b-metric space over Banach algebra $\mathcal{A}$ and P a cone in $\mathcal{A}$. Then the following properties are often used.*

(1) *If $c \in \mathrm{int}P$ and $\theta \preceq a_n \to \theta$ $(n \to \infty)$, then there exists N such that for all $n > N$, we have $a_n \ll c$;*
(2) *If $x \preceq y$ and $y \ll z$, then $x \ll z$;*
(3) *If $\theta \preceq u \ll c$ for each $c \in \mathrm{int}P$, then $u = \theta$;*
(4) *If $u \in P$ and $u \preceq ku$ for some $0 \leqslant k < 1$, then $u = \theta$;*
(5) *If $a \preceq b + c$ for each $c \in \mathrm{int}P$, then $a \preceq b$;*
(6) *Let $\theta \ll c$. If $\theta \preceq d(x_n, x) \preceq b_n$ and $b_n \to \theta$ $(n \to \infty)$, then $d(x_n, x) \ll c$, where $\{x_n\}$, x are a sequence and a given point in X, respectively.*

Lemma 2.5 ([31]). *Let $\mathcal{A}$ be a Banach algebra with a unit e and $x \in \mathcal{A}$, then $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(x)$ satisfies*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $\rho(x) < |\lambda|$, then $\lambda e - x$ is invertible in $\mathcal{A}$, moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 2.6 ([31]). *Let $\mathcal{A}$ be a Banach algebra with a unit e, and $a, b \in \mathcal{A}$. If a commutes with b, then*

$$\rho(a + b) \leqslant \rho(a) + \rho(b), \qquad \rho(ab) \leqslant \rho(a)\rho(b).$$

Definition 2.7 ([34]). An element $(x, y, z) \in X^3$ is said to be a tripled fixed point of the mapping $F : X^3 \to X$ if $F(x, y, z) = x, F(y, z, x) = y$, and $F(z, x, y) = z$.

Note that if (x, y, z) is a tripled fixed point of F, then (y, z, x) and (z, x, y) are tripled fixed points of F too.

Definition 2.8 ([34]). An element $(x, y, z) \in X^3$ is called

(1) a tripled coincidence point of the mappings $F : X^3 \to X$ and $g : X \to X$ if $F(x, y, z) = gx$, $F(y, z, x) = gy, F(z, x, y) = gz$, and (gx, gy, gz) is called a tripled point of coincidence;
(2) a common tripled fixed point of mappings $F : X^3 \to X$ and $g : X \to X$ if $F(x, y, z) = gx = x$, $F(y, z, x) = gy = y$, and $F(z, x, y) = gz = z$.

Definition 2.9 ([2]). The mappings $F : X^3 \to X$ and $g : X \to X$ are called *w*-compatible provided that $gF(x, y, z) = F(gx, gy, gz)$ whenever $F(x, y, z) = gx, F(y, z, x) = gy$ and $F(z, x, y) = gz$.

Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω and let (X, d) be a metric space. A mapping $T : \Omega \to X$ is called Σ-measurable if for any open subset U of X, $T^{-1}(U) = \{\omega : T(\omega) \in U\} \in \Sigma$ (see [16]). In what follows, when we speak of measurability we shall mean Σ-measurability. A mapping $T : \Omega \times X \to X$ is called a random operator if for any $x \in X, T(\cdot, x)$ is measurable. A measurable mapping $\xi : \Omega \to X$ is called a random fixed point of a random operator $T : \Omega \times X \to X$, if $\xi(\omega) = T(\omega, \xi(\omega))$ for every $\omega \in \Omega$.

Definition 2.10 ([19]). Let (X, d) be a separable metric space and (Ω, Σ) be a measurable space. Then $F : \Omega \times X^3 \to X$ and $g : \Omega \times X \to X$ are said to be w-compatible random operators if

$$g(\omega, F(\omega, (x, y, z))) = F(\omega, (g(\omega, x), g(\omega, y), g(\omega, z))),$$

whenever $F(\omega, (x, y, z)) = g(\omega, x), F(\omega, (y, z, x)) = g(\omega, y), F(\omega, (z, x, y)) = g(\omega, z)$ for all $\omega \in \Omega$ and $x, y, z \in X$ are satisfied.

Lemma 2.11 ([39]). *Let P be a cone in a Banach algebra $\mathcal{A}$ and $k \in P$ be a given vector. Let $\{u_n\}$ be a sequence in P. If for each $c_1 \gg \theta$, there exists N_1 such that $u_n \ll c_1$ for all $n > N_1$, then for each $c_2 \gg \theta$, there exists N_2 such that $ku_n \ll c_2$ for all $n > N_2$.*

Now, we state our main results as follows.

3. Main results

In this section, we prove some tripled random coincidence and tripled random fixed point theorems for contractive mappings with several generalized Lipschitz constants in the setting of cone b-metric spaces over Banach algebras by deleting the normality of cones.

Lemma 3.1 ([39]). *Let$\mathcal{A}$ be a Banach algebra and $k \in \mathcal{A}$. If $\rho(k) < 1$, then $\lim_{n\to\infty} \|k^n\| = 0$.*

Remark 3.2. If $\|k\| < 1$, it is natural that $\rho(k) < 1$, yet, the converse is not true.

Lemma 3.3 ([39]). *Let $\mathcal{A}$ be a Banach algebra with a unit e, $\{x_n\}$ a sequence in $\mathcal{A}$. If $\{x_n\}$ converges to x in $\mathcal{A}$, and for any $n \geqslant 1$, $\{x_n\}$ commutes with x, then $\rho(x_n) \to \rho(x)$ as $n \to \infty$*

Theorem 3.4. *Let (X, d) be a separable cone b-metric space over Banach algebra $\mathcal{A}$, P be a cone in $\mathcal{A}$ and (Ω, Σ) be a measurable space. Suppose that the mappings $F : \Omega \times X^3 \to X$ and $g : \Omega \times X \to X$ satisfy the following contractive condition:*

$$\begin{aligned} d(F(\omega, (x, y, z)), F(\omega, (u, v, w))) \preceq {} & [a_1 d(g(\omega, x), F(\omega, (x, y, z))) + a_2 d(g(\omega, y), F(\omega, (y, z, x))) \\ & + a_3 d(g(\omega, z), F(\omega, (z, x, y)))] + [a_4 d(g(\omega, u), F(\omega, (u, v, w))) \\ & + a_5 d(g(\omega, v), F(\omega, (v, w, u))) + a_6 d(g(\omega, w), F(\omega, (w, u, v)))] \\ & + [a_7 d(g(\omega, x), F(\omega, (u, v, w))) + a_8 d(g(\omega, y), F(\omega, (v, w, u))) \\ & + a_9 d(g(\omega, z), F(\omega, (w, u, v)))] + [a_{10} d(g(\omega, u), F(\omega, (x, y, z))) \\ & + a_{11} d(g(\omega, v), F(\omega, (y, z, x))) + a_{12} d(g(\omega, w), F(\omega, (z, x, y)))] \\ & + [a_{13} d(g(\omega, x), g(\omega, u)) + a_{14} d(g(\omega, y), g(\omega, v)) \\ & + a_{15} d(g(\omega, z), g(\omega, w))], \end{aligned} \tag{3.1}$$

for all $x, y, z, u, v, w \in X$, where $a_i \in P$, $a_i a_j = a_j a_i$ $(i, j = 1, \dots, 15)$, a_i are generalized Lipschitz constants with $(s+1)\rho(a_1 + \cdots + a_6) + s(s+1)\rho(a_7 + \cdots + a_{12}) + 2s\rho(a_{13} + a_{14} + a_{15}) < 2$ and $\rho\big(sa_1 + sa_2 + sa_3 + s^2 a_{10} + s^2 a_{11} + s^2 a_{12}\big) < 1$, where $s \geqslant 1$ is a constant. Let $F(\cdot, v), g(\cdot, x)$ be measurable for $v \in X^3$ and $x \in X$, respectively. Suppose that $F(\omega \times X^3) \subseteq g(\omega \times X)$ and $g(\omega \times X)$ is a complete subspace of X for each $\omega \in \Omega$. Then there exist mappings $\xi, \eta, \theta : \Omega \to X$ such that $F(\omega, (\xi(\omega), \eta(\omega), \theta(\omega))) = g(\omega, \xi(\omega))$, $F(\omega, (\eta(\omega), \theta(\omega), \xi(\omega))) = g(\omega, \eta(\omega))$ and $F(\omega, (\theta(\omega), \xi(\omega), \eta(\omega))) = g(\omega, \theta(\omega))$ for all $\omega \in \Omega$, that is, F and g have a tripled random coincidence point.

