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Applications of a novel integral transform to partial differential equations

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Abstract

In this paper, we establish and perfect the dualities among the Laplace transform (LT), Laplace-Carson transform (LCT), Sumudu transform (ST), and a novel integral transform (NIT). In addition, some novel properties of the NIT are explored and the NIT is applied to solve some partial differential equations (PDEs). ©2017 all rights reserved.

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1. Introduction

As the convenient mathematical tool, numerous Integral transforms (ITs) were proposed to focus on the analytic solutions of differential and integral equations involving multiple fields, which included physics, chemistry, as well as economy [7, 18]. For instance, Eltayeb and Kılıçman [9] proved the existence of solution for the wave propagation by Laplace transform (LT). Atangana and Baleanu [4] handled the fractional advection-dispersion equation. Yang et al. [19] derived the exact solution on the problem of fractional LC-electric circuit. Kudinov [13] explored the heat-exchange problem by Laplace-Carson transform (LCT). Kang et al. [11] analyzed American strangle-options problem. Watugala [15] discussed the control engineering problem by Sumudu transform (ST). Atangana solved the Keller-Segel equation in [2] and the fractional Fisher's reaction-diffusion equation in [3]. Recently, a new integral transform (NIT) [17], similar to the LT, LCT, and ST was proposed to find the exact solution of the ordinary differential heat-transfer equation. However, the dualities of the above different ITs and some properties of the NIT are still imperfect. Meanwhile, the NIT has not been applied to solve the PDEs.

In view of the above proposed idea, the brief objective of this paper is focused on:

1. Establishment of the dualities among the NIT, LT, LCT and ST.
2. Discussions about some novel properties of the NIT.
3. Application in solving the PDEs by the NIT.

The remainder of the current paper is arranged as follows. In Section 2, we establish and perfect the

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dualities among the LT, LT, LCT, ST, and NIT, as well as develop a few novel properties of the NIT. In Section 3, we solve some PDEs by the NIT. Finally, in Section 4, the outcomes are summarized.

2. The NIT (new integral transform)

2.1. Definitions and dualities

Firstly, the definitions of the NIT, LT, LCT, and ST are reviewed as follows.

Definition 2.1 ([17])**.** The NIT of the real function $\omega(\lambda)$, $\lambda > 0$, is defined by

$$\Omega(\gamma) = N_I[\omega(\lambda)] = \frac{1}{\gamma}\int_0^\infty \omega(\lambda) e^{-\gamma\lambda} d\lambda, \gamma > 0,$$

where N_I is the NIT operator.

Definition 2.2 ([5])**.** The LT of the real function $\omega(\lambda)$, $\lambda > 0$, is defined by

$$F(\gamma) = L[\omega(\lambda)] = \int_0^\infty \omega(\lambda) e^{-\gamma\lambda} d\lambda, \gamma > 0,$$

where L is the LT operator.

Definition 2.3 ([8])**.** The LCT of the real function $\omega(\lambda)$, $\lambda > 0$, is defined by

$$L_c(\gamma) = C[\omega(\lambda)] = \gamma\int_0^\infty \omega(\lambda) e^{-\gamma\lambda} d\lambda, \gamma > 0,$$

where C is the LCT operator.

Definition 2.4 ([1])**.** The ST of the real function $\omega(\lambda)$, $\lambda > 0$, is defined as

$$\Psi(\gamma) = S[\omega(\lambda)] = \frac{1}{\gamma}\int_0^\infty \omega(\lambda) e^{-\frac{1}{\gamma}\lambda} d\lambda, \gamma > 0,$$

where S is the ST operator.

Comparisons of the above different ITs in their definitions display that much deeper connections occur among them. Belgacem et al. [6] had analyzed the duality between LT and ST in detail. Here, depending on the definitions of the different ITs, we prove and perfect the dualities among the LT, LCT, ST and NIT. The dualities are demonstrated in Figure 1.

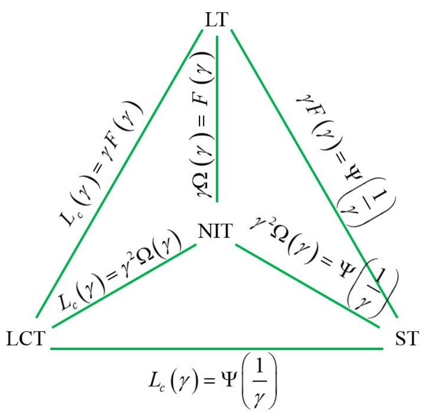

Figure 1: The duality relation graph among the LT, LCT, ST and NIT.

