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Existence of periodic solutions for a class of discrete systems with classical or bounded (ϕ_1, ϕ_2)-Laplacian

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Abstract

In this paper, we investigate the existence of periodic solutions for the nonlinear discrete system with classical or bounded (ϕ_1, ϕ_2)-Laplacian:

$$\begin{cases} \Delta\phi_1\big(\Delta u_1(t-1)\big) + \nabla_{u_1} F\big(t, u_1(t), u_2(t)\big) = 0, \\ \Delta\phi_2\big(\Delta u_2(t-1)\big) + \nabla_{u_2} F\big(t, u_1(t), u_2(t)\big) = 0. \end{cases}$$

By using the saddle point theorem, we obtain that system with classical (ϕ_1, ϕ_2)-Laplacian has at least one periodic solution when F has (p, q)-sublinear growth, and system with bounded (ϕ_1, ϕ_2)-Laplacian has at least one periodic solution when F has sublinear growth. By using the least action principle, we obtain that system with classical or bounded (ϕ_1, ϕ_2)-Laplacian has at least one periodic solution when F has a growth like Lipschitz condition.

Keywords: Discrete systems, (ϕ_1, ϕ_2)-Laplacian, periodic solutions, saddle point theorem, the least action principle.
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1. Introduction and preliminaries

As we all know, critical point theory plays an important role in studying the existence and multiplicity of solutions for various differential equations, for example, nonlinear Schrödinger elliptic partial differential equations, nonlinear Dirac equations, reaction-diffusion equations, Hamiltonian systems (see [1, 16, 19, 21]). In 2003, the pioneering work for applying the critical point theory to discrete equations was given by Guo and Yu in [4] and [3]. Since then, lots of achievements for various types of discrete equations were presented. It is impossible to review them one by one here. We just refer readers to [5, 6, 10, 13, 23, 25, 27–29] and references therein. Recently, in [14] and [15], Mawhin studied a class of nonlinear discrete systems with ϕ-Laplacian which possesses generality. To be precise, he considered the following system:

$$\Delta\phi[\Delta u(n-1)] = \nabla_u F[n, u(n)] + h(n), \quad (n \in \mathbb{Z}), \tag{1.1}$$

where ϕ is a homeomorphism from $X \subset \mathbb{R}^N$ onto $Y \subset \mathbb{R}^N$ and the following three different types of homeomorphisms were discussed:

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(1) classical homeomorphism: when $X = Y = \mathbb{R}^N$, that is, $\phi : \mathbb{R}^N \to \mathbb{R}^N$;

(2) bounded homeomorphism: when $X = \mathbb{R}^N, Y = B_a$, that is, $\phi : \mathbb{R}^N \to B_a$ $(a < +\infty)$;

(3) singular homeomorphism: when $X = B_a, Y = \mathbb{R}^N$, that is, $\phi : B_a \subset \mathbb{R}^N \to \mathbb{R}^N$;

where B_a is a ball with its center at origin and radius a. By virtue of some critical point theorems, Mawhin obtained a series of results on existence and multiplicity of periodic solutions for system (1.1). Motivated by [14] and [15], Wang and our second author in [24] considered the following (ϕ_1, ϕ_2)-Laplacian system:

$$\begin{cases} \Delta\phi_1\big(\Delta u_1(t-1)\big) = \nabla_{u_1} F\big(t, u_1(t), u_2(t)\big) + h_1(t), \\ \Delta\phi_2\big(\Delta u_2(t-1)\big) = \nabla_{u_2} F\big(t, u_1(t), u_2(t)\big) + h_2(t). \end{cases} \tag{1.2}$$

Under the assumption that potential function $F(t, x_1, x_2)$ is periodic about some components of the independent variables (x_1, x_2) and has a (p, q)-sublinear growth and $\phi_m (m = 1, 2)$ are classical or bounded homeomorphisms, by virtue of some abstract critical point theorems in [16] and [11], the authors obtained some multiplicity results of periodic solutions for system (1.2). Moreover, in [30], they also considered the following system:

$$\begin{cases} \Delta\phi_1\big(\Delta u_1(t-1)\big) + \nabla_{u_1} V\big(t, u_1(t), u_2(t)\big) = f_1(t), \\ \Delta\phi_2\big(\Delta u_2(t-1)\big) + \nabla_{u_2} V\big(t, u_1(t), u_2(t)\big) = f_2(t), \end{cases} \tag{1.3}$$

where $\phi_m (m = 1, 2)$ are classical homeomorphisms, functions $V(t, x_1, x_2) : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $f_m : \mathbb{Z} \to \mathbb{R}^N$ $(m = 1, 2)$ satisfy some reasonable growth conditions. By virtue of an abstract critical point theorem in [21], they presented some existence results of homoclinic solutions for system (1.3). In [31], our second author and Wang investigated the following two classes of nonlinear difference systems with classical (ϕ_1, ϕ_2)-Laplacian:

$$\begin{cases} \mu\Delta\Big[\rho_1(t-1)\phi_1\big(\Delta u_1(t-1)\big)\Big] - \mu\rho_3(t)\phi_3(u_1(t)) + \nabla_{u_1} W\big(t, u_1(t), u_2(t)\big) = 0, \\ \mu\Delta\Big[\rho_2(t-1)\phi_2\big(\Delta u_2(t-1)\big)\Big] - \mu\rho_4(t)\phi_4(u_2(t)) + \nabla_{u_2} W\big(t, u_1(t), u_2(t)\big) = 0, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \Delta\left(\gamma_1(t-1)\phi_1\big(\Delta u_1(t-1)\big)\right) - \gamma_3(t)\phi_3(|u_1(t)|) + \nabla_{u_1} F\big(t, u_1(t), u_2(t)\big) = 0, \\ \Delta\left(\gamma_2(t-1)\phi_2\big(\Delta u_2(t-1)\big)\right) - \gamma_4(t)\phi_4(|u_2(t)|) + \nabla_{u_2} F\big(t, u_1(t), u_2(t)\big) = 0, \end{cases}$$

where $\mu \in \mathbb{R}$, $\rho_i : \mathbb{R} \to \mathbb{R}^+$, $\gamma_i : \mathbb{R} \to \mathbb{R}^+$ and ϕ_i, $i = 1, 2, 3, 4$ satisfy some reasonable assumptions. By using a critical point theorem due to Ricceri in [20], they obtained that (1.4) has at least three distinct T-periodic solutions, and by using the Clark's theorem, they obtained a multiplicity result of T-periodic solutions if F satisfies a symmetric condition. It is easy to see the differences between those results in [31] and our results below in this paper.

Motivated by [14, 15, 24] and [30], in this paper, we investigate the existence of T-periodic solutions for the following system with classical or bounded (ϕ_1, ϕ_2)-Laplacian:

$$\begin{cases} \Delta\phi_1\big(\Delta u_1(t-1)\big) + \nabla_{u_1} F\big(t, u_1(t), u_2(t)\big) = 0, \\ \Delta\phi_2\big(\Delta u_2(t-1)\big) + \nabla_{u_2} F\big(t, u_1(t), u_2(t)\big) = 0, \end{cases} \tag{1.5}$$

where Δ is a forward difference operator, $T > 1$ is an integer, $t \in \mathbb{Z}$, $F : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $\phi_m (m = 1, 2)$ satisfy the following condition:

($\mathcal{A}0$) $\phi_m : \mathbb{R}^N \to B_a \subset \mathbb{R}^N (a \in (0, +\infty])$, $m = 1, 2$ are two homeomorphisms which satisfy $\phi_m = \nabla\Phi_m$, $\phi_m(0) = 0$, where $\Phi_m \in C^1(\mathbb{R}^N, [0, +\infty))$ is strictly convex and $\Phi_m(0) = 0$.

Remark 1.1. Assumption ($\mathcal{A}0$) given in [14] is used to define the homeomorphisms $\phi_m (m = 1, 2)$, that is, $\phi_m (m = 1, 2)$ are called classical homeomorphisms, if $a = +\infty$ and are called bounded homeomorphisms, if $a < +\infty$. Moreover, if Φ_m possesses coercion (i.e., $\Phi_m(x) \to +\infty$ as $|x| \to \infty$), then one can find two constants $\delta_m = \min_{|x|=1} \Phi_m(x) > 0, m = 1, 2$ such that

$$\Phi_m(x) \geqslant \delta_m(|x| - 1), \quad x \in \mathbb{R}^N. \tag{1.6}$$

As $\Phi_1(x) = \frac{1}{q}|x|^q$ and $\Phi_2(x) = \frac{1}{p}|x|^p$, where $p > 1$ and $q > 1$, system (1.5) reduces to the following (q, p)-Laplacian difference system:

$$\begin{cases} \Delta\big(|\Delta u_1(t-1)|^{q-2}\Delta u_1(t-1)\big) + \nabla_{u_1}F\big(t, u_1(t), u_2(t)\big) = 0, \\ \Delta\big(|\Delta u_2(t-1)|^{p-2}\Delta u_2(t-1)\big) + \nabla_{u_2}F\big(t, u_1(t), u_2(t)\big) = 0, \end{cases}$$

which can be regarded as a discretization of the following differential system:

$$\begin{cases} \dfrac{d\big(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)\big)}{dt} + \nabla_{u_1}F\big(t, u_1(t), u_2(t)\big) = 0, \\ \dfrac{d\big(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t)\big)}{dt} + \nabla_{u_2}F\big(t, u_1(t), u_2(t)\big) = 0. \end{cases} \tag{1.7}$$

In recent years, there have been some results about periodic solutions for a system like (1.7) (see [8, 9, 17, 18, 26]). In [8], [17] and [18], by using the least action principle and the saddle point theorem, the authors obtained that system like (1.7) has at least one periodic solution. In [26], by using the least action principle, the authors obtained that system like (1.7) has at least one periodic solution and by using the local linking theorem, the authors obtained that system like (1.7) has at least two nonzero periodic solutions. In [9], by using an abstract critical point theorem in [2], the authors obtained that system like (1.7) has infinitely many periodic solutions.

In this paper, some assumptions on potential function F and some proofs are motivated partially by [26] and [25]. In [25], Xue and Tang investigated the following second-order discrete Hamiltonian system:

$$\Delta^2 u(t-1) + \nabla F(t, u(t)) = 0, \quad \forall\, t \in \mathbb{Z}. \tag{1.8}$$

By using the saddle point theorem, they obtained three theorems that system (1.8) has at least one T-periodic solution when F has a subquadratic growth. Here, we only recall two theorems which are related to our paper.

Theorem 1.2 ([25, Theorem 2])**.** *Assume that* $F(t, x)$ *satisfies*

(H_1) *there exists an integer* $T > 0$ *such that* $F(t + T, x) = F(t, x)$ *for all* $(t, x) \in \mathbb{Z} \times \mathbb{R}^N$*;*

(H_2) $\frac{F(t,x)}{|x|^2} \to 0$ *as* $|x| \to \infty$*, for all* $t \in \mathbb{Z}[1, T]$*, where* $\mathbb{Z}[1, T] := \{1, \cdots, T\}$*;*

(H_3) $2F(t, x) - (x, \nabla F(t, x)) \to +\infty$ *as* $|x| \to \infty$*, for all* $t \in \mathbb{Z}[1, T]$.

Then system (1.8) *possesses at least one* T*-periodic solution.*

As a corollary of Theorem 1.2, the authors also presented the following theorem:

Theorem 1.3 ([25, Theorem 3])**.** *Assume that* $F(t, x)$ *satisfies* (H_1)*,*

(H_4) *there are constants* $G > 0$ *and* $0 < \beta < 2$ *such that for all* $(t, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N$ *and* $|x| \geqslant G$*,*

$$(x, \nabla F(t, x)) \leqslant \beta F(t, x);$$

(H_5) $F(t, x) \to +\infty$ *as* $|x| \to \infty$*, for all* $t \in \mathbb{Z}[1, T]$.

Then system (1.8) *has at least one* T*-periodic solution.*

Next we prepare to present our results. For this purpose, we need to make the following three assumptions:

$(\mathcal{A}1)$ there exist constants $d_1 > 0$, $d_2 > 0$, $p > 1$ and $q > 1$ such that

$$\Phi_1(x_1) + \Phi_2(x_2) \geqslant d_1|x_1|^p + d_2|x_2|^q, \quad \forall\, x_1, x_2 \in \mathbb{R}^N;$$

($\mathcal{A}2$) $(\phi_1(x_1), x_1) + (\phi_2(x_2), x_2) \geqslant \min\{p, q\}\big[\Phi_1(x_1) + \Phi_2(x_2)\big], \forall\, x_1, x_2 \in \mathbb{R}^N$;

($\mathcal{A}3$) there exist constants $p^* \in (0, 1]$ and $q^* \in (0, 1]$ such that

$$(\phi_1(x_1), x_1) + (\phi_2(x_2), x_2) \geqslant \min\{p^*, q^*\}\big[\Phi_1(x_1) + \Phi_2(x_2)\big], \quad \forall\, x_1, x_2 \in \mathbb{R}^N.$$

Moreover, we need to fix some notations. For any $s > 1$ and $s' > 1$ with $1/s + 1/s' = 1$, let

$$C(s, s') = \min\left\{\frac{(T-1)^{2s-1}}{T^{s-1}}, \frac{T^{s-1}\Theta(s', s)}{(s'+1)^{s/s'}}\right\},$$

$$\Theta(s', s) = \sum_{t=1}^{T}\left[\left(\frac{t}{T}\right)^{s'+1} + \left(1 - \frac{t}{T} + \frac{1}{T}\right)^{s'+1} - \frac{2}{T^{s'+1}}\right]^{s/s'}.$$

(I) For classical homeomorphism

Theorem 1.4. *Suppose that ($\mathcal{A}0$) with $a = +\infty$, ($\mathcal{A}1$), ($\mathcal{A}2$) and the following conditions hold:*

(F_0) *for every $t \in \mathbb{Z}[1, T]$, $F : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable in (x_1, x_2), and for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$, $(t, x_1, x_2) \to F(t, x_1, x_2)$ is T-periodic in t, where $x_1 = (x_1^{(1)}, \cdots, x_N^{(1)})^\tau$, $x_2 = (x_1^{(2)}, \cdots, x_N^{(2)})^\tau$;*

(F_1)

$$\lim_{|x_1|+|x_2|\to+\infty}\Big[\min\{p, q\}F(t, x_1, x_2) - (\nabla_{x_1}F(t, x_1, x_2), x_1) - (\nabla_{x_2}F(t, x_1, x_2), x_2)\Big] = +\infty,$$

for all $t \in \mathbb{Z}[1, T]$;

(F_2) *there exists a positive constant M_* such that*

$$(\nabla_{x_1}F(t, x_1, x_2), x_1) \geqslant 0, \quad (\nabla_{x_2}F(t, x_1, x_2), x_2) \geqslant 0,$$

for all $(t, x_1, x_2) \in \mathbb{Z}[1, T] \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1| + |x_2| \geqslant M_*$;

(F_3)

$$\lim_{|x_1|+|x_2|\to\infty}\frac{F(t, x_1, x_2)}{|x_1|^p + |x_2|^q} = 0,$$

for all $t \in \mathbb{Z}[1, T]$.