Proof. Let $\Theta = \{\eta : \Omega \to X\}$ be a family of measurable mappings. We construct three sequences of measurable mappings $\{\xi_n\}$, $\{\eta_n\}$, $\{\theta_n\}$ in Θ and three sequences $\{g(\omega, \xi_n(\omega))\}$, $\{g(\omega, \eta_n(\omega))\}$, $\{g(\omega, \theta_n(\omega))\}$ in X as follows.

Let $\xi_0, \eta_0, \theta_0 \in \Theta$. Since $F(\omega, (\xi_0(\omega), \eta_0(\omega), \theta_0(\omega))) \in F(\omega \times X^3) \subseteq g(\omega \times X)$, by a sort of Filippov measurable implicit function theorems (see [37, 38]), there is $\xi_1 \in \Theta$ such that $g(\omega, \xi_1(\omega)) = F(\omega, (\xi_0(\omega), \eta_0(\omega), \theta_0(\omega)))$. Similarly as $F(\omega, (\eta_0(\omega), \theta_0(\omega), \xi_0(\omega))) \in g(\omega \times X)$, there is $\eta_1 \in \Theta$ such that $g(\omega, \eta_1(\omega)) = F(\omega, (\eta_0(\omega), \theta_0(\omega), \xi_0(\omega)))$, $F(\omega, (\theta_0(\omega), \xi_0(\omega), \eta_0(\omega))) \in g(\omega \times X)$, there is $\theta_1 \in \Theta$ such that

$$g(\omega,\theta_1(\omega)) = F(\omega,(\theta_0(\omega),\xi_0(\omega),\eta_0(\omega))).$$

Continuing this process we can construct sequences $\{\xi_n(\omega)\}$, $\{\eta_n(\omega)\}$, and $\{\theta_n(\omega)\}$ in X such that

$$\begin{aligned} g(\omega,\xi_{n+1}(\omega)) &= F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))),\\ g(\omega,\eta_{n+1}(\omega)) &= F(\omega,(\eta_n(\omega),\theta_n(\omega)\xi_n(\omega))),\\ g(\omega,\theta_{n+1}(\omega)) &= F(\omega,(\theta_n(\omega),\xi_n(\omega),\eta_n(\omega))), \end{aligned}$$

for all $n \in \mathbb{N}$. According to (3.1), we have

$$\begin{aligned} d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) &= d(F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\theta_{n-1}(\omega))),F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))))\\ &\preceq [a_1 d(g(\omega,\xi_{n-1}(\omega)),F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\theta_{n-1}(\omega))))\\ &\quad + a_2 d(g(\omega,\eta_{n-1}(\omega)),F(\omega,(\eta_{n-1}(\omega),\theta_{n-1}(\omega),\xi_{n-1}(\omega))))\\ &\quad + a_3 d(g(\omega,\theta_{n-1}(\omega)),F(\omega,(\theta_{n-1}(\omega),\xi_{n-1}(\omega),\eta_{n-1}(\omega))))]\\ &\quad + [a_4 d(g(\omega,\xi_n(\omega)),F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))))\\ &\quad + a_5 d(g(\omega,\eta_n(\omega)),F(\omega,(\eta_n(\omega),\theta_n(\omega),\xi_n(\omega))))\\ &\quad + a_6 d(g(\omega,\theta_n(\omega)),F(\omega,(\theta_n(\omega),\xi_n(\omega),\eta_n(\omega))))]\\ &\quad + [a_7 d(g(\omega,\xi_{n-1}(\omega)),F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))))\\ &\quad + a_8 d(g(\omega,\eta_{n-1}(\omega)),F(\omega,(\eta_n(\omega),\theta_n(\omega),\xi_n(\omega))))\\ &\quad + a_9 d(g(\omega,\theta_{n-1}(\omega)),F(\omega,(\theta_n(\omega),\xi_n(\omega),\eta_n(\omega))))]\\ &\quad + [a_{10} d(g(\omega,\xi_n(\omega)),F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\theta_{n-1}(\omega))))\\ &\quad + a_{11} d(g(\omega,\eta_n(\omega)),F(\omega,(\eta_{n-1}(\omega),\theta_{n-1}(\omega),\xi_{n-1}(\omega))))\\ &\quad + a_{12} d(g(\omega,\theta_n(\omega)),F(\omega,(\theta_{n-1}(\omega),\xi_{n-1}(\omega),\eta_{n-1}(\omega))))]\\ &\quad + [a_{13} d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega)))\\ &\quad + a_{14} d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega)))\\ &\quad + a_{15} d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega)))]. \end{aligned}$$

Further, we have

$$\begin{aligned} &d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega)))\\ &\quad = d(F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\theta_{n-1}(\omega))),F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))))\\ &\quad \preceq [a_1 d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_2 d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega)))\\ &\qquad + a_3 d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega)))] + [a_4 d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega)))\\ &\qquad + a_5 d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))) + a_6 d(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega)))]\\ &\qquad + [a_7 d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_{n+1}(\omega))) + a_8 d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_{n+1}(\omega)))\\ &\qquad + a_9 d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_{n+1}(\omega)))] + [a_{10} d(g(\omega,\xi_n(\omega)),g(\omega,\xi_n(\omega)))\\ &\qquad + a_{11} d(g(\omega,\eta_n(\omega)),g(\omega,\eta_n(\omega))) + a_{12} d(g(\omega,\theta_n(\omega)),g(\omega,\theta_n(\omega)))]\\ &\qquad + [a_{13} d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_{14} d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega)))\\ &\qquad + a_{15} d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega)))]\\ &\quad \preceq [a_1 d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_2 d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega)))\\ &\qquad + a_3 d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega)))] + [a_4 d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega)))\\ &\qquad + a_5 d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))) + a_6 d(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega)))]\\ &\qquad + [sa_7(d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))))\\ &\qquad + sa_8(d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))) + d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))))\\ &\qquad + sa_9(d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega))) + d(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega))))] \end{aligned}$$

$$+ [a_{13}d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) + a_{14}d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))) \\ + a_{15}d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega)))].$$

Hence, we obtain that

$$\begin{aligned} d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) \preceq & [(a_1+sa_7+a_{13})d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) \\ & + (a_2+sa_8+a_{14})d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))) \\ & + (a_3+sa_9+a_{15})d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega)))] \\ & + [(a_4+sa_7)d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) \\ & + (a_5+sa_8)d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))) \\ & + (a_6+sa_9)d(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega)))]. \end{aligned} \tag{3.2}$$

Similarly, we can prove that

$$\begin{aligned} d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))) \preceq & [(a_1+sa_7+a_{13})d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega))) \\ & + (a_2+sa_8+a_{14})d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega))) \\ & + (a_3+sa_9+a_{15})d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega)))] \\ & + [(a_4+sa_7)d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))) \\ & + (a_5+sa_8)d(g(\omega,\theta_n(\omega)),d(g(\omega,\theta_{n+1}(\omega))) \\ & + (a_6+sa_9)d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega)))] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega))) \preceq & [(a_1+sa_7+a_{13})d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega))) \\ & + (a_2+sa_8+a_{14})d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) \\ & + (a_3+sa_9+a_{15})d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega)))] \\ & + [(a_4+sa_7)d(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega))) \\ & + (a_5+sa_8)d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) \\ & + (a_6+sa_9)d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))). \end{aligned} \tag{3.4}$$

Put

$$d_n = d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) + d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))) + d(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega))).$$

Uniting (3.2)-(3.4), ones assert that

$$d_n \preceq (a_1+a_2+a_3+sa_7+sa_8+sa_9+a_{13}+a_{14}+a_{15})d_{n-1} + (a_4+a_5+a_6+sa_7+sa_8+sa_9)d_n. \tag{3.5}$$