A. (The duality of the NIT and LT)

$$\Omega(\gamma) = N_I[\omega(\lambda)] = \frac{1}{\lambda}\int_0^\infty \omega(\lambda)\, e^{-\gamma\lambda} d\lambda = \frac{1}{\gamma} F(\gamma), \gamma > 0. \tag{2.1}$$

B. (The duality of the NIT and LCT)

$$\Omega(\gamma) = N_I[\omega(\lambda)] = \frac{1}{\gamma^2}\left(\gamma\int_0^\infty \omega(\lambda)\, e^{-\gamma\lambda} d\lambda\right) = \frac{1}{\gamma^2} L_c(\gamma), \gamma > 0.$$

C. (The duality of the NIT and ST)

$$\Omega(\gamma) = N_I[\omega(\lambda)] = \frac{1}{\gamma^2}\left(\gamma\int_0^\infty \omega(\lambda)\, e^{-\gamma\lambda} d\lambda\right) = \frac{1}{\gamma^2}\Psi\left(\frac{1}{\gamma}\right), \gamma > 0.$$

As shown in Figure 1, the close duality relation may indicate that the NIT is of similar properties to LT, LCT and ST. Also, the NIT may be equivalent in function to the LT, LCT and ST for getting the analytic solutions of the PDEs. Therefore, we next derive some properties of the NIT and calculate several examples of PDEs.

2.2. *The novel properties of the NIT*

(T1): If $\Omega(\gamma) = N_I[\omega(\lambda)]$, $\gamma > 0$, $\lambda > 0$, then we have

$$N_I\left[\omega^{(n)}(\lambda)\right] = \gamma^n \Omega(\gamma) - \gamma^{n-2}\Omega(0) - \gamma^{n-3}\omega^{(1)}(0) - \cdots - \omega^{(n-2)}(0) - \frac{\omega^{(n-1)}(0)}{\gamma}, \tag{2.2}$$

where, $\omega^{(n)}(\lambda)$ is the n-order derivative of $\omega(\lambda)$.

(T2): If $\Omega(\gamma) = N_I[\omega(\lambda)]$, $\gamma > 0$, $\lambda > 0$, then we obtain

$$\Omega'(\gamma) = -\frac{1}{\gamma} N_I[\omega(\lambda)] - N_I[\lambda\omega(\lambda)],$$

where, $\Omega'(\gamma)$ is the derivative of $\Omega(\gamma)$.

(T3): Supposing that $\omega(\lambda)$, $\lambda > 0$, is a periodic function with period T ($\omega(\lambda + T) = \omega(\lambda)$, ($T > 0$)), we have

$$N_I[\omega(\lambda)] = \frac{1}{\gamma(1 - e^{-\gamma T})}\int_0^T \omega(\lambda) e^{-\gamma\lambda} d\lambda, \gamma > 0.$$

(T4): If $\Omega_1(\gamma) = N_I[\omega_1(\lambda)]$ and $\Omega_2(\gamma) = N_I[\omega_2(\lambda)]$, $\gamma > 0$, $\lambda > 0$, then we have the NIT of convolution [12] as follows:

$$\Omega[\omega_1(\lambda) * \omega_2(\lambda)] = \gamma\Omega_1(\gamma) \times \Omega_2(\gamma),$$

where

$$\omega_1(\lambda) * \omega_2(\lambda) = \int_0^\lambda \omega_1(\theta) \times \varphi_2(\lambda - \theta)\, d\theta.$$

Proof.