Then system (1.5) possesses at least one T-periodic solution.

Corollary 1.5. *Suppose that ($\mathcal{A}0$) with $a = +\infty$, ($\mathcal{A}1$), ($\mathcal{A}2$), (F_0) and (F_2) hold. If*

$(F_1)'$ *there exist constants $L > 0$ and $0 < \beta < \min\{p, q\}$, such that*

$$\beta F(t, x_1, x_2) \geqslant (\nabla_{x_1}F(t, x_1, x_2), x_1) + (\nabla_{x_2}F(t, x_1, x_2), x_2),$$

for all $(t, x_1, x_2) \in \mathbb{Z}[1, T] \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1| + |x_2| \geqslant L$;

$(F_3)'$

$$\lim_{|x_1|+|x_2|\to\infty} F(t, x_1, x_2) = +\infty,$$

for all $(t, x_1, x_2) \in \mathbb{Z}[1, T] \times \mathbb{R}^N \times \mathbb{R}^N$.

Then system (1.5) possesses at least one T-periodic solution.

Remark 1.6. There exist some examples satisfying Theorem 1.4. For example, let $T > 1$, $\Phi_1(x_1) = \frac{1}{p}|x_1|^p$, $\Phi_2(x_2) = \frac{1}{q}|x_2|^q$ and

$$F(t, x_1, x_2) = \left(1 + \sin^2 \frac{\pi}{T} t\right) \ln(1 + |x_1|^p + |x_2|^q).$$

It is easy to verify that the example satisfies Theorem 1.4 if we take $d_1 = \frac{1}{p}$ and $d_2 = \frac{1}{q}$.

Theorem 1.7. *Suppose that* $(\mathcal{A}0)$ *with* $a = +\infty$, $(\mathcal{A}1)$, $(\mathcal{A}2)$, (F_0), (F_1), (F_2) *and the following conditions hold:*

(F_4) *there exists a positive constant* M^* *such that*

$$F(t, x_1, x_2) \geqslant 0, \quad \textit{for all } (t, x_1, x_2) \in \mathbb{Z}[1, T] \times \mathbb{R}^N \times \mathbb{R}^N \textit{ with } |x_1| + |x_2| \geqslant M^*;$$

(F_5)

$$\lim_{|x_1|+|x_2|\to\infty} \frac{F(t, x_1, x_2)}{|x_1|^p + |x_2|^q} < \min\left\{\frac{d_1}{C(p, p')}, \frac{d_2}{C(q, q')}\right\}.$$

Then system (1.5) *possesses at least one* T*-periodic solution.*

Remark 1.8. In Theorem 1.4, Corollary 1.5 and Theorem 1.7, (F_2) can be deleted if $p = q$. One can see the reason in Remark 3.4 below. Thus we claim that Theorem 1.4 and Corollary 1.5 generalize Theorem 1.2 and Theorem 1.3, respectively. In fact, when $p = q = 2$, $\Phi_1(x) = \Phi_2(x) = \frac{1}{2}|x|^2$ and $F(t, x, y) = F(t, y, x)$, system (1.5) reduces to system (1.8) and Theorem 1.4 and Corollary 1.5 become Theorem 1.2 and Theorem 1.3, respectively. Moreover, Theorem 1.7 is still a new result even if system (1.5) reduces to system (1.8), which shows that (F_2) can be weakened to (F_5) if (F_4) holds. There exist examples satisfying Theorem 1.7 but not satisfying Theorem 1.4. For example, let $T > 1$, $\Phi_1(x_1) = \frac{1}{p}|x_1|^p$, $\Phi_2(x_2) = \frac{1}{p}|x_2|^p$ and

$$F(t, x_1, x_2) = \frac{1}{4pC(p, p')}\left(1 + \sin^2 \frac{\pi}{T} t\right) [|x_1|^p + |x_2|^p + \ln(1 + |x_1|^p + |x_2|^p)].$$

It is easy to verify that the example satisfies Theorem 1.7 if we take $p = q$ and $d_1 = d_2 = \frac{1}{p}$.

Theorem 1.9. *Assume that* $F(t, x_1, x_2) \equiv F(x_1, x_2)$ *for all* $t \in \mathbb{Z}[1, T]$, *and* $(\mathcal{A}0)$ *with* $a = +\infty$, $(\mathcal{A}1)$, (F_0) *and the following conditions hold:*

(F_6) *there exist constants* $r_1 \in \left[0, \frac{d_1 p}{C(p,p')}\right)$, $r_2 \in [0, +\infty)$, $r_3 \in \left[0, \frac{d_2 q}{C(q,q')}\right)$, $r_4 \in [0, +\infty)$, $\alpha_0 \in [0, p)$ *and* $\beta_0 \in [0, q)$ *such that*

$$(\nabla_{x_1} F(x_1, x_2) - \nabla_{y_1} F(y_1, y_2), x_1 - y_1) \leqslant r_1|x_1 - y_1|^p + r_2|x_1 - y_1|^{\alpha_0},$$

and

$$(\nabla_{x_2} F(x_1, x_2) - \nabla_{y_2} F(y_1, y_2), x_2 - y_2) \leqslant r_3|x_2 - y_2|^q + r_4|x_2 - y_2|^{\beta_0},$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$;

(F_7)

$$\lim_{|x_1|+|x_2|\to+\infty} F(x_1, x_2) = -\infty,$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

Then system (1.5) *possesses at least one* T*-periodic solution.*

By Theorem 1.9, it is easy to obtain the following corollary.

Corollary 1.10. *Assume that* $F(t, x_1, x_2) \equiv F(x_1, x_2)$ *for all* $t \in \mathbb{Z}[1, T]$, *and* $(\mathcal{A}0)$ *with* $a = +\infty$, $(\mathcal{A}1)$, (F_0), (F_7) *and the following condition holds:*

(F_8) *there exist constants* $r_1 \in \left[0, \frac{d_1 p}{C(p,p')}\right)$, $r_2 \in [0,+\infty)$, $r_3 \in \left[0, \frac{d_2 q}{C(q,q')}\right)$, $r_4 \in [0,+\infty)$, $\alpha_0 \in [0,p)$ *and* $\beta_0 \in [0,q)$ *such that*

$$|\nabla_{x_1} F(x_1,x_2) - \nabla_{y_1} F(y_1,y_2)| \leqslant r_1|x_1-y_1|^{p-1} + r_2|x_1-y_1|^{\alpha_0-1},$$

and

$$|\nabla_{x_2} F(x_1,x_2) - \nabla_{y_2} F(y_1,y_2)| \leqslant r_3|x_2-y_2|^{q-1} + r_4|x_2-y_2|^{\beta_0-1},$$

for all $(x_1,x_2), (y_1,y_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

Then system (1.5) *possesses at least one* T*-periodic solution.*

Remark 1.11. There exist some examples satisfying Theorem 1.9. For example, let $p=2$, $q=\frac{3}{2}$, $\Phi_1(x_1) = \frac{1}{2}|x_1|^2$, $\Phi_2(x_2) = \frac{2}{3}|x_2|^{\frac{3}{2}}$ and

$$F(x_1,x_2) = \frac{3r_1}{4}|x_1|^{\frac{4}{3}} + \frac{3r_2}{4}|x_2|^{\frac{4}{3}} - \frac{r_1}{2}|x_1|^2 - \frac{2r_2}{3}|x_2|^{\frac{3}{2}},$$

where $r_1 \in \left(0, \frac{d_1 p}{C(p,p')}\right)$ and $r_2 \in \left(0, \frac{d_2 p}{C(q,q')}\right)$. Then it is easy to verify that the example satisfies Theorem 1.9 if we take $\alpha_0 = \beta_0 = \frac{4}{3}$, $d_1 = \frac{1}{2}$ and $d_2 = \frac{2}{3}$.

(II) For bounded homeomorphism

Theorem 1.12. *Assume that* $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ *are coercive,* $m = 1,2$, *and* $(\mathcal{A}0)$ *with* $a < +\infty$, $(\mathcal{A}3)$, (F_0), (F_2) *and the following conditions holds:*

(S_1)

$$\lim_{|x_1|+|x_2| \to +\infty} \Big[\min\{p^*,q^*\}F(t,x_1,x_2) - (\nabla_{x_1}F(t,x_1,x_2),x_1) - (\nabla_{x_2}F(t,x_1,x_2),x_2)\Big] = +\infty,$$

for all $t \in \mathbb{Z}[1,T]$;

(S_2)

$$\lim_{|x_1|+|x_2| \to \infty} \frac{F(t,x_1,x_2)}{|x_1|+|x_2|} = 0,$$

for all $t \in \mathbb{Z}[1,T]$.

Then system (1.5) *possesses at least one* T*-periodic solution.*

Corollary 1.13. *Assume that* $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ *are coercive,* $m = 1,2$, *and* $(\mathcal{A}0)$ *with* $a < +\infty$, $(\mathcal{A}3)$, (F_0), (F_2) *and* (S_1) *and the following condition holds:*

$(S_2)'$ *there exist constants* $\theta_1 \in (0,1)$ *and* $\theta_2 \in (0,1)$ *such that*

$$\lim_{|x_1|+|x_2| \to \infty} \frac{F(t,x_1,x_2)}{|x_1|^{\theta_1}+|x_2|^{\theta_2}} < +\infty,$$

for all $t \in \mathbb{Z}[1,T]$.

Then system (1.5) *possesses at least one* T*-periodic solution.*

Remark 1.14. There exist some examples satisfying Theorem 1.12. For example, let $T > 1$, $\Phi_1(x_1) = \sqrt{1+|x_1|^2} - 1$, $\Phi_2(x_2) = \sqrt{2+|x_2|^2} - \sqrt{2}$ and

$$F(t,x_1,x_2) = \left(1 + \sin^2 \frac{\pi}{T} t\right) \ln(1+|x_1|^2+|x_2|^2).$$

It is easy to verify that the example satisfies Theorem 1.12 if we take $p^* = q^* = 1$, $\delta_1 = \sqrt{2}-1$ and $\delta_2 = \sqrt{3}-\sqrt{2}$.

Theorem 1.15. *Assume that* $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ *are coercive,* $m = 1, 2$, *and* $(\mathcal{A}0)$ *with* $a < +\infty$, $(\mathcal{A}3)$, (F_0), (F_2), (S_1), (F_4) *and the following condition holds:*

(S_3) *there exist four constants* $s_0, s_0', s_1, s_1' \in (1, +\infty)$ *with* $\frac{1}{s_0} + \frac{1}{s_0'} = 1$ *and* $\frac{1}{s_1} + \frac{1}{s_1'} = 1$ *such that*

$$\lim_{|x_1|+|x_2|\to\infty} \frac{F(t, x_1, x_2)}{|x_1| + |x_2|} < \min\left\{\frac{\delta_1}{T^{\frac{s_0-1}{s_0}}[C(s_0, s_0')]^{\frac{1}{s_0}}}, \frac{\delta_2}{T^{\frac{s_1-1}{s_1}}[C(s_1, s_1')]^{\frac{1}{s_1}}}\right\},$$

for all $t \in \mathbb{Z}[1, T]$, *where* δ_1 *and* δ_2 *are given in* (1.6).

Then system (1.5) *possesses at least one* T*-periodic solution.*

Remark 1.16. There exist some examples satisfying Theorem 1.15 but not satisfying Theorem 1.12. For example, let $T > 1$, $\Phi_1(x_1) = \sqrt{1 + |x_1|^2} - 1$, $\Phi_2(x_2) = \sqrt{2 + |x_2|^2} - \sqrt{2}$ and

$$F(t, x_1, x_2) = A\left(1 + \sin^2\frac{\pi}{T}t\right)\left[\sqrt{1 + |x_1|^2} + \sqrt{2 + |x_2|^2} + \ln(1 + |x_1|^2 + |x_2|^2)\right],$$

where

$$0 < A < \frac{1}{2}\min\left\{\frac{\sqrt{2} - 1}{T^{\frac{s_0-1}{s_0}}[C(s_0, s_0')]^{\frac{1}{s_0}}}, \frac{\sqrt{3} - \sqrt{2}}{T^{\frac{s_1-1}{s_1}}[C(s_1, s_1')]^{\frac{1}{s_1}}}\right\},$$

and s_0, s_0', s_1 and s_1' are four fixed positive constants with $\frac{1}{s_0} + \frac{1}{s_0'} = 1$ and $\frac{1}{s_1} + \frac{1}{s_1'} = 1$. It is easy to verify that the example satisfies Theorem 1.15 if we take $p^* = q^* = 1$, $\delta_1 = \sqrt{2} - 1$ and $\delta_2 = \sqrt{3} - \sqrt{2}$. Moreover, it is obvious that F does not satisfy (S_2) so that it does not satisfy Theorem 1.12.