Furthermore,

$$\begin{aligned} d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi_n(\omega))) = & d(F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))),F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\theta_{n-1}(\omega)))) \\ \preceq & [a_1 d(g(\omega,\xi_n(\omega)),F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega)))) \\ & + a_2 d(g(\omega,\eta_n(\omega),F(\omega,(\eta_n(\omega),\theta_n(\omega),\xi_n(\omega)))) \\ & + a_3 d(g(\omega,\theta_n(\omega)),F(\omega,(\theta_n(\omega),\xi_n(\omega),\eta_n(\omega))))] \\ & + [a_4 d(g(\omega,\xi_{n-1}(\omega)),F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\theta_{n-1}(\omega)))) \\ & + a_5 d(g(\omega,\eta_{n-1}(\omega)),F(\omega,(\eta_{n-1}(\omega),\theta_{n-1}(\omega),\xi_{n-1}(\omega)))) \\ & + a_6 d(g(\omega,\theta_{n-1}(\omega)),F(\omega,(\theta_{n-1}(\omega),\xi_{n-1}(\omega),\eta_{n-1}(\omega))))] \\ & + [a_7 d(g(\omega,\xi_n(\omega)),F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\theta_{n-1}(\omega)))) \end{aligned}$$

$$
\begin{aligned}
&+ a_8 d(g(\omega,\eta_n(\omega)), F(\omega,(\eta_{n-1}(\omega),\theta_{n-1}(\omega),\xi_{n-1}(\omega))))\\
&+ a_9 d(g(\omega,\theta_n(\omega)), F(\omega,(\theta_{n-1}(\omega),\xi_{n-1}(\omega),\eta_{n-1}(\omega))))]\\
&+ [a_{10} d(g(\omega,\xi_{n-1}(\omega)), F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))))\\
&+ a_{11} d(g(\omega,\eta_{n-1}(\omega)), F(\omega,(\eta_n(\omega),\theta_n(\omega),\xi_n(\omega))))\\
&+ a_{12} d(g(\omega,\theta_{n-1}(\omega)), F(\omega,(\theta_n(\omega),\xi_n(\omega),\eta_n(\omega))))]\\
&+ [a_{13} d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n-1}(\omega)))\\
&+ a_{14} d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n-1}(\omega)))\\
&+ a_{15} d(g(\omega,\theta_n(\omega)), g(\omega,\theta_{n-1}(\omega)))].
\end{aligned}
$$

Then

$$
\begin{aligned}
&d(g(\omega,\xi_{n+1}(\omega)), g(\omega,\xi_n(\omega)))\\
&\quad= d(F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))), F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega),\theta_{n-1}(\omega))))\\
&\quad\preceq [a_1 d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) + a_2 d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n+1}(\omega)))\\
&\qquad+ a_3 d(g(\omega,\theta_n(\omega)), g(\omega,\theta_{n+1}(\omega)))] + [a_4 d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega)))\\
&\qquad+ a_5 d(g(\omega,\eta_{n-1}(\omega)), g(\omega,\eta_n(\omega))) + a_6 d(g(\omega,\theta_{n-1}(\omega)), g(\omega,\theta_n(\omega)))]\\
&\qquad+ [a_7 d(g(\omega,\xi_n(\omega)), g(\omega,\xi_n(\omega))) + a_8 d(g(\omega,\eta_n(\omega)), g(\omega,\eta_n(\omega)))\\
&\qquad+ a_9 d(g(\omega,\theta_n(\omega)), g(\omega,\theta_n(\omega)))] + [a_{10} d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_{n+1}(\omega)))\\
&\qquad+ a_{11} d(g(\omega,\eta_{n-1}(\omega)), g(\omega,\eta_{n+1}(\omega))) + a_{12} d(g(\omega,\theta_{n-1}(\omega)), g(\omega,\theta_{n+1}(\omega)))]\\
&\qquad+ [a_{13} d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n-1}(\omega))) + a_{14} d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n-1}(\omega)))\\
&\qquad+ a_{15} d(g(\omega,\theta_n(\omega)), g(\omega,\theta_{n-1}(\omega)))]\\
&\quad\preceq [a_1 d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) + a_2 d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n+1}(\omega)))\\
&\qquad+ a_3 d(g(\omega,\theta_n(\omega)), g(\omega,\theta_{n+1}(\omega)))] + [a_4 d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega)))\\
&\qquad+ a_5 d(g(\omega,\eta_{n-1}(\omega)), g(\omega,\eta_n(\omega))) + a_6 d(g(\omega,\theta_{n-1}(\omega)), g(\omega,\theta_n(\omega)))]\\
&\qquad+ [sa_{10}(d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) + d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))))\\
&\qquad+ sa_{11}(d(g(\omega,\eta_{n-1}(\omega)), g(\omega,\eta_n(\omega))) + d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n+1}(\omega))))\\
&\qquad+ [sa_{12}(d(g(\omega,\theta_{n-1}(\omega)), g(\omega,\theta_n(\omega))) + d(g(\omega,\theta_n(\omega)), g(\omega,\theta_{n+1}(\omega))))]\\
&\qquad+ [a_{13} d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega))) + a_{14} d(g(\omega,\eta_{n-1}(\omega)), g(\omega,\eta_n(\omega)))\\
&\qquad+ a_{15} d(g(\omega,\theta_{n-1}(\omega)), g(\omega,\theta_n(\omega)))].
\end{aligned}
$$

Accordingly, it is clear that

$$
\begin{aligned}
d(g(\omega,\xi_{n+1}(\omega)), g(\omega,\xi_n(\omega))) \preceq\ & [(a_4 + sa_{10} + a_{13}) d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega)))\\
&+ (a_5 + sa_{11} + a_{14}) d(g(\omega,\eta_{n-1}(\omega)), g(\omega,\eta_n(\omega)))\\
&+ (a_6 + sa_{12} + a_{15}) d(g(\omega,\theta_{n-1}(\omega)), g(\omega,\theta_n(\omega)))]\\
&+ [(a_1 + sa_{10}) d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega)))\\
&+ (a_2 + sa_{11}) d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n+1}(\omega)))\\
&+ (a_3 + sa_{12}) d(g(\omega,\theta_n(\omega)), g(\omega,\theta_{n+1}(\omega)))].
\end{aligned} \tag{3.6}
$$

Similarly, we can prove that

$$
\begin{aligned}
d(g(\omega,\eta_{n+1}(\omega)), g(\omega,\eta_n(\omega))) \preceq\ & [(a_4 + sa_{10} + a_{13}) d(g(\omega,\eta_{n-1}(\omega)), g(\omega,\eta_n(\omega)))\\
&+ (a_5 + sa_{11} + a_{14}) d(g(\omega,\theta_{n-1}(\omega)), g(\omega,\eta_n(\omega)))\\
&+ (a_6 + sa_{12} + a_{15}) d(g(\omega,\xi_{n-1}(\omega)), g(\omega,\xi_n(\omega)))]\\
&+ [(a_1 + sa_{10}) d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n+1}(\omega)))\\
&+ (a_2 + sa_{11}) d(g(\omega,\theta_n(\omega)), g(\omega,\theta_{n+1}(\omega)))\\
&+ (a_3 + sa_{12}) d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega)))]
\end{aligned} \tag{3.7}
$$

and

$$\begin{aligned} d(g(\omega,\theta_{n+1}(\omega)),g(\omega,\theta_n(\omega))) \preceq {} & [(a_4+sa_{10}+a_{13})d(g(\omega,\theta_{n-1}(\omega)),g(\omega,\theta_n(\omega))) \\ & +(a_5+sa_{11}+a_{14})d(g(\omega,\xi_{n-1}(\omega)),g(\omega,\xi_n(\omega))) \\ & +(a_6+sa_{12}+a_{15})d(g(\omega,\eta_{n-1}(\omega)),g(\omega,\eta_n(\omega)))] \\ & +[(a_1+sa_{10})d(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega))) \\ & +(a_2+sa_{11})d(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega))) \\ & +(a_3+sa_{12})d(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega))). \end{aligned} \tag{3.8}$$