(T1):

$$\begin{aligned} N_I\left[\omega^{(n)}(\lambda)\right] &= \frac{1}{\gamma}\int_0^\infty \omega^{(n)}(\lambda)\, e^{-\gamma\lambda} d\lambda \\ &= \frac{1}{\gamma}\left[\omega^{(n-1)}(\lambda)\, e^{-\gamma\lambda}\right]\Big|_0^\infty + \gamma N_I\left[\omega^{(n-1)}(\lambda)\right] \\ &= -\frac{\omega^{(n-1)}(0)}{\gamma} + \gamma N_I\left[\omega^{(n-1)}(\lambda)\right]. \end{aligned} \tag{2.3}$$

Noting the recurrence relation in Eq. (2.3), we have

$$N_I\left[\omega^{(n)}(\lambda)\right] = \gamma^n \Omega(\gamma) - \gamma^{n-2}\omega(0) - \gamma^{n-3}\omega^{(1)}(0) - \cdots - \omega^{(n-2)}(0) - \frac{\omega^{(n-1)}(0)}{\gamma}.$$

(T2): Employing the subsection integration, we have

$$\Omega'(\gamma) = \left[\int_0^{\infty} \frac{1}{\gamma}\omega(\lambda)\, e^{-\gamma\lambda} d\lambda\right]' = -\frac{1}{\gamma} N_I [\omega(\lambda)] - N_I [\lambda\omega(\lambda)].$$

(T3):

$$N_I [\omega(\lambda)] = \frac{1}{\gamma}\int_0^{\infty} \omega(\lambda)\, e^{-\gamma\lambda} d\lambda = \frac{1}{\gamma}\left[\int_0^{T} \omega(\lambda)\, e^{-\gamma\lambda} d\lambda + \int_T^{\infty} \omega(\lambda)\, e^{-\gamma\lambda} d\lambda\right].$$

Let $u = \lambda - T$. Then, we obtain

$$\int_T^{\infty} \omega(\lambda)\, e^{-\gamma\lambda} d\lambda = \int_0^{\infty} \omega(u)\, e^{-\gamma(u+T)} du = e^{-\gamma T}\int_0^{\infty} \omega(u)\, e^{-\gamma u} du$$

and

$$N_I [\omega(\lambda)] = \frac{1}{\gamma}\int_0^{\infty} \omega(\lambda)\, e^{-\gamma\lambda} d\lambda = \frac{1}{\gamma}\left[\int_0^{T} \omega(\lambda)\, e^{-\gamma\lambda} d\lambda + e^{-\gamma T}\int_0^{\infty} \omega(u)\, e^{-\gamma u} du\right]$$

$$= \frac{1}{\gamma}\int_0^{T} \omega(\lambda)\, e^{-\gamma\lambda} d\lambda + e^{-\gamma T} N_I [\omega(\lambda)].$$

Finally, we have

$$N_I [\omega(\lambda)] = \frac{1}{\gamma(1 - e^{-\gamma T})}\int_0^{T} \omega(\lambda) e^{-\gamma\lambda} d\lambda.$$

(T4):

$$N_I [\omega_1(\lambda) * \omega_2(\lambda)] = \frac{1}{\gamma}\int_0^{\infty} (\omega_1(\lambda) * \omega_2(\lambda))\, e^{-\gamma\lambda} d\lambda = \frac{1}{\gamma}\int_0^{\infty}\left[\int_0^{\lambda} \omega_1(\theta) \times \omega_2(\lambda - \theta)\, d\theta\right] e^{-\gamma\lambda} d\lambda$$

$$= \frac{1}{\gamma}\int_0^{\infty} \omega_1(\theta)\left[\int_{\theta}^{\infty} \omega_2(\lambda - \theta) e^{-\gamma\lambda} d\lambda\right] d\theta.$$

Let $u = \lambda - \theta$. Then, we have

$$\int_{\theta}^{\infty} \varphi_2(\lambda - \theta) e^{-\gamma\lambda} d\lambda = \int_0^{\infty} \omega_2(u) e^{-\gamma(\theta + u)} du = \gamma e^{-\gamma\theta}\Omega_2(\gamma).$$

Finally, we have

$$\Omega [\omega_1(\lambda) * \omega_2(\lambda)] = \gamma\Omega_1(\gamma) \times \Omega_2(\gamma).$$

□

3. Solving the partial differential equations by the NIT

Example 3.1. Let us consider the following diffusion equation in a semi-infinite domain [10]:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2}, \quad x > 0, \quad t > 0, \tag{3.1}$$

where k is the thermal diffusivity.

The initial-value condition (IC) and boundary-value conditions (BC) are

$$\begin{cases} u(x,0) = 0 \quad (\text{IC}), \\ u(0,t) = c, \quad u(\infty,t) = BV \quad (\text{BC}), \end{cases} \tag{3.2}$$

respectively, where c is a real constant and BV is bounded.