Theorem 1.17. *Assume that* $F(t, x_1, x_2) \equiv F(x_1, x_2)$ *for all* $t \in \mathbb{Z}[1, T]$, $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ *are coercive,* $m = 1, 2$, *and* $(\mathcal{A}0)$ *with* $a < +\infty$, (F_0), (F_7) *and the following condition holds:*

(S_4) *there exist constants* $s_2, s_2', s_3, s_3' \in (1, +\infty)$ *with* $\frac{1}{s_2} + \frac{1}{s_2'} = 1$ *and* $\frac{1}{s_3} + \frac{1}{s_3'} = 1$, $r_1 \in \left[0, \frac{\delta_1}{T^{\frac{s_2-1}{s_2}}[C(s_2,s_2')]^{\frac{1}{s_2}}}\right)$, $r_2 \in [0, +\infty)$, $r_3 \in \left[0, \frac{\delta_2}{T^{\frac{s_3-1}{s_3}}[C(s_3,s_3')]^{\frac{1}{s_3}}}\right)$, $r_4 \in [0, +\infty)$, $\mu_0 \in [0, 1)$ *and* $\nu_0 \in [0, 1)$ *such that*

$$(\nabla_{x_1}F(x_1, x_2) - \nabla_{y_1}F(y_1, y_2), x_1 - y_1) \leqslant r_1|x_1 - y_1| + r_2|x_1 - y_1|^{\mu_0},$$

and

$$(\nabla_{x_2}F(x_1, x_2) - \nabla_{y_2}F(y_1, y_2), x_2 - y_2) \leqslant r_3|x_2 - y_2| + r_4|x_2 - y_2|^{\nu_0},$$

for all $t \in \mathbb{Z}[1, T]$ *and all* $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

Then system (1.5) *has at least one* T*-periodic solution.*

By Theorem 1.17, it is easy to obtain the following corollary.

Corollary 1.18. *Assume that* $F(t, x_1, x_2) \equiv F(x_1, x_2)$ *for all* $t \in \mathbb{Z}[1, T]$, $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ *are coercive,* $m = 1, 2$ *and* $(\mathcal{A}0)$ *with* $a < +\infty$, (F_0), (F_7) *and the following condition holds:*

(S_5) *there exist constants* $s_2, s_2', s_3, s_3' \in (1, +\infty)$ *with* $\frac{1}{s_2} + \frac{1}{s_2'} = 1$ *and* $\frac{1}{s_3} + \frac{1}{s_3'} = 1$, $l_1 \in \left(0, \frac{\delta_1}{T^{\frac{s_2-1}{s_2}}[C(s_2,s_2')]^{\frac{1}{s_2}}}\right)$ *and* $l_2 \in \left(0, \frac{\delta_2}{T^{\frac{s_3-1}{s_3}}[C(s_3,s_3')]^{\frac{1}{s_3}}}\right)$ *such that*

$$|\nabla_{x_1}F(x_1, x_2) - \nabla_{y_1}F(y_1, y_2)| \leqslant l_1,$$

and

$$|\nabla_{x_2}F(x_1,x_2)-\nabla_{y_2}F(y_1,y_2)|\leqslant l_2,$$

for all (x_1,x_2), $(y_1,y_2)\in\mathbb{R}^N\times\mathbb{R}^N$.

Then system (1.5) *possesses at least one* T*-periodic solution.*

Remark 1.19. There exist some examples satisfying Corollary 1.18. For example, let $T>1$, $\Phi_1(x_1)=\sqrt{1+|x_1|^2}-1$, $\Phi_2(x_2)=\sqrt{2+|x_2|^2}-\sqrt{2}$ and

$$F(x_1,x_2)=-\frac{l}{2}\ln(2+|x_1|^2+|x_2|^2),$$

where $l\in\left(0,\min\left\{\frac{\sqrt{2}-1}{T^{\frac{s_2-1}{s_2}}[C(s_2,s_2')]^{\frac{1}{s_2}}},\frac{\sqrt{3}-\sqrt{2}}{T^{\frac{s_3-1}{s_3}}[C(s_3,s_3')]^{\frac{1}{s_3}}}\right\}\right)$, and s_2, s_2', s_3 and s_3' are four fixed positive constants with $\frac{1}{s_2}+\frac{1}{s_2'}=1$ and $\frac{1}{s_3}+\frac{1}{s_3'}=1$. Then it is easy to verify that the example satisfies Corollary 1.18 if we take $l_1=l_2=l$, $\delta_1=\sqrt{2}-1$ and $\delta_2=\sqrt{3}-\sqrt{2}$.

2. Preliminaries

Let

$$E_T=\{v:=\{v(t)\}\mid v(t+T)=v(t),v(t)\in\mathbb{R}^N,t\in\mathbb{Z}\}.$$

It is easy to see that E_T has NT dimensions. For $v\in E_T$, set

$$\|v\|_{[r]}=\left(\sum_{t=1}^{T}|v(t)|^r\right)^{1/r},\quad r>1\quad\text{and}\quad\|v\|_\infty=\max_{t\in\mathbb{Z}[1,T]}|v(t)|.$$

It is obvious that

$$\|v\|_\infty\leqslant\|v\|_{[r]}\leqslant T^{\frac{1}{r}}\|v\|_\infty.\tag{2.1}$$

For $1<s<+\infty$, for $v\in E_T$, define

$$\|v\|_s=\left(\sum_{t=1}^{T}|\Delta v(t)|^s+\sum_{t=1}^{T}|v(t)|^s\right)^{1/s}.$$

Then there exist two positive constants D_1,D_2 such that

$$\|u_1\|_{[p]}\leqslant\|u_1\|_p\leqslant D_1\|u_1\|_{[p]},\qquad\|u_2\|_{[q]}\leqslant\|u_2\|_q\leqslant D_2\|u_2\|_{[q]},\tag{2.2}$$

for all $u_1,u_2\in E_T$.

Let $E=E_T\times E_T$. For $u=(u_1,u_2)^\tau\in E$, define

$$\|u\|=\|u_1\|_p+\|u_2\|_q.$$

Note that for any $v\in E_T$, it can be expressed as $v=\bar{v}+\tilde{v}$, where $\bar{v}=\frac{1}{T}\sum_{t=1}^{T}v(t)$ and $\tilde{v}$ satisfies that $\sum_{t=1}^{T}v(t)=0$. Hence, for any $u\in E$, $u=(u_1,u_2)^\tau=(\bar{u}_1+\tilde{u}_1,\bar{u}_2+\tilde{u}_2)^\tau=(\bar{u}_1,\bar{u}_2)^\tau+(\tilde{u}_1,\tilde{u}_2)^\tau$. Define W and Y by

$$W=\left\{u=(u_1,u_2)^\tau\in E\,\middle|\,u_m(1)=\cdots=u_m(T)=\frac{1}{T}\sum_{t=1}^{T}u_m(t),m=1,2\right\},$$

and

$$Y=\left\{(u_1,u_2)^\tau\in E\,\middle|\,\sum_{t=1}^{T}u_m(t)=0,m=1,2\right\}.$$

Then, $E = W \bigoplus Y$ and for any $u \in E$, it can be expressed as $u = \overline{u} + \widetilde{u}$, where $\overline{u} = (\overline{u}_1, \overline{u}_2)^\tau \in W$ and $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2)^\tau \in Y$. Furthermore, $u_m = \overline{u}_m + \widetilde{u}_m$, where $\overline{u}_m = \frac{1}{T}\sum_{t=1}^{T} u_m(t)$, and $\Delta\widetilde{u}_m = \Delta u_m$, $m = 1, 2$.

For $u = (u_1, u_2)^\tau \in Y$, define

$$\|\Delta u\| = \|\Delta u_1\|_{[p]} + \|\Delta u_2\|_{[q]}, \quad \forall u \in Y,$$

which is also a norm on Y. Since Y is finite dimensional space, so the norm $\|\Delta u\|$ is equivalent to the norm $\|u\|$ in Y.

Lemma 2.1 ([27]). *Let $u = (u_1, u_2)^\tau \in Y$. Then for any $s > 1$ and $s' > 1$ with $1/s + 1/s' = 1$, we have*

$$\sum_{t=1}^{T} |u_m(t)|^s \leqslant C(s, s') \sum_{t=1}^{T} |\Delta u_m(t)|^s, \quad m = 1, 2.$$

Remark 2.2. By Lemma 2.1 and a simple calculation, it is easy to obtain that

$$\|u_1\|_p^p \leqslant \left(C(p, p') + 1\right) \|\Delta u_1\|_{[p]}^p, \quad \|u_2\|_q^q \leqslant \left(C(q, q') + 1\right) \|\Delta u_2\|_{[q]}^q, \quad \forall u = (u_1, u_2)^\tau \in Y. \tag{2.3}$$

Moreover, for any $u = (u_1, u_2)^\tau \in E$, by Lemma 2.1, we have

$$\begin{aligned}
\|u_m\|_s &= \left(\sum_{t=1}^{T} |\Delta u_m(t)|^s + \sum_{t=1}^{T} |u_m(t)|^s\right)^{1/s} \\
&= \left(\sum_{t=1}^{T} |\Delta u_m(t)|^s + \sum_{t=1}^{T} |\bar{u}_m + \tilde{u}_m(t)|^s\right)^{1/s} \\
&\leqslant \left(\sum_{t=1}^{T} |\Delta u_m(t)|^s + 2^{s-1}\sum_{t=1}^{T} |\bar{u}_m|^s + 2^{s-1}\sum_{t=1}^{T} |\tilde{u}_m(t)|^s\right)^{1/s} \\
&\leqslant \left[2^{s-1}T|\bar{u}_m|^s + (1 + 2^{s-1}C(s, s'))\sum_{t=1}^{T} |\Delta u_m(t)|^s\right]^{1/s} \\
&\leqslant 2^{\frac{s-1}{s}}T^{\frac{1}{s}}|\bar{u}_m| + (1 + 2^{s-1}C(s, s'))^{1/s}\left[\sum_{t=1}^{T} |\Delta u_m(t)|^s\right]^{1/s} && (2.4) \\
&\leqslant 2^{\frac{s-1}{s}}T^{\frac{1}{s}}|\bar{u}_m| + (1 + 2^{s-1}C(s, s'))^{1/s}\sum_{t=1}^{T} |\Delta u_m(t)|, && (2.5)
\end{aligned}$$

where $m = 1$ (or 2) and $s > 1$. Hence, if $\|u\| \to \infty$, then $\|u_1\|_p \to \infty$ or $\|u_2\|_q \to \infty$ and so $|\bar{u}_m| + \sum_{t=1}^{T} |\Delta u_m(t)|^s \to \infty$, $m = 1$ (or 2) and $s = p$ (or q).

Lemma 2.3 ([24]). *For any $u = (u_1, u_2)^\tau$, $v = (v_1, v_2)^\tau \in E$, the following two equalities hold:*

$$-\sum_{t=1}^{T} (\Delta\phi_1(\Delta u_1(t-1)), v_1(t)) = \sum_{t=1}^{T} (\phi_1(\Delta u_1(t)), \Delta v_1(t)),$$

$$-\sum_{t=1}^{T} (\Delta\phi_2(\Delta u_2(t-1)), v_2(t)) = \sum_{t=1}^{T} (\phi_2(\Delta u_2(t)), \Delta v_2(t)).$$

Lemma 2.4 ([24]). *Let $L : \mathbb{Z}[1, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $(t, x_1, x_2, y_1, y_2) \to L(t, x_1, x_2, y_1, y_2)$ and assume that L is continuously differentiable in (x_1, x_2, y_1, y_2) for all $t \in \mathbb{Z}[1, T]$. Then the functional $\mathcal{J} : E \to \mathbb{R}$ defined by*

$$\mathcal{J}(u) = \mathcal{J}(u_1, u_2) = \sum_{t=1}^{T} L(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)),$$

is continuously differentiable on E *and for all* $u, v \in E$*, we have*

$$\begin{aligned}\langle \mathcal{J}'(u), v\rangle &= \langle \mathcal{J}'(u_1,u_2),(v_1,v_2)\rangle\\ &= \sum_{t=1}^{T}\big[(D_{x_1}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),v_1(t))\\ &\quad + (D_{y_1}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),\Delta v_1(t))\\ &\quad + (D_{x_2}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),v_2(t))\\ &\quad + (D_{y_2}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),\Delta v_2(t))\big].\end{aligned}$$

Let

$$L(t,x_1,x_2,y_1,y_2) = \Phi_1(y_1) + \Phi_2(y_2) - F(t,x_1,x_2).$$

Then

$$\mathcal{J}(u) = \mathcal{J}(u_1,u_2) = \sum_{t=1}^{T}\big[\Phi_1(\Delta u_1(t)) + \Phi_2(\Delta u_2(t)) - F(t,u_1(t),u_2(t))\big].$$

By $(\mathcal{A}0)$, (F_0) and Lemma 2.3, we have

$$\begin{aligned}\langle \mathcal{J}'(u), v\rangle &= \langle \mathcal{J}'(u_1,u_2),(v_1,v_2)\rangle\\ &= \sum_{t=1}^{T}\big[(\phi_1(\Delta u_1(t)),\Delta v_1(t)) + (\phi_2(\Delta u_2(t)),\Delta v_2(t))\\ &\quad - (\nabla_{u_1}F(t,u_1(t),u_2(t)),v_1(t)) - (\nabla_{u_2}F(t,u_1(t),u_2(t)),v_2(t))\big]\\ &= -\sum_{t=1}^{T}\big[(\Delta\phi_1(\Delta u_1(t-1)),v_1(t)) + (\Delta\phi_2(\Delta u_2(t-1)),v_2(t))\\ &\quad + (\nabla_{u_1}F(t,u_1(t),u_2(t)),v_1(t)) + (\nabla_{u_2}F(t,u_1(t),u_2(t)),v_2(t))\big],\end{aligned}$$

and then it is easy to obtain that critical point of $\mathcal{J}$ in E is T-periodic solution of system (1.5).