Uniting (3.6)-(3.8), one gets that

$$\begin{aligned} d_n \preceq {} & (a_4+a_5+a_6+sa_{10}+sa_{11}+sa_{12}+a_{13}+a_{14}+a_{15})d_{n-1} \\ & +(a_1+a_2+a_3+sa_{10}+sa_{11}+sa_{12})d_n. \end{aligned} \tag{3.9}$$

By using (3.5) and (3.9), it is easy to see that

$$\begin{aligned} 2d_n \preceq {} & (a_1+a_2+a_3+a_4+a_5+a_6+sa_7+sa_8+sa_9+sa_{10}+sa_{11} \\ & +sa_{12}+2(a_{13}+a_{14}+a_{15}))d_{n-1}+(a_1+a_2+a_3+a_4 \\ & +a_5+a_6+sa_7+sa_8+sa_9+sa_{10}+sa_{11}+sa_{12})d_n. \end{aligned}$$

Put $k_1 = a_{13}+a_{14}+a_{15}$ and $k=\sum_{i=1}^{6} a_i + \sum_{i=7}^{12} sa_i$, then

$$(2e-k)d_n \preceq (2k_1+k)d_{n-1}. \tag{3.10}$$

Because of $(s+1)\rho\Big(\sum_{i=1}^{6} a_i\Big)+s(s+1)\rho\Big(\sum_{i=7}^{12} sa_i\Big)+2s\rho(k_1) < 2$ and $s \geqslant 1$, it is clear that

$$\rho\Big(\sum_{i=1}^{6} a_i + \sum_{i=7}^{12} sa_i\Big) \leqslant \rho\Big(\sum_{i=1}^{6} a_i\Big) + s\rho\Big(\sum_{i=7}^{12} a_i\Big) < 2.$$

Then by Lemma 2.5 and Lemma 2.6, it follows that $2e-k$ is invertible. Furthermore, $(2e-k)^{-1} = \sum\limits_{i=0}^{\infty} \frac{k^i}{2^{i+1}}$. Multiplying in both side of (3.10) by $(2e-k)^{-1}$, we obtain

$$d_n \preceq (2e-k)^{-1}(2k_1+k)d_{n-1}. \tag{3.11}$$

Denote $h = (2e-k)^{-1}(2k_1+k)$, then by (3.11) we get

$$d_n \preceq hd_{n-1} \preceq \cdots \preceq h^n d_0. \tag{3.12}$$

Note by Lemma 2.6 that

$$\rho\Big(\sum_{i=0}^{n} \frac{k^i}{2^{i+1}}\Big) \leqslant \sum_{i=0}^{n} \rho\Big(\frac{k^i}{2^{i+1}}\Big) \leqslant \sum_{i=0}^{n} \frac{[\rho(k)]^i}{2^{i+1}},$$

so by Lemma 3.3 it leads to

$$\rho\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big) \leqslant \sum_{i=0}^{\infty} \frac{[\rho(k)]^i}{2^{i+1}}.$$

Because $a_i a_j = a_j a_i$ $(i,j=1,\dots,15)$ implies k_1 commutes k, we have

$$\begin{aligned} (2e-k)^{-1}(2k_1+k) &= \Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big)(2k_1+k) \\ &= 2\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big)k_1 + \sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} \end{aligned}$$

$$= 2k_1\Big(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}}\Big)+k\Big(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}}\Big)$$
$$= (2k_1+k)\Big(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}}\Big) = (2k_1+k)(2e-k)^{-1},$$

that is to say, $(2e-k)^{-1}$ commutes with $2k_1+k$. Note that $(s+1)\rho\Big(\sum_{i=1}^{6}a_i\Big)+s(s+1)\rho\Big(\sum_{i=7}^{12}sa_i\Big)+2s\rho(k_1)<2$ means $2s\rho(k_1)+(s+1)\rho(k)<2$, then by Lemma 2.6 we gain

$$\begin{aligned}\rho(h)=\rho\big((2e-k)^{-1}(2k_1+k)\big)\leqslant\rho\big((2e-k)^{-1}\big)\rho(2k_1+k)&\leqslant\rho\Big(\sum_{i=0}^{\infty}\frac{k^i}{2^{i+1}}\Big)\big(2\rho(k_1)+\rho(k)\big)\\ &\leqslant\Big(\sum_{i=0}^{\infty}\frac{[\rho(k)]^i}{2^{i+1}}\Big)\big(2\rho(k_1)+\rho(k)\big)\\ &\leqslant\frac{1}{2-\rho(k)}\big(2\rho(k_1)+\rho(k)\big)<\frac{1}{s}\leqslant 1,\end{aligned}$$

which establishes that $e-h$ is invertible and $\|h^n\|\to 0$ as $n\to\infty$. Thus for all $m>n\geqslant 1$, ones have

$$\begin{aligned}d(g(\omega,\xi_n(\omega)),g(\omega,\xi_m(\omega)))&\preceq sd(g(\omega,\xi_n(\omega)),g(\omega,\xi_{n+1}(\omega)))\\&\quad+s^2d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi_{n+2}(\omega)))\\&\quad\vdots\\&\quad+s^{m-n}d(g(\omega,\xi_{m-1}(\omega)),g(\omega,\xi_m(\omega))),\\ d(g(\omega,\eta_n(\omega)),g(\omega,\eta_m(\omega)))&\preceq sd(g(\omega,\eta_n(\omega)),g(\omega,\eta_{n+1}(\omega)))\\&\quad+s^2d(g(\omega,\eta_{n+1}(\omega)),g(\omega,\eta_{n+2}(\omega)))\\&\quad\vdots\\&\quad+s^{m-n}d(g(\omega,\eta_{m-1}(\omega)),g(\omega,\eta_m(\omega))),\\ d(g(\omega,\theta_n(\omega)),g(\omega,\theta_m(\omega)))&\preceq sd(g(\omega,\theta_n(\omega)),g(\omega,\theta_{n+1}(\omega)))\\&\quad+s^2d(g(\omega,\theta_{n+1}(\omega)),g(\omega,\theta_{n+2}(\omega)))\\&\quad\vdots\\&\quad+s^{m-n}d(g(\omega,\theta_{m-1}(\omega)),g(\omega,\theta_m(\omega))).\end{aligned}$$

Now, by (3.12) and $s\rho(h)<1$, it follows that

$$\begin{aligned}&d(g(\omega,\xi_n(\omega)),g(\omega,\xi_m(\omega)))+d(g(\omega,\eta_n(\omega)),g(\omega,\eta_m(\omega)))+d(g(\omega,\theta_n(\omega)),g(\omega,\theta_m(\omega)))\\&\quad\preceq sd_n+s^2d_{n+1}+\cdots+s^{m-n}d_{m-1}\\&\quad\preceq sh^nd_0+s^2h^{n+1}d_0+\cdots+s^{m-n}h^{m-1}d_0\\&\quad=(sh^n+s^2h^{n+1}+\cdots+s^{m-n}h^{m-1})d_0\\&\quad=sh^n(e+sh+(sh)^2+\cdots+(sh)^{m-n-1})d_0\preceq(e-sh)^{-1}sh^nd_0.\end{aligned}\tag{3.13}$$

Owing to

$$\|(e-sh)^{-1}sh^nd_0\|\leqslant\|(e-sh)^{-1}\|s\|h^n\|\|d_0\|\to 0\ \ (n\to\infty),$$

we have $(e-sh)^{-1}sh^nd_0\to\theta\ \ (n\to\infty)$.