By applying Eq. (2.2), when $n = 1$, the one-dimensional NIT of Eq. (3.1) with respect to t is performed as follows:

$$\mu\Omega(x,\gamma) - \frac{u(x,0)}{\gamma} = k\Omega^{(2)}(x,\gamma). \tag{3.3}$$

Correspondingly, the boundary conditions in Eq. (3.2) become

$$\begin{cases} \Omega(0,\gamma) = \frac{c}{\gamma^2}, \\ \Omega(\infty,\gamma) = BV. \end{cases} \tag{3.4}$$

Substitution of $u(x,0) = 0$ into Eq. (3.3) results in

$$\gamma\Omega(x,\gamma) = k\Omega^{(2)}(x,\gamma). \tag{3.5}$$

Then, using the eigenvalues of Eq. (3.5) given by [14] $\pm\sqrt{\gamma/k}$, we have

$$\Omega(x,\gamma) = C_1 e^{\sqrt{\frac{\gamma}{k}}x} + C_2 e^{-\sqrt{\frac{\gamma}{k}}x}. \tag{3.6}$$

Additionally, considering the boundary conditions (3.4), we obtain

$$\begin{cases} C_1 + C_2 = \frac{c}{\gamma^2}, \\ C_1 = 0. \end{cases} \tag{3.7}$$

Substituting Eq. (3.7) into Eq. (3.6), we have

$$\Omega(x,\gamma) = \frac{c}{\gamma^2} e^{-\sqrt{\frac{\gamma}{k}}x}. \tag{3.8}$$

Furthermore, from the literature [7, 10], the LT of $c \times \operatorname{erfc}\left(x\big/\left(2\sqrt{kt}\right)\right)$ with respect to t is as follows:

$$F(\gamma) = L\left[c \times \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)\right] = \frac{c}{\gamma} e^{-\sqrt{\frac{\gamma}{k}}x},$$

where the complementary error function is [7]:

$$\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt.$$

Using the duality of the NIT and LT in Eq. (2.1), the NIT of $c \times \operatorname{erfc}\left(x\big/\left(2\sqrt{kt}\right)\right)$ is obtained:

$$\Omega(\gamma) = N_I\left[c \times \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)\right] = \frac{1}{\gamma} F(\gamma) = \frac{c}{\gamma^2} e^{-\sqrt{\frac{\gamma}{k}}x}.$$

Finally, we calculate the inverse NIT for Eq. (3.8) with respect to t and obtain:

$$u(x,t) = c \times \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right). \tag{3.9}$$

Eq. (3.9) is the same as the solution of diffusion equation by employing the LT in [10] with special initial and boundary conditions.

Example 3.2. Let us solve the following equation with an initial value:

$$\frac{\partial u(x,t)}{\partial x} = 2\frac{\partial u(x,t)}{\partial t} + u, \tag{3.10}$$

where, $x > 0, t > 0$, and $u(x,t)$ is bounded.

The IC is

$$u(x,0) = 6e^{-3x}.$$

Taking the NIT of Eq. (3.10) with respect to t, we obtain the ordinary differential equation

$$\Omega'(x,\gamma) - (2\gamma+1)\Omega(x,\gamma) = -\frac{12}{\gamma}e^{-3x}. \tag{3.11}$$

Applying the integral factor (IF) method, we have the solution of Eq. (3.11):

$$\Omega(x,\gamma) = \frac{6}{\gamma(\gamma+2)}e^{-3x} + Ce^{(2\gamma+1)x}, \tag{3.12}$$

where, the IF is $e^{-(2\mu+1)x}$.

Because $\Omega(x,\gamma)$ is bounded, C is 0.

Computing the inverse NIT of Eq. (3.12), we have

$$u(x,t) = 6e^{-2t} \times e^{-3x} = 6e^{-2t-3x}. \tag{3.13}$$

Eq. (3.13), by contrast, has same result by ST method in literature [16]. From Examples 3.1 and 3.2, we can see that the NIT is efficient to solve partial differential equations like LT or ST.

4. Conclusions

In this work, we graphically illustrated the dualities among the LT,LCT, ST, and NIT. The close duality relations indicate that the NIT may be similar in properties to LT, LCT, and ST, as well as be equivalent in function to get the analytic solution of the PDEs. Derivations of several properties for the NIT and the examples of PDEs support the above conclusions. The NIT, as a new integral transform, can be applied to solve many partial differential equations like LT, LCT, or ST effectively.

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