Definition 2.5 ([16])**.** Let E be a real Banach space and for $\varphi \in C^1(E,\mathbb{R})$, we say that φ satisfies the (PS) condition, if any sequence $(u_n) \subset E$ for which $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Next, we introduce some abstract critical point theorems which will be used to prove our main results.

Lemma 2.6 ([19])**.** *Let* $X = X_1 \bigoplus X_2$*, where* X *is a real Banach space and* $X_1 \neq \{0\}$ *and is finite dimensional. Suppose* $\varphi \in C^1(X,\mathbb{R})$*, satisfies (PS) condition, and*

(I_1) *there is a constant* α *and a bounded neighborhood* D *of* 0 *in* X_1 *such that* $\varphi|_{\partial D} \leqslant \alpha$*, and*

(I_2) *there is a constant* $\beta > \alpha$ *such that* $\varphi|_{X_2} \geqslant \beta$*.*

Then φ *possesses a critical value* $c \geqslant \beta$*. Moreover* c *can be characterized as*

$$c = \inf_{h\in\Gamma}\max_{u\in\overline{D}} \varphi(h(u)),$$

where

$$\Gamma = \{h \in C(\overline{D},X) | h = id \text{ on } \partial D\}.$$

Remark 2.7. As we all know, under the weaker condition (C) than (PS), a deformation lemma holds true. We say that $\{u_n\}$ is a (C) sequence for φ, if $\{u_n\}$ is bounded and $(1+\|u_n\|)\|\varphi'(u_n)\| \to 0$, as $n \to \infty$, and φ satisfies (C) condition, if any (C) sequence for φ has a convergent subsequence.

Lemma 2.8 ([12]). *Assume that $\varphi \in C^1(X, \mathbb{R})$ is bounded from below (above) and satisfies the (PS) condition. Then*

$$c = \inf_{u \in X} \varphi(u) \quad (c = \sup_{u \in X} \varphi(u)),$$

is a critical value of φ.

3. Proofs for classical homeomorphism

Lemma 3.1. *Assume that $(\mathcal{A}0)$ with $a = +\infty$, $(\mathcal{A}1)$, $(\mathcal{A}2)$, (F_0), (F_1) and (F_5) (or (F_3)) hold. Then $\mathcal{J}$ satisfies the (C) condition.*

Proof. The proof is motivated by [7]. Suppose that $\{u_n = (u_1^{(n)}, u_2^{(n)})\}$ is a (C) sequence for $\mathcal{J}$, that is,

$$\mathcal{J}(u_n) \text{ is bounded} \quad \text{and} \quad (1 + \|u_n\|)\|\mathcal{J}'(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

Then there exists a positive constant C_0 such that

$$|\mathcal{J}(u_n)| \leqslant C_0, \qquad (1 + \|u_n\|)\|\mathcal{J}'(u_n)\| \leqslant C_0, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Then we claim that $\{u_n\}$ is bounded, that is, both $\{u_1^{(n)}\}$ and $\{u_2^{(n)}\}$ are bounded. Otherwise, without loss of generality, we assume that $\{u_1^{(n)}\}$ is unbounded. Then there exists a subsequence of $\{u_1^{(n)}\}$, still denoted by $\{u_1^{(n)}\}$, such that $\|u_1^{(n)}\|_p \to \infty$. Let $z_1^{(n)} = \frac{u_1^{(n)}}{\|u_1^{(n)}\|_p}$. Then $\|z_1^{(n)}\|_p = 1$. Hence there exists a convergent subsequence $\{z_1^{(n_k)}\}$ of $\{z_1^{(n)}\}$ such that $z_1^{(n_k)} \to z_1^*$ for some $z_1^* \in E_T$. Correspondingly, we choose a subsequence $\{u_2^{(n_k)}\}$ of $\{u_2^{(n)}\}$, which has the same index (n_k) as $\{u_1^{(n_k)}\}$. Then there exist the following two cases.

(i) $\{u_2^{(n_k)}\}$ is unbounded.

For this case, there exists a subsequence of $\{u_2^{(n_k)}\}$, still denoted by $\{u_2^{(n_k)}\}$, such that $\|u_2^{(n_k)}\|_q \to \infty$. Let $z_2^{(n_k)} = \frac{u_2^{(n_k)}}{\|u_2^{(n_k)}\|_q}$. Then $\|z_2^{(n_k)}\|_q = 1$. Hence there exists a convergent subsequence $\{z_2^{(n_{k_j})}\}$ of $\{z_2^{(n_k)}\}$ such that $z_2^{(n_{k_j})} \to z_2^*$ for some $z_2^* \in E_T$ as $j \to \infty$. Correspondingly, it is obvious that $z_1^{(n_{k_j})} \to z_1^*$ as $j \to \infty$. Then (2.1) and (2.2) imply that

$$z_1^{(n_{k_j})}(t) \to z_1^*(t), \qquad z_2^{(n_{k_j})}(t) \to z_2^*(t), \text{ as } j \to \infty, \quad \forall t \in \mathbb{Z}[1, T], \tag{3.2}$$

and it is easy to see that

$$\|u_1^{(n_{k_j})}\|_p \to \infty, \qquad \|u_2^{(n_{k_j})}\|_q \to \infty, \quad \text{as } j \to \infty. \tag{3.3}$$

Moreover, it follows from (F_5) or (F_3) that there exist constants $G_1 > 0$ and $0 < \varepsilon < \min\left\{\frac{d_1}{C(p,p')}, \frac{d_2}{C(q,q')}\right\}$ such that

$$F(t, x_1, x_2) \leqslant \varepsilon(|x_1|^p + |x_2|^q),$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1| + |x_2| \geqslant G_1$. Then by (F_0), there exists a positive constant C_1 such that

$$F(t, x_1, x_2) \leqslant \varepsilon(|x_1|^p + |x_2|^q) + C_1, \tag{3.4}$$

where $C_1 = \max\{|F(t, x_1, x_2)| \,|\, t \in \mathbb{Z}[1, T], |x_1| \leqslant G_1, |x_2| \leqslant G_1\}$. Then by $(\mathcal{A}1)$, (3.1) and (3.4), for the sequence $\{u_{n_{k_j}} = (u_1^{(n_{k_j})}, u_2^{(n_{k_j})})\}$, we have

$$\begin{aligned}
&\frac{C_0}{\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q}\\
&\geqslant \frac{\mathcal{J}(u_1^{(n_{k_j})},u_2^{(n_{k_j})})}{\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q}\\
&=\frac{\sum_{t=1}^T\left[\Phi_1(\Delta u_1^{(n_{k_j})}(t))+\Phi_2(\Delta u_2^{(n_{k_j})}(t))-F(t,u_1^{(n_{k_j})}(t),u_2^{(n_{k_j})}(t))\right]}{\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q}\\
&\geqslant \frac{\min\{d_1,d_2\}\sum_{t=1}^T\left[|\Delta u_1^{(n_{k_j})}(t)|^p+|\Delta u_2^{(n_{k_j})}(t)|^q\right]-\sum_{t=1}^T\left[\varepsilon|u_1^{(n_{k_j})}(t)|^p+\varepsilon|u_2^{(n_{k_j})}(t)|^q+C_1\right]}{\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q}\\
&=\frac{\min\{d_1,d_2\}(\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q)-\min\{d_1,d_2\}\sum_{t=1}^T(|u_1^{(n_{k_j})}(t)|^p+|u_2^{(n_{k_j})}(t)|^q)}{\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q}\\
&\quad-\frac{\sum_{t=1}^T\left[\varepsilon|u_1^{(n_{k_j})}(t)|^p+\varepsilon|u_2^{(n_{k_j})}(t)|^q+C_1\right]}{\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q}\\
&\geqslant \min\{d_1,d_2\}-(\min\{d_1,d_2\}+\varepsilon)\left[\frac{\sum_{t=1}^T|u_1^{(n_{k_j})}(t)|^p}{\|u_1^{(n_{k_j})}\|_p^p}+\frac{\sum_{t=1}^T|u_2^{(n_{k_j})}(t)|^q}{\|u_2^{(n_{k_j})}\|_q^q}\right.\\
&\quad\left.+\frac{C_1T}{\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q}\right]\\
&=\min\{d_1,d_2\}-(\min\{d_1,d_2\}+\varepsilon)\left[\sum_{t=1}^T|z_1^{(n_{k_j})}(t)|^p+\sum_{t=1}^T|z_2^{(n_{k_j})}(t)|^q\right.\\
&\quad\left.+\frac{C_1T}{\|u_1^{(n_{k_j})}\|_p^p+\|u_2^{(n_{k_j})}\|_q^q}\right].
\end{aligned}\tag{3.5}$$

Let $j\to\infty$. Then it follows from (3.2), (3.3) and (3.5) that

$$\sum_{t=1}^T|z_1^*(t)|^p+\sum_{t=1}^T|z_2^*(t)|^q\geqslant\frac{\min\{d_1,d_2\}}{\min\{d_1,d_2\}+\varepsilon}>0,$$

which implies that there exists a nonempty set $\Omega_0\subset\mathbb{Z}[1,T]$ such that

$$|z_1^*(t)|+|z_2^*(t)|>0,\quad \forall t\in\Omega_0.$$

Without loss of generality, we assume that $|z_1^*(t)|>0$ for all $t\in\Omega_{01}$, where Ω_{01} is a nonempty set of Ω_0. Then the definition of $z_1^*(t)$, together with (3.3), implies that $|u_1^{(n_{k_j})}(t)|\to\infty$ as $j\to\infty$, for all $t\in\Omega_{01}$, which shows that

$$|u_1^{(n_{k_j})}(t)|+|u_2^{(n_{k_j})}(t)|\to\infty,\quad \text{as } j\to\infty,\ \forall t\in\Omega_{01}.\tag{3.6}$$

Moreover, (F_1) implies that there exists a positive constant G_2 such that

$$\min\{p,q\}F(t,x_1,x_2)-(\nabla_{x_1}F(t,x_1,x_2),x_1)-(\nabla_{x_2}F(t,x_1,x_2),x_2)\geqslant 0,$$

for all $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ with $|x_1|+|x_2|\geqslant G_2$ and all $t\in\mathbb{Z}[1,T]$. Since F is continuously differential, there is a positive constant C_2 such that

$$\left|\min\{p,q\}F(t,x_1,x_2)-(\nabla_{x_1}F(t,x_1,x_2),x_1)-(\nabla_{x_2}F(t,x_1,x_2),x_2)\right|\leqslant C_2,$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1| + |x_2| \leqslant G_2$ and all $t \in \mathbb{Z}[1, T]$. So

$$\min\{p, q\}F(t, x_1, x_2) - (\nabla_{x_1} F(t, x_1, x_2), x_1) - (\nabla_{x_2} F(t, x_1, x_2), x_2) \geqslant -C_2, \tag{3.7}$$

for all $(t, x_1, x_2) \in \mathbb{Z}[1, T] \times \mathbb{R}^N \times \mathbb{R}^N$. Thus it follows from $(\mathcal{A}2)$, (3.1) and (3.7) that

$$\begin{aligned}
&\big(\min\{p, q\} + 1\big)C_0 \\
&\geqslant (1 + \|u_{n_{k_j}}\|)\|\mathcal{J}'(u_{n_{k_j}})\| - \min\{p, q\}\mathcal{J}(u_{n_{k_j}}) \\
&\geqslant \langle \mathcal{J}'(u_1^{(n_{k_j})}, u_2^{(n_{k_j})}), (u_1^{(n_{k_j})}, u_2^{(n_{k_j})})\rangle - \min\{p, q\}\mathcal{J}(u_1^{(n_{k_j})}, u_2^{(n_{k_j})}) \\
&\geqslant \sum_{t=1}^{T} \big[\min\{p, q\}F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)) - (\nabla_{u_1} F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)), u_1^{(n_{k_j})}(t)) \\
&\quad - (\nabla_{u_2} F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)), u_2^{(n_{k_j})}(t))\big] \\
&= \sum_{t\in\Omega_{01}} \big[\min\{p, q\}F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)) - (\nabla_{u_1} F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)), u_1^{(n_{k_j})}(t)) \\
&\quad - (\nabla_{u_2} F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)), u_2^{(n_{k_j})}(t))\big] \\
&\quad + \sum_{t\in\mathbb{Z}[1,T]/\Omega_{01}} \big[\min\{p, q\}F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)) - (\nabla_{u_1} F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)), u_1^{(n_{k_j})}(t)) \\
&\quad - (\nabla_{u_2} F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)), u_2^{(n_{k_j})}(t))\big] \\
&\geqslant \sum_{t\in\Omega_{01}} \big[\min\{p, q\}F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)) - (\nabla_{u_1} F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)), u_1^{(n_{k_j})}(t)) \\
&\quad - (\nabla_{u_2} F(t, u_1^{(n_{k_j})}(t), u_2^{(n_{k_j})}(t)), u_2^{(n_{k_j})}(t))\big] - C_2T,
\end{aligned}$$

which, together with (3.6), contradicts (F_1). Hence, $\{u_1^{(n)}\}$ is bounded.