According to Lemma 2.4, and for any $c\gg\theta$, there exists N_0 such that for all $n>N_0$, $(e-sh)^{-1}sh^nd_0\ll c$. Furthermore, from (3.13) and for any $m>n>N_0$, it follows that

$$(g(\omega,\xi_n(\omega)),g(\omega,\xi_m(\omega)))+d(g(\omega,\eta_n(\omega)),g(\omega,\eta_m(\omega)))+d(g(\omega,\theta_n(\omega)),g(\omega,\theta_m(\omega)))\ll c,$$

which implies that

$$d(g(\omega,\xi_n(\omega)),g(\omega,\xi_m(\omega)))\ll c,\, d(g(\omega,\eta_n(\omega)),g(\omega,\eta_m(\omega)))\ll c,\, d(g(\omega,\theta_n(\omega)),g(\omega,\theta_m(\omega)))\ll c.$$

Hence, $\{g(\omega,\xi_n(\omega))\}$, $\{g(\omega,\eta_n(\omega))\}$, $\{g(\omega,\theta_n(\omega))\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist $\xi^*(\omega),\eta^*(\omega)$, and $\theta^*(\omega)\in X$ for all $\omega\in\Omega$ such that

$$g(\omega,\xi_n(\omega))\to g(\omega,\xi^*(\omega)),\, g(\omega,\eta_n(\omega))\to g(\omega,\eta^*(\omega)),\, g(\omega,\theta_n(\omega))\to g(\omega,\theta^*(\omega))\quad \text{as } n\to\infty.$$

Moreover, note that

$$\begin{aligned}
&d(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi^*(\omega)))\\
&\preceq s(d(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi_{n+1}(\omega)))+d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega))))\\
&= sd(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))))+d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega))))\\
&\preceq s[a_1d(g(\omega,\xi^*(\omega)),F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))))+a_2d(g(\omega,\eta^*(\omega)),F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))))\\
&\quad +a_3d(g(\omega,\theta^*(\omega)),F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))))]+s[a_4d(g(\omega,\xi_n(\omega)),F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))))\\
&\quad +a_5d(g(\omega,\eta_n(\omega)),F(\omega,(\eta_n(\omega),\theta_n(\omega),\xi_n(\omega))))+a_6d(g(\omega,\theta_n(\omega)),F(\omega,(\theta_n(\omega),\xi_n(\omega),\eta_n(\omega))))]\\
&\quad +s[a_7d(g(\omega,\xi^*(\omega)),F(\omega,(\xi_n(\omega),\eta_n(\omega),\theta_n(\omega))))+a_8d(g(\omega,\eta^*(\omega)),F(\omega,(\eta_n(\omega),\theta_n(\omega),\xi_n(\omega))))\\
&\quad +a_9d(g(\omega,\theta^*(\omega)),F(\omega,(\theta_n(\omega),\xi_n(\omega),\eta_n(\omega))))]+s[a_{10}(g(\omega,\xi_n(\omega)),F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))))\\
&\quad +a_{11}d(g(\omega,\eta_n(\omega)),F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))))+a_{12}d(g(\omega,\theta_n(\omega)),F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))))]\\
&\quad +s[a_{13}d(g(\omega,\xi^*(\omega)),g(\omega,\xi_n(\omega)))+a_{14}d(g(\omega,\eta^*(\omega)),g(\omega,\eta_n(\omega)))\\
&\quad +a_{15}d(g(\omega,\theta^*(\omega)),g(\omega,\theta_n(\omega)))]+sd(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega)))\\
&\preceq s[a_1d(g(\omega,\xi^*(\omega)),F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))))+a_2d(g(\omega,\eta^*(\omega)),F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))))\\
&\quad +a_3d(g(\omega,\theta^*(\omega)),F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))))]+s[sa_4d(g(\omega,\xi_n(\omega)),g(\omega,\xi^*(\omega)))\\
&\quad +sa_4d(g(\omega,\xi^*(\omega))),g(\omega,\xi_{n+1}(\omega)))+sa_5d(g(\omega,\eta_n(\omega)),g(\omega,\eta^*(\omega)))\\
&\quad +sa_5d(g(\omega,\eta^*(\omega))),g(\omega,\eta_{n+1}(\omega)))+sa_6d(g(\omega,\theta_n(\omega)),g(\omega,\theta^*(\omega)))\\
&\quad +sa_6d(g(\omega,\theta^*(\omega))),g(\omega,\theta_{n+1}(\omega)))]+s[a_7d(g(\omega,\xi^*(\omega)),g(\omega,\xi_{n+1}(\omega)))\\
&\quad +a_8d(g(\omega,\eta^*(\omega))),g(\omega,\eta_{n+1}(\omega)))+a_9d(g(\omega,\theta^*(\omega))),g(\omega,\theta_{n+1}(\omega)))]\\
&\quad +s[sa_{10}g(\omega,\xi_n(\omega)),g(\omega,\xi^*(\omega)))+sa_{10}g(\omega,\xi^*(\omega)),F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))))\\
&\quad +sa_{11}g(\omega,\eta_n(\omega)),g(\omega,\eta^*(\omega)))+sa_{11}g(\omega,\eta^*(\omega)),F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))))\\
&\quad +sa_{12}g(\omega,\theta_n(\omega)),g(\omega,\theta^*(\omega)))+sa_{12}g(\omega,\theta^*(\omega)),F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))))]\\
&\quad +s[a_{13}d(g(\omega,\xi^*(\omega)),g(\omega,\xi_n(\omega)))+a_{14}d(g(\omega,\eta^*(\omega)),g(\omega,\eta_n(\omega)))\\
&\quad +a_{15}d(g(\omega,\theta^*(\omega)),g(\omega,\theta_n(\omega)))]+sd(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega))).
\end{aligned}$$

Hence, ones acquire that

$$\begin{aligned}
&d(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi^*(\omega)))\\
&\preceq (sa_1+s^2a_{10})d(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi^*(\omega)))\\
&\quad +(sa_2+s^2a_{11})d(F(\omega,\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))),g(\omega,\eta^*(\omega)))\\
&\quad +(sa_3+s^2a_{12})d(F(\omega,\theta^*(\omega),\xi*(\omega),\eta^*(\omega))),g(\omega,\theta^*(\omega)))\\
&\quad +(s^2a_4+s^2a_{10}+sa_{13})d(g(\omega,\xi_n(\omega)),g(\omega,\xi^*(\omega)))\\
&\quad +(s^2a_4+sa_7+s)d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega)))\\
&\quad +(s^2a_5+s^2a_{11}+sa_{14})d(g(\omega,\eta_n(\omega)),g(\omega,\eta^*(\omega)))\\
&\quad +(s^2a_6+s^2a_{12}+sa_{15})d(g(\omega,\theta_n(\omega)),g(\omega,\theta^*(\omega)))\\
&\quad +(s^2a_5+sa_8)d(g(\omega,\eta_{n+1}(\omega)),g(\omega,\eta^*(\omega)))\\
&\quad +(s^2a_6+sa_9)d(g(\omega,\theta_{n+1}(\omega)),g(\omega,\theta^*(\omega))).
\end{aligned}\tag{3.14}$$

Similarly, it is easily obtain that

$$\begin{aligned}
&d(F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))),g(\omega,\eta^*(\omega)))\\
&\quad\preceq (sa_1+s^2a_{10})d(F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))),g(\omega,\eta^*(\omega)))\\
&\qquad+(sa_2+s^2a_{11})d(F(\omega,\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))),g(\omega,\theta^*(\omega)))\\
&\qquad+(sa_3+s^2a_{12})d(F(\omega,\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi^*(\omega)))\\
&\qquad+(s^2a_4+s^2a_{10}+sa_{13})d(g(\omega,\eta_n(\omega)),g(\omega,\eta^*(\omega)))\\
&\qquad+(s^2a_4+sa_7+s)d(g(\omega,\eta_{n+1}(\omega)),g(\omega,\eta^*(\omega)))\\
&\qquad+(s^2a_5+s^2a_{11}+sa_{14})d(g(\omega,\theta_n(\omega)),g(\omega,\theta^*(\omega)))\\
&\qquad+(s^2a_6+s^2a_{12}+sa_{15})d(g(\omega,\xi_n(\omega)),g(\omega,\xi^*(\omega)))\\
&\qquad+(s^2a_5+sa_8)d(g(\omega,\theta_{n+1}(\omega)),g(\omega,\theta^*(\omega)))\\
&\qquad+(s^2a_6+sa_9)d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega))),
\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
&d(F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))),g(\omega,\theta^*(\omega)))\\
&\quad\preceq (sa_1+s^2a_{10})d(F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))),g(\omega,\theta^*(\omega)))\\
&\qquad+(sa_2+s^2a_{11})d(F(\omega,\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi^*(\omega)))\\
&\qquad+(sa_3+s^2a_{12})d(F(\omega,\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))),g(\omega,\eta^*(\omega)))\\
&\qquad+(s^2a_4+s^2a_{10}+sa_{13})d(g(\omega,\theta_n(\omega)),g(\omega,\theta^*(\omega)))\\
&\qquad+(s^2a_4+sa_7+s)d(g(\omega,\theta_{n+1}(\omega)),g(\omega,\theta^*(\omega)))\\
&\qquad+(s^2a_5+s^2a_{11}+sa_{14})d(g(\omega,\xi_n(\omega)),g(\omega,\xi^*(\omega)))\\
&\qquad+(s^2a_6+s^2a_{12}+sa_{15})d(g(\omega,\eta_n(\omega)),g(\omega,\eta^*(\omega)))\\
&\qquad+(s^2a_5+sa_8)d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega)))\\
&\qquad+(s^2a_6+sa_9)d(g(\omega,\eta_{n+1}(\omega)),g(\omega,\eta^*(\omega))).
\end{aligned}\tag{3.16}$$

Put

$$\begin{aligned}
\delta=\;&d(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi^*(\omega)))+d(F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))),g(\omega,\eta^*(\omega)))\\
&+d(F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))),g(\omega,\theta^*(\omega))).
\end{aligned}$$

On view of (3.14)-(3.16), we get

$$\begin{aligned}
\delta\preceq\;&(sa_1+sa_2+sa_3+s^2a_{10}+s^2a_{11}+s^2a_{12})\delta\\
&+(s^2a_4+s^2a_5+s^2a_6+s^2a_{10}+s^2a_{11}+s^2a_{12}+sa_{13}+sa_{14}+sa_{15})d(g(\omega,\xi_n(\omega)),g(\omega,\xi^*(\omega)))\\
&+(s^2a_4+s^2a_5+s^2a_6+s^2a_{10}+s^2a_{11}+s^2a_{12}+sa_{13}+sa_{14}+sa_{15})d(g(\omega,\eta_n(\omega)),g(\omega,\eta^*(\omega)))\\
&+(s^2a_4+s^2a_5+s^2a_6+s^2a_{10}+s^2a_{11}+s^2a_{12}+sa_{13}+sa_{14}+sa_{15})d(g(\omega,\theta_n(\omega)),g(\omega,\theta^*(\omega)))\\
&+(s^2a_4+s^2a_5+s^2a_6+s+sa_7+sa_8+sa_9)d(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega)))\\
&+(s^2a_4+s^2a_5+s^2a_6+s+sa_7+sa_8+sa_9)d(g(\omega,\eta_{n+1}(\omega)),g(\omega,\eta^*(\omega)))\\
&+(s^2a_4+s^2a_5+s^2a_6+s+sa_7+sa_8+sa_9)d(g(\omega,\theta_{n+1}(\omega)),g(\omega,\theta^*(\omega))).
\end{aligned}$$

Then

$$\delta\preceq(e-A)^{-1}Bd(g(\omega,\xi_n(\omega)),g(\omega,\xi^*(\omega)))+(e-A)^{-1}Bd(g(\omega,\eta_n(\omega)),g(\omega,\eta^*(\omega)))$$

$$+ (e-A)^{-1}Bd(g(\omega,\theta_n(\omega)),g(\omega,\theta^*(\omega))) + (e-A)^{-1}Cd(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega)))$$
$$+ (e-A)^{-1}Cd(g(\omega,\eta_{n+1}(\omega)),g(\omega,\eta^*(\omega))) + (e-A)^{-1}Cd(g(\omega,\theta_{n+1}(\omega)),g(\omega,\theta^*(\omega))),$$

where $A = sa_1 + sa_2 + sa_3 + s^2a_{10} + s^2a_{11} + s^2a_{12}$, $B = s^2a_4 + s^2a_5 + s^2a_6 + s^2a_{10} + s^2a_{11} + s^2a_{12} + sa_{13} + sa_{14} + sa_{15}$, $C = s^2a_4 + s^2a_5 + s^2a_6 + s + sa_7 + sa_8 + sa_9$, $\rho(A) < 1$. Since $g(\omega,\xi_n(\omega)) \to g(\omega,\xi^*(\omega))$, $g(\omega,\eta_n(\omega)) \to g(\omega,\eta^*(\omega))$, $g(\omega,\theta_n(\omega)) \to g(\omega,\theta^*(\omega))$, then by Lemma 2.11, it follows that for any $c \gg \theta$, there exists N_0 such that for $n > N_0$, we have

$$(e-A)^{-1}Bd(g(\omega,\xi_n(\omega)),g(\omega,\xi^*(\omega))) \ll \frac{c}{6},$$
$$(e-A)^{-1}Bd(g(\omega,\eta_n(\omega)),g(\omega,\eta^*(\omega))) \ll \frac{c}{6},$$
$$(e-A)^{-1}Bd(g(\omega,\theta_n(\omega)),g(\omega,\theta^*(\omega))) \ll \frac{c}{6},$$
$$(e-A)^{-1}Cd(g(\omega,\xi_{n+1}(\omega)),g(\omega,\xi^*(\omega))) \ll \frac{c}{6},$$
$$(e-A)^{-1}Cd(g(\omega,\eta_{n+1}(\omega)),g(\omega,\eta^*(\omega))) \ll \frac{c}{6},$$
$$(e-A)^{-1}Cd(g(\omega,\theta_{n+1}(\omega)),g(\omega,\theta^*(\omega))) \ll \frac{c}{6}.$$

Hence,

$$\delta \ll \frac{c}{6}+\frac{c}{6}+\frac{c}{6}+\frac{c}{6}+\frac{c}{6}+\frac{c}{6} = c.$$

Now, according to Lemma 2.4, it follows that $\delta = \theta$, that is,

$$d(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi^*(\omega))) + d(F(\omega,(\eta^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\eta^*(\omega)))$$
$$+ d(F(\omega,(\theta^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\theta^*(\omega))) = \theta,$$

which implies that

$$d(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\xi^*(\omega))) = \theta,$$
$$d(F(\omega,(\eta^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\eta^*(\omega))) = \theta,$$
$$d(F(\omega,(\theta^*(\omega),\eta^*(\omega),\theta^*(\omega))),g(\omega,\theta^*(\omega))) = \theta.$$

Thus,

$$g(\omega,\xi^*(\omega)) = F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),$$
$$g(\omega,\eta^*(\omega)) = F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))),$$
$$g(\omega,\theta^*(\omega)) = F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))).$$

Therefore $(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))$ is a tripled coincidence point of F and g for all $\omega \in \Omega$. □

Corollary 3.5. *Let (X, d) be a separable cone b-metric space over Banach algebra $\mathcal{A}$ and P be a cone in $\mathcal{A}$, $s \geqslant 1$ be a constant, and (Ω, Σ) be a measurable space. Suppose that the mappings $F : \Omega \times X^3 \to X$, $g : \Omega \times X \to X$ satisfy the following contractive condition:*

$$d(F(\omega,(x,y,z)),F(\omega,(u,v,w))) \preceq kd(g(\omega,x),g(\omega,u)) + ld(g(\omega,y),g(\omega,v)) + td(g(\omega,z),g(\omega,w)),$$

for all $x,y,z,u,v,w \in X$, where $k,l,t \in P$ are generalized Lipschitz constants with $\rho(k+l+t) < \frac{1}{s}$, $F(.,v)$, $g(.,x)$ are measurable for $v \in X^3$ and $x \in X$, respectively, $F(\omega \times X^3) \subseteq g(\omega \times X)$ and $g(\omega \times X)$ is complete subspace of X for each $\omega \in \Omega$, then there are mappings $\xi,\eta,\theta : \Omega \to X$, such that $F(\omega,(\xi(\omega),\eta(\omega),\theta(\omega))) = g(\omega,\xi(\omega))$, $F(\omega,(\eta(\omega),\theta(\omega),\xi(\omega))) = g(\omega,\eta(\omega))$, $F(\omega,(\theta(\omega),\xi(\omega),\eta(\omega))) = g(\omega,\theta(\omega))$ for all $\omega \in \Omega$, that is F and g have a tripled random coincidence point.