(ii) $\{u_2^{(n_k)}\}$ is bounded

For this case, we consider the subsequence $\{(u_1^{(n_k)}, u_2^{(n_k)})\}$. Then we have

$$\|u_1^{(n_k)}\|_p \to \infty, \quad \text{as } k \to \infty, \quad \text{and} \quad \|u_2^{(n_k)}\|_q \leqslant C_3,$$

for a constant $C_3 > 0$. Similar to the argument in (3.5), it is easy to obtain that

$$\begin{aligned}
\frac{C_0}{\|u_1^{(n_k)}\|_p^p} &\geqslant \frac{\mathcal{J}(u_1^{(n_k)}, u_2^{(n_k)})}{\|u_1^{(n_k)}\|_p^p + \|u_2^{(n_k)}\|_q^q} \\
&\geqslant \min\{d_1, d_2\} - (\min\{d_1, d_2\} + \varepsilon)\left[\frac{\sum_{t=1}^T |u_1^{(n_k)}(t)|^p}{\|u_1^{(n_k)}\|_p^p} + \frac{\sum_{t=1}^T |u_2^{(n_k)}(t)|^q}{\|u_1^{(n_k)}\|_p^p} + \frac{C_1T}{\|u_1^{(n_k)}\|_p^p}\right] \\
&\geqslant \min\{d_1, d_2\} - (\min\{d_1, d_2\} + \varepsilon)\left[\sum_{t=1}^T |z_1^{(n_k)}(t)|^p + \frac{C_3^q}{\|u_1^{(n_k)}\|_p^p} + \frac{C_1T}{\|u_1^{(n_k)}\|_p^p}\right].
\end{aligned} \tag{3.8}$$

Letting $k \to \infty$ in (3.8) implies that

$$\sum_{t=1}^T |z_1^*(t)|^p > 0.$$

The remainder of the argument is the same as case (i) with replacing Ω_{01} with Ω_0 and replacing n_{k_j} with n_k. Hence $\{u_1^{(n)}\}$ is also bounded for this case.

Similarly, it is easy to show that $\{u_2^{(n)}\}$ is also bounded, so that $\{u_n\}$ is bounded. Since all topologies in finite dimensional space are equivalent, $\{u_n\}$ has a convergent subsequence. The proof is complete. □

Lemma 3.2. *Assume that* $(\mathcal{A}0)$ *with* $a = +\infty$, (F_0), (F_1), (F_2) *and* (F_3) *hold. Then* $\mathcal{J}(u) \to -\infty$ *as* $\|u\| \to \infty$ *in* W.

Proof. It is obvious that $\Delta u_m = 0,\ m = 1,2$, for all $u = (u_1, u_2) \in W$ so that $\Phi_m(\Delta u_m) = 0,\ m = 1,2$. Then

$$\mathcal{J}(u) = \mathcal{J}(u_1, u_2) = -\sum_{t=1}^{T} F(t, u_1(t), u_2(t)), \quad \forall u \in W. \tag{3.9}$$

It follows from (F_1) that for any given $\mathcal{E} > 0$, there exists a constant $M_0(\mathcal{E}) > 0$ such that

$$\min\{p,q\}F(t, r_p x_1, r_q x_2) - (\nabla_{r_p x_1} F(t, r_p x_1, r_q x_2), r_p x_1) - (\nabla_{r_q x_2} F(t, r_p x_1, r_q x_2), r_q x_2) \geqslant \mathcal{E}, \tag{3.10}$$

for all $(r, x_1, x_2) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$ with $|r_p x_1| + |r_q x_2| > M_0(\mathcal{E})$, where $r_p := r^{\frac{\min\{p,q\}}{p}}$, $r_q := r^{\frac{\min\{p,q\}}{q}}$. Then by (F_2) and (3.10), we have

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{F(t, r_p x_1, r_q x_2)}{r^{\min\{p,q\}}}\right)\\
&= \frac{r^{\min\{p,q\}}(\nabla_{r_p x_1} F(t, r_p x_1, r_q x_2), \frac{\min\{p,q\}}{p} r^{\frac{\min\{p,q\}}{p}-1} x_1)}{r^{2\min\{p,q\}}}\\
&\quad + \frac{r^{\min\{p,q\}}(\nabla_{r_q x_2} F(t, r_p x_1, r_q x_2), \frac{\min\{p,q\}}{q} r^{\frac{\min\{p,q\}}{q}-1} x_2)}{r^{2\min\{p,q\}}} - \frac{\min\{p,q\} r^{\min\{p,q\}-1} F(t, r_p x_1, r_q x_2)}{r^{2\min\{p,q\}}}\\
&= \frac{r(\nabla_{r_p x_1} F(t, r_p x_1, r_q x_2), \frac{\min\{p,q\}}{p} r^{\frac{\min\{p,q\}}{p}-1} x_1) + r(\nabla_{r_q x_2} F(t, r_p x_1, r_q x_2), \frac{\min\{p,q\}}{q} r^{\frac{\min\{p,q\}}{q}-1} x_2)}{r^{\min\{p,q\}+1}}\\
&\quad - \frac{\min\{p,q\}F(t, r_p x_1, r_q x_2)}{r^{\min\{p,q\}+1}}\\
&= \frac{\frac{\min\{p,q\}}{p}(\nabla_{r_p x_1} F(t, r_p x_1, r_q x_2), r_p x_1) + \frac{\min\{p,q\}}{q}(\nabla_{r_q x_2} F(t, r_p x_1, r_q x_2), r_q x_2)}{r^{\min\{p,q\}+1}}\\
&\quad - \frac{\min\{p,q\}F(t, r_p x_1, r_q x_2)}{r^{\min\{p,q\}+1}} && (3.11)\\
&\leqslant \frac{(\nabla_{r_p x_1} F(t, r_p x_1, r_q x_2), r_p x_1) + (\nabla_{r_q x_2} F(t, r_p x_1, r_q x_2), r_q x_2) - \min\{p,q\}F(t, r_p x_1, r_q x_2)}{r^{\min\{p,q\}+1}} && (3.12)\\
&\leqslant -\frac{\mathcal{E}}{r^{\min\{p,q\}+1}} = \frac{d}{dr}\left(\frac{\mathcal{E}}{\min\{p,q\} r^{\min\{p,q\}}}\right),
\end{aligned}$$

for all $(r, x_1, x_2) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$ with $|r_p x_1| + |r_q x_2| > M_0(\mathcal{E}) + M_*$. For any given $r > 1$ and all given $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|r_p x_1| + |r_q x_2| > M_0(\mathcal{E}) + M_*$, we integrate the above inequality from 1 to r and then obtain that

$$\frac{F(t, r_p x_1, r_q x_2)}{r^{\min\{p,q\}}} - F(t, x_1, x_2) \leqslant \frac{\mathcal{E}}{\min\{p,q\} r^{\min\{p,q\}}} - \frac{\mathcal{E}}{\min\{p,q\}}. \tag{3.13}$$

It follows from (F_3) that for any given $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\begin{aligned}
\left|\frac{F(t, r_p x_1, r_q x_2)}{r^{\min\{p,q\}}}\right| &= \left|\frac{F(t, r_p x_1, r_q x_2)}{r^{\min\{p,q\}}(|x_1|^p + |x_2|^q)}(|x_1|^p + |x_2|^q)\right|\\
&= \left|\frac{F(t, r_p x_1, r_q x_2)}{|r_p x_1|^p + |r_q x_2|^q}\right|(|x_1|^p + |x_2|^q) && (3.14)\\
&\to 0, \quad \text{as } r \to \infty.
\end{aligned}$$

Equations (3.13) and (3.14) imply that

$$F(t, x_1, x_2) \geqslant \frac{\mathcal{E}}{\min\{p, q\}}, \tag{3.15}$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1| + |x_2| > M_0(\mathcal{E}) + M_*$, which together with the arbitrary of $\mathcal{E}$, shows that

$$\lim_{|x_1|+|x_2| \to +\infty} F(t, x_1, x_2) = +\infty. \tag{3.16}$$

As $\|u\| \to \infty$ in W, by (2.1) and (2.2), it is easy to see that there exists a nonempty subset of $\mathbb{Z}[1, T]$, denoted by Ω_1, such that $|u_1(t)| + |u_2(t)| \to \infty$ for all $t \in \Omega_1$, and $|u_1(t)| + |u_2(t)|$ is bounded for $t \in \mathbb{Z}[1, T]\backslash\Omega_1$. Hence by (3.16), we have

$$\lim_{|u_1(t)|+|u_2(t)| \to +\infty} F(t, u_1(t), u_2(t)) = +\infty,$$

for all $t \in \Omega_1$. Moreover, the continuity of F implies that $\sum_{t \in \mathbb{Z}[1,T]\backslash\Omega_1} F(t, u_1(t), u_2(t))$ is bounded. Hence

$$\sum_{t=1}^{T} F(t, u_1(t), u_2(t)) = \sum_{t \in \Omega_1} F(t, u_1(t), u_2(t)) + \sum_{t \in \mathbb{Z}[1,T]\backslash\Omega_1} F(t, u_1(t), u_2(t)) \to +\infty, \text{ as } \|u\| \to \infty \text{ in } W,$$

which, together with (3.9) implies that $\mathcal{J}(u) \to -\infty$ as $\|u\| \to \infty$ in W. The proof is complete. □

Lemma 3.3. *Assume that* $(\mathcal{A}0)$ *with* $a = +\infty$, (F_0), (F_1), (F_2) *and* (F_4) *hold. Then* $\mathcal{J}(u) \to -\infty$ *as* $\|u\| \to \infty$ *in* W.

Proof. Similar to the argument in Lemma 3.2, by (F_1) and (F_2), (3.13) holds. Then it follows from (F_4) and (3.13) that

$$-F(t, x_1, x_2) \leqslant \frac{F(t, r_p x_1, r_q x_2)}{r^{\min\{p,q\}}} - F(t, x_1, x_2) \leqslant \frac{\mathcal{E}}{\min\{p, q\} r^{\min\{p,q\}}} - \frac{\mathcal{E}}{\min\{p, q\}}, \tag{3.17}$$

for any given $r > 1$ and all given $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|r_p x_1| + |r_q x_2| > M_0(\mathcal{E}) + M_* + M^*$. Let $r \to \infty$ in (3.17). Then (3.15) holds. The remainder of the proof is the same as Lemma 3.2. □

Remark 3.4. (F_2) plays the role in deducing the inequality (3.12) from (3.11). However, if $p = q$, then $r_p = r_q = r$ and (3.12) is the same as (3.11). Hence (F_2) is not necessary in Lemma 3.2 and Lemma 3.3.

Lemma 3.5. *Assume that* $(\mathcal{A}0)$ *with* $a = +\infty$, $(\mathcal{A}1)$, (F_0) *and* (F_5) *(or* (F_3)*) hold. Then* $\mathcal{J}(u) \to +\infty$ *as* $\|u\| \to \infty$ *in* Y.

Proof. It follows from $(\mathcal{A}1)$, Lemma 2.1, (2.3) and (3.4) that

$$\begin{aligned}
\mathcal{J}(u) &= \mathcal{J}(u_1, u_2) \\
&= \sum_{t=1}^{T} \left[\Phi_1(\Delta u_1(t)) + \Phi_2(\Delta u_2(t)) - F(t, u_1(t), u_2(t))\right] \\
&\geqslant d_1 \sum_{t=1}^{T} |\Delta u_1(t)|^p + d_2 \sum_{t=1}^{T} |\Delta u_2(t)|^q - \varepsilon \sum_{t=1}^{T} (|u_1(t)|^p + |u_2(t)|^q) - C_1 T \\
&\geqslant d_1 \sum_{t=1}^{T} |\Delta u_1(t)|^p + d_2 \sum_{t=1}^{T} |\Delta u_2(t)|^q \\
&\quad - \varepsilon \cdot C(p, p') \sum_{t=1}^{T} |\Delta u_1(t)|^p - \varepsilon \cdot C(q, q') \sum_{t=1}^{T} |\Delta u_2(t)|^q - C_1 T
\end{aligned}$$

$$
\begin{aligned}
&= (d_1 - \varepsilon \cdot C(p,p'))\|\Delta u_1\|_{[p]}^p + (d_2 - \varepsilon \cdot C(q,q'))\|\Delta u_2\|_{[q]}^q - C_1 T \\
&\geqslant \min\left\{d_1 - \varepsilon \cdot C(p,p'), d_2 - \varepsilon \cdot C(q,q')\right\}\left[\|\Delta u_1\|_{[p]}^p + \|\Delta u_2\|_{[q]}^q\right] - C_1 T \\
&\geqslant \min\left\{d_1 - \varepsilon \cdot C(p,p'), d_2 - \varepsilon \cdot C(q,q')\right\}\left[\frac{1}{C(p,p')+1}\|u_1\|_p^p + \frac{1}{C(q,q')+1}\|u_2\|_q^q\right] - C_1 T,
\end{aligned}
$$

for $u \in Y$. Note that $0 < \varepsilon < \min\left\{\frac{d_1}{C(p,p')}, \frac{d_2}{C(q,q')}\right\}$. The above inequality implies that $\mathcal{J}(u) \to +\infty$ as $\|u\| \to \infty$ in Y. The proof is complete. □