Corollary 3.6. *Let (X, d) be a separable cone b-metric space over Banach algebra $\mathcal{A}$, P be a cone in $\mathcal{A}$ and (Ω, Σ) be a measurable space. Suppose that the mappings $F : \Omega \times X^3 \to X$, $g : \Omega \times X \to X$ satisfy the following contractive condition:*

$$d(F(\omega,(x,y,z)),F(\omega,(u,v,w))) \preceq kd(g(\omega,x),F(\omega,(u,v,w))) + ld(g(\omega,u),F(\omega,(x,y,z))),$$

for all $x, y, z, u, v, w \in X$, where $k, l \in P$ are generalized Lipschitz constants with $\rho(k+l) < \frac{2}{s(s+1)}$ and $\rho(l) < \frac{1}{s^2}$, $F(\cdot, v), g(\cdot, x)$ are measurable for $v \in X^3$ and $x \in X$, respectively, $F(\omega \times X^3) \subseteq g(\omega \times X)$ and $g(\omega \times X)$ is complete subspace of X for each $\omega \in \Omega$, then there are mappings $\xi, \eta, \theta : \Omega \to X$, such that $F(\omega,(\xi(\omega),\eta(\omega),\theta(\omega))) = g(\omega,\xi(\omega))$, $F(\omega,(\eta(\omega),\theta(\omega),\xi(\omega))) = g(\omega,\eta(\omega))$, $F(\omega,(\theta(\omega),\xi(\omega),\eta(\omega))) = g(\omega,\theta(\omega))$ for all $\omega \in \Omega$, that is, F and g have a tripled random coincidence point.

The conditions of Theorem 3.4 are not enough to prove the existence of a common tripled fixed point for the mappings F and g. By restricting to w-compatibility for F and g, we obtain the following theorem.

Theorem 3.7. *In addition to hypotheses of Theorem 3.4, if F and g are w-compatible, then F and g have a unique tripled common fixed point. Moreover, a tripled common random fixed point of F and g is of the form $(\xi^*(\omega), \xi^*(\omega), \xi^*(\omega)) \in X$ for all $\omega \in \Omega$.*

Proof. By Theorem 3.4, F and g have tripled random coincidence point $(\xi^*(\omega), \eta^*(\omega), \theta^*(\omega))$. Then $(g(\omega, \xi^*(\omega)), g(\omega, \eta^*(\omega)), g(\omega, \theta^*(\omega)))$ is a tripled random point of coincidence of F and g such that

$$\begin{aligned}
g(\omega,\xi^*(\omega)) &= F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),\\
g(\omega,\eta^*(\omega)) &= F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))),\\
g(\omega,\theta^*(\omega)) &= F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))).
\end{aligned}$$

First, we shall show that the tripled random point of coincidence is unique. Suppose that F and g have another tripled random point of coincidence $(g(\omega, \xi^{**}(\omega)), g(\omega, \eta^{**}(\omega)), g(\omega, \theta^{**}(\omega)))$ such that

$$\begin{aligned}
g(\omega,\xi^{**}(\omega)) &= F(\omega,(\xi^{**}(\omega),\eta^{**}(\omega),\theta^{**}(\omega))),\\
g(\omega,\eta^{**}(\omega)) &= F(\omega,(\eta^{**}(\omega),\theta^{**}(\omega),\xi^{**}(\omega))),\\
g(\omega,\theta^{**}(\omega)) &= F(\omega,(\theta^{**}(\omega),\xi^{**}(\omega),\eta^{**}(\omega))),
\end{aligned}$$

where $(\xi^{**}(\omega), \eta^{**}(\omega), \theta^{**}(\omega)) \in X^3$ for all $\omega \in \Omega$. Then we have

$$\begin{aligned}
d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) &= d(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))),F(\omega,(\xi^{**}(\omega),\eta^{**}(\omega),\theta^{**}(\omega))))\\
&\preceq [a_1 d(g(\omega,\xi^*(\omega)),F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))))\\
&\quad + a_2 d(g(\omega,\eta^*(\omega)),F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))))\\
&\quad + a_3 d(g(\omega,\theta^*(\omega)),F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))))]\\
&\quad + [a_4 d(g(\omega,\xi^{**}(\omega)),F(\omega,(\xi^{**}(\omega),\eta^{**}(\omega),\theta^{**}(\omega))))\\
&\quad + a_5 d(g(\omega,\eta^{**}(\omega)),F(\omega,(\eta^{**}(\omega),\theta^{**}(\omega),\xi^{**}(\omega))))\\
&\quad + a_6 d(g(\omega,\theta^{**}(\omega)),F(\omega,(\theta^{**}(\omega),\xi^{**}(\omega),\eta^{**}(\omega))))]\\
&\quad + [a_7 d(g(\omega,\xi^*(\omega)),F(\omega,(\xi^{**}(\omega),\eta^{**}(\omega),\theta^{**}(\omega))))\\
&\quad + a_8 d(g(\omega,\eta^*(\omega)),F(\omega,(\eta^{**}(\omega),\theta^{**}(\omega),\xi^{**}(\omega))))\\
&\quad + a_9 d(g(\omega,\theta^*(\omega)),F(\omega,(\theta^{**}(\omega),\xi^{**}(\omega),\eta^{**}(\omega))))]\\
&\quad + [a_{10} d(g(\omega,\xi^{**}(\omega)),F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega))))\\
&\quad + a_{11} d(g(\omega,\eta^{**}(\omega)),F(\omega,(\eta^*(\omega),\theta^*(\omega),\xi^*(\omega))))\\
&\quad + a_{12} d(g(\omega,\theta^{**}(\omega)),F(\omega,(\theta^*(\omega),\xi^*(\omega),\eta^*(\omega))))]\\
&\quad + [a_{13} d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) + a_{14} d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega)))
\end{aligned}$$

$$\begin{aligned}
&+ a_{15}d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega)))] \\
&= [a_1 d(g(\omega,\xi^*(\omega)),g(\omega,\xi^*(\omega))) + a_2 d(g(\omega,\eta^*(\omega)),g(\omega,\eta^*(\omega))) \\
&\quad + a_3 d(g(\omega,\theta^*(\omega)),g(\omega,\theta^*(\omega)))] + [a_4 d(g(\omega,\xi^{**}(\omega)),g(\omega,\xi^{**}(\omega))) \\
&\quad + a_5 d(g(\omega,\eta^{**}(\omega)),g(\omega,\eta^{**}(\omega))) + a_6 d(g(\omega,\theta^{**}(\omega)),g(\omega,\theta^{**}(\omega)))] \\
&\quad + [a_7 d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) + a_8 d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) \\
&\quad + a_9 d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega)))] + [a_{10} d(g(\omega,\xi^{**}(\omega)),g(\omega,\xi^*(\omega))) \\
&\quad + a_{11} d(g(\omega,\eta^{**}(\omega)),g(\omega,\eta^*(\omega))) + a_{12} d(g(\omega,\theta^{**}(\omega)),g(\omega,\theta^*(\omega))) \\
&\quad + [a_{13} d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) + a_{14} d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) \\
&\quad + a_{15} d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega)))].
\end{aligned}$$

Hence,

$$\begin{aligned}
d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) \preceq\ & (a_7+a_{10}+a_{13})d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) \\
&+ (a_8+a_{11}+a_{14})d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) \\
&+ (a_9+a_{12}+a_{15})d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega))).
\end{aligned} \tag{3.17}$$

Similarly, we have

$$\begin{aligned}
d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) \preceq\ & (a_7+a_{10}+a_{13})d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) \\
&+ (a_8+a_{11}+a_{14})d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega))) \\
&+ (a_9+a_{12}+a_{15})d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))),
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega))) \preceq\ & (a_7+a_{10}+a_{13})d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega))) \\
&+ (a_8+a_{11}+a_{14})d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) \\
&+ (a_9+a_{12}+a_{15})d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))).
\end{aligned} \tag{3.19}$$

By combining (3.17)-(3.19), we get

$$\begin{aligned}
&d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) + d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) + d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega))) \\
&\quad \preceq (a_7+\cdots+a_{15})(d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) + d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) \\
&\qquad + d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega)))).
\end{aligned}$$

Set $\alpha = a_7 + \cdots + a_{15}$, and

$$\gamma = d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) + d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) + d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega))),$$

we have $\gamma \preceq \alpha\gamma \preceq \cdots \preceq \alpha^n\gamma$. Now that

$$\rho(\alpha) \leqslant \rho(a_7+\cdots+a_{12}) + \rho(a_{13}+a_{14}+a_{15}) < 1,$$

which leads to $\alpha^n \to \theta$ $(n \to \infty)$, we claim that, for each $c \gg \theta$, there exists $n_0(c)$ such that $\alpha^n \ll c$ $(n > n_0(c))$. Consequently by Lemma 2.11,

$$d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) + d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) + d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega))) = \theta.$$