Proof of Theorem 1.4. Let $X = E$, $X_1 = W$, $X_2 = Y$ and $\varphi = \mathcal{J}$. Then by Lemmas 3.1, 3.2, 3.5, 2.6 and Remark 2.7, $\mathcal{J}$ possesses a critical value c and then $\mathcal{J}$ possesses a critical point u^*. The proof is complete. □

Proof of Corollary 1.5. It follows from $(F_1)'$ and $(F_3)'$ that

$$
\begin{aligned}
&\min\{p,q\}F(t,x_1,x_2) - (\nabla_{x_1}F(t,x_1,x_2),x_1) - (\nabla_{x_2}F(t,x_1,x_2),x_2) \\
&\quad = (\min\{p,q\} - \beta)F(t,x_1,x_2) + \left[\beta F(t,x_1,x_2) - (\nabla_{x_1}F(t,x_1,x_2),x_1) - (\nabla_{x_2}F(t,x_1,x_2),x_2)\right] \\
&\quad \geqslant (\min\{p,q\} - \beta)F(t,x_1,x_2) \\
&\quad \to +\infty, \quad \text{as } |x_1| + |x_2| \to \infty,
\end{aligned}
$$

for all $t \in \mathbb{Z}[1,T]$. Thus (F_1) holds. Moreover, by $(F_1)'$, we claim that there exist two positive constants C_4 and C_5 such that

$$F(t,x_1,x_2) \leqslant C_4|x_1|^\beta + C_4|x_2|^\beta + C_5, \tag{3.18}$$

for all $t \in \mathbb{Z}[1,T]$. In fact, similar to the argument in [22], for any given $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and all $t \in \mathbb{Z}[1,T]$, we define $y : \mathbb{R}^+ \to \mathbb{R}$ by

$$y(s) = F(t,sx_1,sx_2),$$

and let

$$Q(s) = y'(s) - \frac{\beta}{s}y(s). \tag{3.19}$$

It is easy to solve the equation (3.19) and obtain

$$y(s) = s^\beta\left(\int_1^s r^{-\beta}Q(r)dr + F(t,x_1,x_2)\right). \tag{3.20}$$

Then by $(F_1)'$, we have

$$Q(s) = \frac{1}{s}\left[(\nabla_{sx_1}F(t,sx_1,sx_2),sx_1) + (\nabla_{sx_2}F(t,sx_1,sx_2),sx_2) - \beta F(t,sx_1,sx_2)\right] \leqslant 0, \tag{3.21}$$

for all $s \geqslant L/(|x_1| + |x_2|)$. Then on one hand, it follows from (3.20) and (3.21) that

$$
\begin{aligned}
y\left(\frac{L}{|x_1|+|x_2|}\right) &= F\left(t, \frac{Lx_1}{|x_1|+|x_2|}, \frac{Lx_2}{|x_1|+|x_2|}\right) \\
&= \left(\frac{L}{|x_1|+|x_2|}\right)^\beta\left(\int_1^{\frac{L}{|x_1|+|x_2|}} r^{-\beta}Q(r)dr + F(t,x_1,x_2)\right) \\
&\geqslant \left(\frac{L}{|x_1|+|x_2|}\right)^\beta F(t,x_1,x_2),
\end{aligned} \tag{3.22}
$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1| + |x_2| > L$. On the other hand, it follows from (F_0) that there exists a positive constant C_6 such that

$$y\left(\frac{L}{|x_1|+|x_2|}\right) = F\left(t, \frac{Lx_1}{|x_1|+|x_2|}, \frac{Lx_2}{|x_1|+|x_2|}\right)$$
$$\leqslant \max\{|F(t,x_1,x_2)||t \in \mathbb{Z}[1,T], |x_1| \leqslant L, |x_2| \leqslant L\} := C_6. \tag{3.23}$$

Then (3.22), (3.23) and (F_0) imply that (3.18) holds. By $(F_3)'$, there exists $M_1 > 0$ such that

$$F(t,x_1,x_2) > 0, \quad \forall (x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N \text{ with } |x_1|+|x_2| > M_1.$$

Then by (3.18), we have

$$0 < \frac{F(t,x_1,x_2)}{|x_1|^p+|x_2|^q} \leqslant \frac{C_4(|x_1|^\beta+|x_2|^\beta)+C_5}{|x_1|^p+|x_2|^q},$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1|+|x_2| > M_1$. Since $\beta < \min\{p,q\}$, we have

$$\lim_{|x_1|+|x_2|\to\infty} \frac{F(t,x_1,x_2)}{|x_1|^p+|x_2|^q} = 0.$$

So (F_3) holds. Thus the proof is complete. □

Proof of Theorem 1.7. Let $X = E$, $X_1 = W$, $X_2 = Y$ and $\varphi = \mathcal{J}$. Then by Lemmas 3.1, 3.3, 3.5, 2.6 and Remark 2.7, $\mathcal{J}$ possesses a critical value c and then $\mathcal{J}$ possesses a critical point u^*. The proof is complete. □

Proof of Theorem 1.9. For $\alpha_0 \in (0,p)$ and $\beta_0 \in (0,q)$, it follows from (F_6), Lemma 2.1 and Hölder inequality that

$$\begin{aligned}
\sum_{t=1}^{T} \left|F(u_1(t),\overline{u}_2) - F(\overline{u}_1,\overline{u}_2)\right| &= \sum_{t=1}^{T}\int_0^1 (\nabla_{x_1}F(\overline{u}_1+s\tilde{u}_1(t),\overline{u}_2),\tilde{u}_1(t))ds \\
&= \sum_{t=1}^{T}\int_0^1 \frac{1}{s}(\nabla_{x_1}F(\overline{u}_1+s\tilde{u}_1(t),\overline{u}_2)-\nabla_{y_1}F(\overline{u}_1,\overline{u}_2),s\tilde{u}_1(t))ds \\
&\leqslant \frac{r_1}{p}\sum_{t=1}^{T}|\tilde{u}_1(t)|^p + \frac{r_2}{\alpha_0}\sum_{t=1}^{T}|\tilde{u}_1(t)|^{\alpha_0} \\
&\leqslant \frac{r_1}{p}\sum_{t=1}^{T}|\tilde{u}_1(t)|^p + \frac{r_2}{\alpha_0}\left(\sum_{t=1}^{T}|\tilde{u}_1(t)|^p\right)^{\frac{\alpha_0}{p}} T^{\frac{p-\alpha_0}{p}} \\
&\leqslant \frac{r_1}{p}C(p,p')\sum_{t=1}^{T}|\Delta u_1(t)|^p + \frac{r_2}{\alpha_0}[C(p,p')]^{\frac{\alpha_0}{p}}\left(\sum_{t=1}^{T}|\Delta u_1(t)|^p\right)^{\frac{\alpha_0}{p}} T^{\frac{p-\alpha_0}{p}},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{t=1}^{T} \left|F(u_1(t),u_2(t)) - F(u_1(t),\overline{u}_2)\right| &= \sum_{t=1}^{T}\int_0^1 (\nabla_{x_2}F(u_1(t),\overline{u}_2+s\tilde{u}_2(t)),\tilde{u}_2(t))ds \\
&= \sum_{t=1}^{T}\int_0^1 \frac{1}{s}(\nabla_{x_2}F(u_1(t),\overline{u}_2+s\tilde{u}_2(t))-\nabla_{y_2}F(\overline{u}_1,\overline{u}_2),s\tilde{u}_2(t))ds \\
&\leqslant \frac{r_3}{q}\sum_{t=1}^{T}|\tilde{u}_2(t)|^q + \frac{r_4}{\beta_0}\sum_{t=1}^{T}|\tilde{u}_2(t)|^{\beta_0} \\
&\leqslant \frac{r_3}{q}\sum_{t=1}^{T}|\tilde{u}_2(t)|^q + \frac{r_4}{\beta_0}\left(\sum_{t=1}^{T}|\tilde{u}_2(t)|^q\right)^{\frac{\beta_0}{q}} T^{\frac{q-\beta_0}{q}}
\end{aligned}$$

$$\leqslant \frac{r_3}{q} C(q,q') \sum_{t=1}^{T} |\Delta u_2(t)|^q + \frac{r_4}{\beta_0} [C(q,q')]^{\frac{\beta_0}{q}} \left(\sum_{t=1}^{T} |\Delta u_2(t)|^q \right)^{\frac{\beta_0}{q}} T^{\frac{q-\beta_0}{q}},$$

for all $u = (u_1, u_2)^\tau \in E$. Hence, we have

$$\begin{aligned} \mathcal{J}(u_1,u_2) &= \sum_{t=1}^{T} [\Phi_1(\Delta u_1(t)) + \Phi_2(\Delta u_2(t)) - F(u_1(t),u_2(t))] \\ &\geqslant d_1 \sum_{t=1}^{T} |\Delta u_1(t)|^p + d_2 \sum_{t=1}^{T} |\Delta u_2(t)|^q - \sum_{t=1}^{T} [F(u_1(t),u_2(t)) - F(u_1(t),\overline{u}_2)] \\ &\quad - \sum_{t=1}^{T} [F(u_1(t),\overline{u}_2) - F(\overline{u}_1,\overline{u}_2)] - \sum_{t=1}^{T} F(\overline{u}_1,\overline{u}_2) \\ &\geqslant \left(d_1 - \frac{r_1}{p} \cdot C(p,p') \right) \sum_{t=1}^{T} |\Delta u_1(t)|^p + \left(d_2 - \frac{r_3}{q} \cdot C(q,q') \right) \sum_{t=1}^{T} |\Delta u_2(t)|^q \\ &\quad - \frac{r_2}{\alpha_0} [C(p,p')]^{\frac{\alpha_0}{p}} \left(\sum_{t=1}^{T} |\Delta u_1(t)|^p \right)^{\frac{\alpha_0}{p}} T^{\frac{p-\alpha_0}{p}} - TF(\overline{u}_1,\overline{u}_2) \\ &\quad - \frac{r_4}{\beta_0} [C(q,q')]^{\frac{\beta_0}{q}} \left(\sum_{t=1}^{T} |\Delta u_2(t)|^q \right)^{\frac{\beta_0}{q}} T^{\frac{q-\beta_0}{q}}. \end{aligned} \tag{3.24}$$

Note that $r_1 \in \left[0, \frac{d_1 p}{C(p,p')}\right)$, $r_3 \in \left[0, \frac{d_2 q}{C(q,q')}\right)$, $\alpha_0 \in (0,p)$ and $\beta_0 \in (0,q)$. Equation (2.4) implies that if $\|u\| \to \infty$, then $|\overline{u}_m| + \sum_{t=1}^{T} |\Delta u_m(t)|^s \to \infty$, $m = 1$ (or 2) and $s = p$ (or q). So (3.24) and (F_7) imply that

$$\mathcal{J}(u_1,u_2) \to +\infty, \quad \text{as } \|(u_1,u_2)\| \to \infty. \tag{3.25}$$

If $\alpha_0 = 0$ or $\beta_0 = 0$, from the above argument, it is easy to see that (3.25) also holds. (3.25) implies that $\mathcal{J}$ is bounded from below and (PS) condition holds. Let $X = E$ and $\varphi = \mathcal{J}$. Then by Lemma 2.8, it is easy to know that $\mathcal{J}$ has at least one critical point u^* such that

$$\mathcal{J}(u^*) = c = \inf_{u \in E} \mathcal{J}(u).$$

Thus the proof is complete. □

4. Proofs for bounded homeomorphism

Lemma 4.1. *Assume that $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ are coercive, $m = 1,2$, and $(\mathcal{A}0)$ with $a < +\infty$, $(\mathcal{A}3)$, (F_0), (S_1) and (S_3) (or (S_2)) hold. Then $\mathcal{J}$ satisfies the (C) condition.*

Proof. Similar to the proof of Lemma 3.1, we need to consider two cases:

(i) $\{u_2^{(n_k)}\}$ is unbounded, and

(ii) $\{u_2^{(n_k)}\}$ is bounded.

For case (i), by the same argument as Lemma 3.1, we obtain that (3.2) and (3.3) hold for some subsequence $\{(u_1^{(n_{k_j})}, u_2^{(n_{k_j})})\}$ of $\{(u_1^{(n_k)}, u_2^{(n_k)})\}$. It follows from (S_3) or (S_2) that there exist constants G_3 and $0 < \varepsilon < \min\left\{ \frac{\delta_1}{T^{\frac{s_0-1}{s_0}} [C(s_0,s_0')]^{\frac{1}{s_0}}}, \frac{\delta_2}{T^{\frac{s_1-1}{s_1}} [C(s_1,s_1')]^{\frac{1}{s_1}}} \right\}$ such that

$$F(t,x_1,x_2) \leqslant \varepsilon(|x_1| + |x_2|),$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1| + |x_2| \geqslant G_3$. Then by (F_0), there exists a positive constant C_7 such that

$$F(t, x_1, x_2) \leqslant \varepsilon(|x_1| + |x_2|) + C_7, \tag{4.1}$$

where $C_7 = \max\{|F(t, x_1, x_2)||t \in \mathbb{Z}[1, T], |x_1| \leqslant G_3, |x_2| \leqslant G_3\}$. Then by (1.6), (3.1) and (4.1), for the sequence $\{u_{n_{k_j}} = (u_1^{(n_{k_j})}, u_2^{(n_{k_j})})\}$, we have

$$\begin{aligned}
&\frac{C_0}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q} \\
&\geqslant \frac{\mathcal{J}(u_1^{(n_{kj})}, u_2^{(n_{kj})})}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q} \\
&= \frac{\sum_{t=1}^{T}[\Phi_1(\Delta u_1^{(n_{kj})}(t)) + \Phi_2(\Delta u_2^{(n_{kj})}(t))] - F(t, u_1^{(n_{kj})}(t), u_2^{(n_{kj})}(t))}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q} \\
&\geqslant \frac{\min\{\delta_1, \delta_2\}\left(\sum_{t=1}^{T}|\Delta u_1^{(n_{kj})}(t)| + \sum_{t=1}^{T}|\Delta u_2^{(n_{kj})}(t)|\right) - \sum_{t=1}^{T}[\varepsilon|u_1^{(n_{kj})}(t)| + \varepsilon|u_2^{(n_{kj})}(t)| + C_7]}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q} \\
&\quad - \frac{(\delta_1 + \delta_2)T}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q} \\
&\geqslant \frac{\min\{\delta_1, \delta_2\}(\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q) - \min\{\delta_1, \delta_2\}\left(\sum_{t=1}^{T}|u_1^{(n_{kj})}(t)| + \sum_{t=1}^{T}|u_2^{(n_{kj})}(t)|\right)}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q} \\
&\quad - \frac{\sum_{t=1}^{T}[\varepsilon|u_1^{(n_{kj})}(t)| + \varepsilon|u_2^{(n_{kj})}(t)| + C_7] + (\delta_1 + \delta_2)T}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q} \\
&\geqslant \min\{\delta_1, \delta_2\} - (\min\{\delta_1, \delta_2\} + \varepsilon)\left[\frac{\sum_{t=1}^{T}|u_1^{(n_{kj})}(t)|}{\|u_1^{(n_{kj})}\|_p} + \frac{\sum_{t=1}^{T}|u_2^{(n_{kj})}(t)|}{\|u_2^{(n_{kj})}\|_q} + \frac{(C_7 + \delta_1 + \delta_2)T}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q}\right] \\
&= \min\{\delta_1, \delta_2\} - (\min\{\delta_1, \delta_2\} + \varepsilon)\left[\sum_{t=1}^{T}|z_1^{(n_{kj})}(t)| + \sum_{t=1}^{T}|z_2^{(n_{kj})}(t)| + \frac{(C_7 + \delta_1 + \delta_2)T}{\|u_1^{(n_{kj})}\|_p + \|u_2^{(n_{kj})}\|_q}\right].
\end{aligned} \tag{4.2}$$

Let $j \to \infty$. Then it follows from (3.2), (3.3) and (4.2) that

$$\sum_{t=1}^{T}|z_1^*(t)| + \sum_{t=1}^{T}|z_2^*(t)| \geqslant \frac{\min\{\delta_1, \delta_2\}}{\min\{\delta_1, \delta_2\} + \varepsilon} > 0,$$

which implies that there exists a nonempty set $\Omega_2 \subset \mathbb{Z}[1, T]$ such that

$$|z_1^*(t)| + |z_2^*(t)| > 0, \quad \forall t \in \Omega_2.$$

Without loss of generality, we assume that $|z_1^*(t)| > 0$ for all $t \in \Omega_{21}$, where Ω_{21} is a nonempty set of Ω_0. The reminder of the argument is the same as Lemma 3.1 with replacing $\min\{p, q\}$ with $\min\{p^*, q^*\}$, replacing $\min\{d_1, d_2\}$ with $\min\{\delta_1, \delta_2\}$ and replacing Ω_{01} with Ω_{21}.

For case (ii), similar to Lemma 3.1, we consider the subsequence $\{(u_1^{(n_k)}, u_2^{(n_k)})\}$. Then we have

$$\|u_1^{(n_k)}\|_p \to \infty, \quad \text{as } k \to \infty, \text{ and } \quad \|u_2^{(n_k)}\|_q \leqslant C_8,$$

for some constant $C_8 > 0$. Similar to the argument in (4.2), together with Hölder inequality, it is easy to obtain that

$$\begin{aligned}\frac{C_0}{\|u_1^{(n_k)}\|_p} &\geqslant \frac{\mathcal{J}(u_1^{(n_k)},u_2^{(n_k)})}{\|u_1^{(n_k)}\|_p+\|u_2^{(n_k)}\|_q}\\ &\geqslant \min\{\delta_1,\delta_2\}-(\min\{\delta_1,\delta_2\}+\varepsilon)\left[\frac{\sum_{t=1}^{T}|u_1^{(n_k)}(t)|}{\|u_1^{(n_k)}\|_p}+\frac{\sum_{t=1}^{T}|u_2^{(n_k)}(t)|}{\|u_1^{(n_k)}\|_p}+\frac{(C_7+\delta_1+\delta_2)T}{\|u_1^{(n_k)}\|_p}\right] \quad (4.3)\\ &\geqslant \min\{\delta_1,\delta_2\}-(\min\{\delta_1,\delta_2\}+\varepsilon)\left[\sum_{t=1}^{T}|z_1^{(n_k)}(t)|+\frac{T^{\frac{q-1}{q}}C_8}{\|u_1^{(n_k)}\|_p}+\frac{(C_7+\delta_1+\delta_2)T}{\|u_1^{(n_k)}\|_p}\right].\end{aligned}$$

Let $k\to\infty$. (4.3) implies that

$$\sum_{t=1}^{T}|z_1^{*}(t)|>0.$$

The remainder of the argument is the same as case (i) with replacing Ω_{21} with Ω_2 and replacing n_{k_j} with n_k. Finally, by the same argument as Lemma 3.1, it is easy to complete the proof. □

Lemma 4.2. *Assume that* $(\mathcal{A}0)$ *with* $a<+\infty$, (F_0), (F_2), (S_1) *and* (S_2) *hold. Then* $\mathcal{J}(u)\to-\infty$ *as* $\|u\|\to\infty$ *in* W.

Proof. Note that $p^*,q^*\in(0,1]$. Similar to the argument in Lemma 3.2 with replacing $\min\{p,q\}$ with $\min\{p^*,q^*\}$ and r_p and r_q with r_{p^*} and r_{q^*}, respectively, where $r_{p^*}=r_{q^*}=r^{\min\{p^*,q^*\}}$, it follows from (F_2) and (S_1) that for any given $\mathcal{E}>0$, there exists a positive constant $M_1(\mathcal{E})$ such that

$$\begin{aligned}&\frac{d}{dr}\left(\frac{F(t,r_{p^*}x_1,r_{q^*}x_2)}{r^{\min\{p^*,q^*\}}}\right)\\ &\quad=\frac{\min\{p^*,q^*\}(\nabla_{r_{p^*}x_1}F(t,r_{p^*}x_1,r_{q^*}x_2),r_{p^*}x_1)+\min\{p^*,q^*\}(\nabla_{r_{q^*}x_2}F(t,r_{p^*}x_1,r_{q^*}x_2),r_{q^*}x_2)}{r^{\min\{p^*,q^*\}+1}}\\ &\quad\quad-\frac{\min\{p^*,q^*\}F(t,r_{p^*}x_1,r_{q^*}x_2)}{r^{\min\{p^*,q^*\}+1}} \quad (4.4)\\ &\quad\leqslant\frac{(\nabla_{r_{p^*}x_1}F(t,r_{p^*}x_1,r_{q^*}x_2),r_{p^*}x_1)+(\nabla_{r_{q^*}x_2}F(t,r_{p^*}x_1,r_{q^*}x_2),r_{q^*}x_2)}{r^{\min\{p^*,q^*\}+1}}\\ &\quad\quad-\frac{\min\{p^*,q^*\}F(t,r_{p^*}x_1,r_{q^*}x_2)}{r^{\min\{p^*,q^*\}+1}} \quad (4.5)\\ &\quad\leqslant-\frac{\mathcal{E}}{r^{\min\{p^*,q^*\}+1}}=\frac{d}{dr}\left(\frac{\mathcal{E}}{\min\{p^*,q^*\}r^{\min\{p^*,q^*\}}}\right),\end{aligned}$$

for all $(r,x_1,x_2)\in\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N$ with $|r_{p^*}x_1|+|r_{q^*}x_2|>M_1(\mathcal{E})+M_*$. For any given $r>1$ and all given $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ with $|r_{p^*}x_1|+|r_{q^*}x_2|>M_1(\mathcal{E})+M_*$, we integrate the above inequality from 1 to r and then obtain that

$$\frac{F(t,r_{p^*}x_1,r_{q^*}x_2)}{r^{\min\{p^*,q^*\}}}-F(t,x_1,x_2)\leqslant\frac{\mathcal{E}}{\min\{p^*,q^*\}r^{\min\{p^*,q^*\}}}-\frac{\mathcal{E}}{\min\{p^*,q^*\}}. \quad (4.6)$$

By (S_2), for any given $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$,

$$\begin{aligned}\left|\frac{F(t,r_{p^*}x_1,r_{q^*}x_2)}{r^{\min\{p^*,q^*\}}}\right|&=\left|\frac{F(t,r_{p^*}x_1,r_{q^*}x_2)}{r^{\min\{p^*,q^*\}}(|x_1|+|x_2|)}(|x_1|+|x_2|)\right|\\ &=\left|\frac{F(t,r_{p^*}x_1,r_{q^*}x_2)}{|r_{p^*}x_1|+|r_{q^*}x_2|}\right|(|x_1|+|x_2|) \quad (4.7)\\ &\to 0,\quad \text{as } r\to\infty.\end{aligned}$$

Then (4.6) and (4.7) imply that

$$F(t,x_1,x_2) \geqslant \frac{\mathcal{E}}{\min\{p^*,q^*\}}, \tag{4.8}$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x_1|+|x_2| > M_1(\mathcal{E})+M_*$. Starting from (4.8) instead of (3.15), we can complete the proof in the same way as in Lemma 3.2. □

Lemma 4.3. *Assume that* $(\mathcal{A}0)$ *with* $a < +\infty$, (F_0), (F_2), (S_1) *and* (F_4) *hold. Then* $\mathcal{J}(u) \to -\infty$ *as* $\|u\| \to \infty$ *in* W.

Proof. Similar to the argument in Lemma 4.2, by (S_1) and (F_2), (4.6) holds. Then it follows from (F_4) and (4.6) that

$$-F(t,x_1,x_2) \leqslant \frac{F(t,r_{p^*}x_1,r_{q^*}x_2)}{r^{\min\{p^*,q^*\}}} - F(t,x_1,x_2) \leqslant \frac{\mathcal{E}}{\min\{p^*,q^*\}r^{\min\{p^*,q^*\}}} - \frac{\mathcal{E}}{\min\{p^*,q^*\}}, \tag{4.9}$$

for any given $r > 1$ and all given $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|r_p^* x_1|+|r_q^* x_2| > M_1(\mathcal{E})+M_*+M^*$. Let $r \to \infty$ in (4.9). Then (4.8) holds. The remainder of the proof is the same as Lemma 4.2. □

Remark 4.4. (F_2) plays the role in deducing the inequality (4.5) from (4.4). However, if $p^* = q^* = 1$, then $r_{p^*} = r_{q^*} = r$ and (4.5) is the same as (4.4). Hence (F_2) is not necessary in Lemma 4.2 and Lemma 4.3.

Lemma 4.5. *Assume that* $(\mathcal{A}0)$, (F_0) *and* (S_2) *(or* (S_3)*) hold and* $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ *are coercive,* $m = 1,2$*. Then* $\mathcal{J}(u) \to +\infty$ *as* $\|u\| \to \infty$ *in* Y.

Proof. Note that $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ are coercive, $m = 1,2$, and $s_0 > 1, s_1 > 1$. Then it follows from (1.6), (4.1), Hölder inequality and Lemma 2.1 that

$$\begin{aligned}
\mathcal{J}(u) &= \mathcal{J}(u_1,u_2) \\
&= \sum_{t=1}^{T} \left[\Phi_1(\Delta u_1(t)) + \Phi_2(\Delta u_2(t)) - F(t,u_1(t),u_2(t))\right] \\
&\geqslant \delta_1 \sum_{t=1}^{T} |\Delta u_1(t)| + \delta_2 \sum_{t=1}^{T} |\Delta u_2(t)| - \varepsilon \sum_{t=1}^{T} (|u_1(t)|+|u_2(t)|) - (C_7+\delta_1+\delta_2)T \\
&\geqslant \delta_1 \sum_{t=1}^{T} |\Delta u_1(t)| + \delta_2 \sum_{t=1}^{T} |\Delta u_2(t)| \\
&\quad - \varepsilon T^{\frac{s_0-1}{s_0}} \left(\sum_{t=1}^{T} |u_1(t)|^{s_0}\right)^{\frac{1}{s_0}} - \varepsilon T^{\frac{s_1-1}{s_1}} \left(\sum_{t=1}^{T} |u_2(t)|^{s_1}\right)^{\frac{1}{s_1}} - (C_7+\delta_1+\delta_2)T \\
&\geqslant \delta_1 \sum_{t=1}^{T} |\Delta u_1(t)| + \delta_2 \sum_{t=1}^{T} |\Delta u_2(t)| - (C_7+\delta_1+\delta_2)T \\
&\quad - \varepsilon T^{\frac{s_0-1}{s_0}} [C(s_0,s_0')]^{\frac{1}{s_0}} \left(\sum_{t=1}^{T} |\Delta u_1(t)|^{s_0}\right)^{\frac{1}{s_0}} - \varepsilon T^{\frac{s_1-1}{s_1}} [C(s_1,s_1')]^{\frac{1}{s_1}} \left(\sum_{t=1}^{T} |\Delta u_2(t)|^{s_1}\right)^{\frac{1}{s_1}} \\
&\geqslant \delta_1 \sum_{t=1}^{T} |\Delta u_1(t)| + \delta_2 \sum_{t=1}^{T} |\Delta u_2(t)| - (C_7+\delta_1+\delta_2)T \\
&\quad - \varepsilon T^{\frac{s_0-1}{s_0}} [C(s_0,s_0')]^{\frac{1}{s_0}} \sum_{t=1}^{T} |\Delta u_1(t)| - \varepsilon T^{\frac{s_1-1}{s_1}} [C(s_1,s_1')]^{\frac{1}{s_1}} \sum_{t=1}^{T} |\Delta u_2(t)|
\end{aligned}$$

$$= \left(\delta_1 - \varepsilon T^{\frac{s_0-1}{s_0}}[C(s_0,s_0')]^{\frac{1}{s_0}}\right)\sum_{t=1}^{T}|\Delta u_1(t)| + \left(\delta_2 - \varepsilon T^{\frac{s_1-1}{s_1}}[C(s_1,s_1')]^{\frac{1}{s_1}}\right)\sum_{t=1}^{T}|\Delta u_2(t)|$$
$$-(C_7+\delta_1+\delta_2)T$$
$$\geqslant \left(\delta_1 - \varepsilon T^{\frac{s_0-1}{s_0}}[C(s_0,s_0')]^{\frac{1}{s_0}}\right)\left(\sum_{t=1}^{T}|\Delta u_1(t)|^p\right)^{\frac{1}{p}}$$
$$+\left(\delta_2 - \varepsilon T^{\frac{s_1-1}{s_1}}[C(s_1,s_1')]^{\frac{1}{s_1}}\right)\left(\sum_{t=1}^{T}|\Delta u_2(t)|^q\right)^{\frac{1}{q}} - (C_7+\delta_1+\delta_2)T,$$

for $u \in Y$. Note that $0 < \varepsilon < \min\left\{\frac{\delta_1}{T^{\frac{s_0-1}{s_0}}[C(s_0,s_0')]^{\frac{1}{s_0}}}, \frac{\delta_2}{T^{\frac{s_1-1}{s_1}}[C(s_1,s_1')]^{\frac{1}{s_1}}}\right\}$. The above inequality implies that $\mathcal{J}(u) \to +\infty$ as $\|u\| \to \infty$ in Y. The proof is complete. □