Hence,

$$\begin{aligned}
d(g(\omega,\xi^*(\omega)),g(\omega,\xi^{**}(\omega))) &= \theta, \\
d(g(\omega,\eta^*(\omega)),g(\omega,\eta^{**}(\omega))) &= \theta,
\end{aligned}$$

$$d(g(\omega,\theta^*(\omega)),g(\omega,\theta^{**}(\omega)))=\theta,$$

that is,

$$g(\omega,\xi^*(\omega))=g(\omega,\xi^{**}(\omega)),\quad g(\omega,\eta^*(\omega))=g(\omega,\eta^{**}(\omega)),\quad g(\omega,\theta^*(\omega))=g(\omega,\theta^{**}(\omega)),\tag{3.20}$$

which implies the uniqueness of the tripled random point of coincidence of F and g. By a similar way, someone can prove that

$$\begin{aligned}g(\omega,\xi^*(\omega))&=g(\omega,\eta^{**}(\omega)),\\ g(\omega,\eta^*(\omega))&=g(\omega,\theta^{**}(\omega)),\\ g(\omega,\theta^*(\omega))&=g(\omega,\xi^{**}(\omega)),\end{aligned}\tag{3.21}$$

$$\begin{aligned}g(\omega,\xi^*(\omega))&=g(\omega,\theta^{**}(\omega)),\\ g(\omega,\eta^*(\omega))&=g(\omega,\xi^{**}(\omega)),\\ g(\omega,\theta^*(\omega))&=g(\omega,\eta^{**}(\omega)).\end{aligned}\tag{3.22}$$

In view of (3.20)-(3.22), one can assert

$$g(\omega,\xi^*(\omega))=g(\omega,\eta^*(\omega))=g(\omega,\theta^*(\omega)).$$

In other words, the unique tripled random point of coincidence of F and g is $(g(\omega,\xi^*(\omega)),g(\omega,\eta^*(\omega)),g(\omega,\theta^*(\omega)))$. Let $u(\omega)=g(\omega,\xi^*(\omega))=F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega)))$. Since F and g are w-compatible, then we have

$$\begin{aligned}g(\omega,u(\omega))=g(g(\omega,\xi^*(\omega)))&=g(F(\omega,(\xi^*(\omega),\eta^*(\omega),\theta^*(\omega)))\\&=F(\omega,(g(\omega,\xi^*(\omega)),g(\omega,\eta^*(\omega)),g(\omega,\theta^*(\omega))))\\&=F(\omega,(u(\omega),u(\omega),u(\omega))).\end{aligned}$$

Thus $(g(\omega,u(\omega)),g(\omega,u(\omega)),g(\omega,u(\omega)))$ is a tripled random point of coincidence. We also have $(u(\omega),u(\omega),u(\omega))$ is a tripled random point of coincidence. Note that the uniqueness of the tripled random point of coincidence implies that $g(\omega,u(\omega))=u(\omega)$. Therefore

$$u(\omega)=g(\omega,u(\omega))=F(\omega,(u(\omega),u(\omega),u(\omega))).$$

Hence $(u(\omega),u(\omega),u(\omega))$ is the unique tripled common random fixed point of F and g for all $\omega\in\Omega$. This completes the proof.
Putting $g(\omega,\cdot)=I(\omega,\cdot)$ (identity mapping) in Theorem 3.4, we obtain the following result. □

Theorem 3.8. *Let (X,d) be a complete separable cone b-metric space over Banach algebra $\mathcal{A}$, P be a cone in $\mathcal{A}$, and (Ω,Σ) be a measurable space. Suppose that the mapping $F:\Omega\times X^3\to X$ satisfies the following contractive condition:*

$$\begin{aligned}d(F(\omega,(x,y,z)),F(\omega,(u,v,w)))\preceq{}&[a_1d(x(\omega),F(\omega,(x,y,z)))+a_2d(y(\omega),F(\omega,(y,z,x)))\\&+a_3d(z(\omega),F(\omega,(z,x,y)))]+[a_4d(u(\omega),F(\omega,(u,v,w)))\\&+a_5d(v(\omega),F(\omega,(v,w,u)))+a_6d(w(\omega),F(\omega,(w,u,v)))]\\&+[a_7d(x(\omega),F(\omega,(u,v,w)))+a_8d(y(\omega),F(\omega,(v,w,u)))\\&+a_9d(z(\omega),F(\omega,(w,u,v)))]+[a_{10}d(u(\omega),F(\omega,(x,y,z)))\\&+a_{11}d(v(\omega),F(\omega,(y,z,x)))+a_{12}d(w(\omega),F(\omega,(z,x,y)))]\\&+[a_{13}d(x(\omega),u(\omega))+a_{14}d(y(\omega),v(\omega))\\&+a_{15}d(z(\omega),w(\omega))],\end{aligned}$$

where $a_i \in P, a_i a_j = a_j a_i$ $(i, j = 1, \dots, 15)$, a_i *are generalized Lipschitz constants with* $(s+1)\rho(a_1 + \cdots + a_6) + s(s+1)\rho(a_7 + \cdots + a_{12}) + 2s\rho(a_{13} + a_{14} + a_{15}) < 2$ *and* $\rho\big(sa_1 + sa_2 + sa_3 + s^2a_{10} + s^2a_{11} + s^2a_{12}\big) < 1$, $F(\cdot, \nu), g(\cdot, x)$ *are measurable for all* $\nu \in X^3$ *and* $x \in X$, *then* F *has a unique tripled random fixed point* $(\xi(\omega), \xi(\omega), \xi(\omega)) \in \Omega \times X^3$.

Remark 3.9. Our main results mainly generalize the recent results. In fact, they never consider the normality of cones, which may offer us more applications since there exist lots of non-normal cones (see [30]). Moreover, we establish the contractive mappings with several generalized Lipschitz constants, where the constants are all vectors but not usual real constants. Thus they are different from some ordinary results and more interesting.

Remark 3.10. Our results refer to the setting of cone b-metric space over Banach algebra and quite meaningful, since there exist many cone b-metric spaces over Banach algebras but they are not cone metric spaces over Banach algebras. Hence, our spaces are more valuable than some previous results.

Remark 3.11. Our theorems deal not only with common fixed point results with random process, but also with them from usual coupled fixed point to tripled fixed point. Therefore, our results greatly improve and extend some results in the literature (see [7, 14]).

Remark 3.12. Our results are mainly related to tripled random coincidence point and common fixed point results of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras. Our tripled random coincidence point and common fixed point results cannot reduce to the counterparts of the results with one variable. In other words, the method of [29] cannot be utilized to our main results. This is because the generalized Lipschitz constants from our results are vectors. Moreover, the multiplication of the vectors do not satisfy the combinative law. Hence we cannot use a method of reducing our tripled results to the respective results for mappings with one variable.

Remark 3.13. According to [22], some fixed point results in C^*-algebra-valued metric spaces are direct consequences of their standard metric counterparts. However, our results are never the corresponding results from the usual metric spaces. In fact, it is well-known that C^*-algebras are the special Banach algebra. Hence, C^*-algebra-valued metric space is the special cone metric space over Banach algebra. Because of the more general character, many results from C^*-algebra-valued metric spaces cannot be extended to cone metric spaces over Banach algebras. Further, based on [27], we claim that the fixed point results in cone metric spaces over Banach algebras cannot reduce to the cases of metric spaces. In addition, the results from this paper are established on cone b-metric spaces over Banach algebras, whereas, b-metric has no continuity regarding their variables. That is, when $x_n \to x$, $y_n \to y$, but $d(x_n, y_n) \nrightarrow d(x, y)$ as $n \to \infty$. However, the usual metric has the continuity. Accordingly, regardless of some fixed point results in C^*-algebra-valued metric spaces can be obtained from the counterpart of the usual metric spaces (see [22]), but our results in cone b-metric spaces over Banach algebras cannot be gotten from the respective metric cases based on the discontinuity problems.

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