Proof of Theorem 1.12. Let $X = E$, $X_1 = W$, $X_2 = Y$ and $\varphi = \mathcal{J}$. Then by Lemmas 4.1, 4.2, 4.5, 2.6 and Remark 2.7, $\mathcal{J}$ possesses a critical value c and hence $\mathcal{J}$ possesses a critical point u^*. The proof is complete. □

Proof of Corollary 1.13. Note that $\theta_1 \in (0,1)$, $\theta_2 \in (0,1)$ and

$$\lim_{|x_1|+|x_2|\to\infty}\frac{F(t,x_1,x_2)}{|x_1|+|x_2|} = \lim_{|x_1|+|x_2|\to\infty}\frac{F(t,x_1,x_2)}{|x_1|^{\theta_1}+|x_2|^{\theta_2}}\cdot\frac{|x_1|^{\theta_1}+|x_2|^{\theta_2}}{|x_1|+|x_2|}.$$

Hence $(S_2)'$ implies that (S_2) holds. The proof is complete. □

Proof of Theorem 1.15. Let $X = E$, $X_1 = W$, $X_2 = Y$ and $\varphi = \mathcal{J}$. Then by Lemmas 4.1, 4.3, 4.5, 2.6 and Remark 2.7, $\mathcal{J}$ possesses a critical value c and hence $\mathcal{J}$ possesses a critical point u^*. The proof is complete. □

For $u = (u_1,u_2)^\tau \in E$, define
$$\|u\|_{[E]} = \|u_1\|_{s_2} + \|u_2\|_{s_3}.$$
Then $\|u\|_{[E]}$ is equivalent to $\|u\|$.

Proof of Theorem 1.17. For $\mu_0 \in (0,1)$ and $\nu_0 \in (0,1)$, it follows from (S_4), Hölder inequality and Lemma 2.1 that

$$\begin{aligned}
\sum_{t=1}^{T}\left|F(u_1(t),\overline{u}_2) - F(\overline{u}_1,\overline{u}_2)\right| &= \sum_{t=1}^{T}\int_0^1(\nabla_{x_1}F(\overline{u}_1+s\widetilde{u}_1(t),\overline{u}_2),\widetilde{u}_1(t))ds\\
&= \sum_{t=1}^{T}\int_0^1\frac{1}{s}(\nabla_{x_1}F(\overline{u}_1+s\widetilde{u}_1(t),\overline{u}_2)-\nabla_{x_1}F(\overline{u}_1,\overline{u}_2),s\widetilde{u}_1(t))ds\\
&\leqslant r_1\sum_{t=1}^{T}|\widetilde{u}_1(t)| + \frac{r_2}{\mu_0}\sum_{t=1}^{T}|\widetilde{u}_1(t)|^{\mu_0}\\
&\leqslant r_1 T^{\frac{s_2-1}{s_2}}\left(\sum_{t=1}^{T}|\widetilde{u}_1(t)|^{s_2}\right)^{1/s_2} + \frac{r_2}{\mu_0}T^{\frac{s_2-\mu_0}{s_2}}\left(\sum_{t=1}^{T}|\widetilde{u}_1(t)|^{s_2}\right)^{\frac{\mu_0}{s_2}}\\
&\leqslant r_1 T^{\frac{s_2-1}{s_2}}[C(s_2,s_2')]^{\frac{1}{s_2}}\left(\sum_{t=1}^{T}|\Delta u_1(t)|^{s_2}\right)^{1/s_2}\\
&\quad + \frac{r_2}{\mu_0}T^{\frac{s_2-\mu_0}{s_2}}[C(s_2,s_2')]^{\frac{\mu_0}{s_2}}\left(\sum_{t=1}^{T}|\Delta u_1(t)|^{s_2}\right)^{\frac{\mu_0}{s_2}}
\end{aligned}$$

$$\leqslant r_1 T^{\frac{s_2-1}{s_2}}[C(s_2,s_2')]^{\frac{1}{s_2}}\sum_{t=1}^{T}|\Delta u_1(t)|+\frac{r_2}{\mu_0}T^{\frac{s_2-\mu_0}{s_2}}[C(s_2,s_2')]^{\frac{\mu_0}{s_2}}\sum_{t=1}^{T}|\Delta u_1(t)|^{\mu_0},$$

and

$$\begin{aligned}\sum_{t=1}^{T}\left|F(u_1(t),u_2(t))-F(u_1(t),\overline{u}_2)\right| &= \sum_{t=1}^{T}\int_0^1(\nabla_{x_2}F(u_1(t),\overline{u}_2+s\widetilde{u}_2(t)),\widetilde{u}_2(t))ds\\ &= \sum_{t=1}^{T}\int_0^1\frac{1}{s}(\nabla_{x_2}F(u_1(t),\overline{u}_2+s\widetilde{u}_2(t))-\nabla_{x_2}F(\overline{u}_1,\overline{u}_2),s\widetilde{u}_2(t))ds\\ &\leqslant r_3\sum_{t=1}^{T}|\widetilde{u}_2(t)|+\frac{r_4}{\nu_0}\sum_{t=1}^{T}|\widetilde{u}_2(t)|^{\nu_0}\\ &\leqslant r_3T^{\frac{s_3-1}{s_3}}\left(\sum_{t=1}^{T}|\widetilde{u}_2(t)|^{s_3}\right)^{1/s_3}+\frac{r_4}{\nu_0}T^{\frac{s_3-\nu_0}{s_3}}\left(\sum_{t=1}^{T}|\widetilde{u}_2(t)|^{s_3}\right)^{\frac{\nu_0}{s_3}}\\ &\leqslant r_3T^{\frac{s_3-1}{s_3}}[C(s_3,s_3')]^{\frac{1}{s_3}}\left(\sum_{t=1}^{T}|\Delta u_2(t)|^{s_3}\right)^{1/s_3}\\ &\quad+\frac{r_4}{\nu_0}T^{\frac{s_3-\nu_0}{s_3}}[C(s_3,s_3')]^{\frac{\nu_0}{s_3}}\left(\sum_{t=1}^{T}|\Delta u_2(t)|^{s_3}\right)^{\frac{\nu_0}{s_3}}\\ &\leqslant r_3T^{\frac{s_3-1}{s_3}}[C(s_3,s_3')]^{\frac{1}{s_3}}\sum_{t=1}^{T}|\Delta u_2(t)|+\frac{r_4}{\nu_0}T^{\frac{s_3-\nu_0}{s_3}}[C(s_3,s_3')]^{\frac{\nu_0}{s_3}}\sum_{t=1}^{T}|\Delta u_2(t)|^{\nu_0},\end{aligned}$$

for all $u=(u_1,u_2)^\tau\in E$. Note that Φ_m are coercive, $m=1,2$. Then by (1.6) we have

$$\begin{aligned}\mathcal{J}(u_1,u_2) &= \sum_{t=1}^{T}[\Phi_1(\Delta u_1(t))+\Phi_2(\Delta u_2(t))-F(u_1(t),u_2(t))]\\ &\geqslant \delta_1\sum_{t=1}^{T}|\Delta u_1(t)|+\delta_2\sum_{t=1}^{T}|\Delta u_2(t)|-\sum_{t=1}^{T}[F(u_1(t),u_2(t))-F(u_1(t),\overline{u}_2)]\\ &\quad-\sum_{t=1}^{T}[F(u_1(t),\overline{u}_2)-F(\overline{u}_1,\overline{u}_2)]-\sum_{t=1}^{T}F(\overline{u}_1,\overline{u}_2)-(\delta_1+\delta_2)T\\ &\geqslant \delta_1\sum_{t=1}^{T}|\Delta u_1(t)|+\delta_2\sum_{t=1}^{T}|\Delta u_2(t)|-r_1T^{\frac{s_2-1}{s_2}}[C(s_2,s_2')]^{\frac{1}{s_2}}\sum_{t=1}^{T}|\Delta u_1(t)|\\ &\quad-\frac{r_2}{\mu_0}T^{\frac{s_2-\mu_0}{s_2}}[C(s_2,s_2')]^{\frac{\mu_0}{s_2}}\sum_{t=1}^{T}|\Delta u_1(t)|^{\mu_0}-r_3T^{\frac{s_3-1}{s_3}}[C(s_3,s_3')]^{\frac{1}{s_3}}\sum_{t=1}^{T}|\Delta u_2(t)|\\ &\quad-\frac{r_4}{\nu_0}T^{\frac{s_3-\nu_0}{s_3}}[C(s_3,s_3')]^{\frac{\nu_0}{s_3}}\sum_{t=1}^{T}|\Delta u_2(t)|^{\nu_0}-(\delta_1+\delta_2)T-TF(\overline{u}_1,\overline{u}_2)\\ &= \left[\delta_1-r_1T^{\frac{s_2-1}{s_2}}[C(s_2,s_2')]^{\frac{1}{s_2}}\right]\sum_{t=1}^{T}|\Delta u_1(t)|+\left[\delta_2-r_3T^{\frac{s_3-1}{s_3}}[C(s_3,s_3')]^{\frac{1}{s_3}}\right]\sum_{t=1}^{T}|\Delta u_2(t)|\\ &\quad-\frac{r_2}{\mu_0}T^{\frac{s_2-\mu_0}{s_2}}[C(s_2,s_2')]^{\frac{\mu_0}{s_2}}\sum_{t=1}^{T}|\Delta u_1(t)|^{\mu_0}-\frac{r_4}{\nu_0}T^{\frac{s_3-\nu_0}{s_3}}[C(s_3,s_3')]^{\frac{\nu_0}{s_3}}\sum_{t=1}^{T}|\Delta u_2(t)|^{\nu_0}\\ &\quad-(\delta_1+\delta_2)T-TF(\overline{u}_1,\overline{u}_2).\end{aligned}\tag{4.10}$$

Note that $r_1 \in \left[0, \frac{\delta_1}{T^{\frac{s_2-1}{s_2}}[C(s_2,s_2')]^{\frac{1}{s_2}}}\right)$, $r_3 \in \left[0, \frac{\delta_2}{T^{\frac{s_3-1}{s_3}}[C(s_3,s_3')]^{\frac{1}{s_3}}}\right)$, and if $\|u\| \to \infty$, then $\|u\|_{[E]} \to \infty$ so that $|\bar{u}_m| + \sum_{t=1}^{T} |\Delta u_m(t)| \to \infty$ $(m = 1, 2)$ by (2.5). If $\mu_0 \in (0, 1)$ and $\nu_0 \in (0, 1)$, then by Hölder inequality we have

$$\sum_{t=1}^{T} |\Delta u_1(t)|^{\mu_0} \leqslant T^{1-\mu_0} \left(\sum_{t=1}^{T} |\Delta u_1(t)| \right)^{\mu_0}, \quad \text{and} \quad \sum_{t=1}^{T} |\Delta u_2(t)|^{\nu_0} \leqslant T^{1-\nu_0} \left(\sum_{t=1}^{T} |\Delta u_2(t)| \right)^{\nu_0},$$

which, together with (4.10), implies that

$$\mathcal{J}(u_1, u_2) \to +\infty, \quad \text{as } \|(u_1, u_2)\| \to \infty. \tag{4.11}$$

If $\mu_0 = 0$ or $\nu_0 = 0$, from the above argument, it is easy to see that (4.11) also holds. Hence $\mathcal{J}$ is bounded from below and (PS) condition holds. Let $X = E$ and $\varphi = \mathcal{J}$. Then by Lemma 2.8, it is easy to know that $\mathcal{J}$ has at least one critical point u^* such that

$$\mathcal{J}(u^*) = c = \inf_{u \in E} \mathcal{J}(u).$$

Thus the proof is complete. □

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