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Contraction principles in M_s-metric spaces

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Abstract

In this paper, we give an interesting extension of the partial S-metric space which was introduced [N. Mlaiki, Univers. J. Math. Math. Appl., **5** (2014), 109–119] to the M_s-metric space. Also, we prove the existence and uniqueness of a fixed point for a self-mapping on an M_s-metric space under different contraction principles.

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1. Introduction

Many researchers over the years proved many interesting results on the existence of a fixed point for a self-mapping on different types of metric spaces, for example, see [1, 2, 4, 8, 10–12, 14–16]. The idea behind this paper was inspired by the work of Asadi et al. in [7]. He gave a more general extension of almost any metric space with two dimensions, and that is not just by defining the self "distance" in a metric as in partial metric spaces [3, 5, 6, 13, 17], but he assumed that is not necessary that the self "distance" is less than the value of the metric between two different elements.

In [9], an extension of S-metric spaces to a partial S-metric spaces was introduced. Also, it was shown that every S-metric space is a partial S-metric space, but not every partial S-metric space is an S-metric space. In our paper, we introduce the concept of M_s-metric spaces which is an extension of the partial S-metric spaces in which we will prove some fixed point results.

First, we remind the reader definition of a partial S-metric space.

Definition 1.1. [9] Let X be a nonempty set. A partial S-metric on X is a function $S_p : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, t \in X$:

(i) $x = y$ if and only if $S_p(x, x, x) = S_p(y, y, y) = S_p(x, x, y)$;

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(ii) $S_p(x,y,z) \leqslant S_p(x,x,t) + S_p(y,y,t) + S_p(z,z,t) - S_p(t,t,t)$;

(iii) $S_p(x,x,x) \leqslant S_p(x,y,z)$;

(iv) $S_p(x,x,y) = S_p(y,y,x)$.

The pair (X, S_p) is called a partial S-metric space.

Next, we give the definition of an M_s-metric space, but first we introduce the following notations.

Notations.

1. $m_{s_{x,y,z}} := \min\{m_s(x,x,x), m_s(y,y,y), m_s(z,z,z)\}$;
2. $M_{s_{x,y,z}} := \max\{m_s(x,x,x), m_s(y,y,y), m_s(z,z,z)\}$.

Definition 1.2. An M_s-metric on a nonempty set X is a function $m_s : X^3 \to \mathbb{R}^+$ such that for all $x, y, z, t \in X$, the following conditions are satisfied:

1. $m_s(x,x,x) = m_s(y,y,y) = m_s(x,x,y)$ if and only if $x = y$;
2. $m_{s_{x,y,z}} \leqslant m_s(x,y,z)$;
3. $m_s(x,x,y) = m_s(y,y,x)$;
4. $(m_s(x,y,z) - m_{s_{x,y,z}}) \leqslant (m_s(x,x,t) - m_{s_{x,x,t}}) + (m_s(y,y,t) - m_{s_{y,y,t}}) + (m_s(z,z,t) - m_{s_{z,z,t}})$.

The pair (X, m_s) is called an M_s-metric space. Notice that the condition $m_s(x,x,x) = m_s(y,y,y) = m_s(z,z,z) = m_s(x,y,z) \Leftrightarrow x = y = z$ implies (1) above.

It is straightforward to verify that every partial S-metric space is an M_s-metric space but the converse is not true. The following example is an M_s-metric which is not a partial S-metric space.

Example 1.3. Let $X = \{1, 2, 3\}$ and define the M_s-metric space m_s on X by $m_s(1,2,3) = 6$, $m_s(1,1,2) = m_s(2,2,1) = m_s(1,1,1) = 8$, $m_s(1,1,3) = m_s(3,3,1) = m_s(3,3,2) = m_s(2,2,3) = 7$, $m_s(2,2,2) = 9$, and $m_s(3,3,3) = 5$. It is not difficult to see that (X, m_s) is an M_s-metric space, but since $m_s(1,1,1) \nleqslant m_s(1,2,3)$ we deduce that m_s is not a partial S-metric space.

Definition 1.4. Let (X, m_s) be an M_s-metric space. Then:

1. A sequence $\{x_n\}$ in X converges to a point x if and only if
$$\lim_{n\to\infty} (m_s(x_n, x_n, x) - m_{s x_n, x_n, x}) = 0.$$
2. A sequence $\{x_n\}$ in X is said to be M_s-Cauchy sequence if and only if
$$\lim_{n,m\to\infty} (m_s(x_n, x_n, x_m) - m_{s x_n, x_n, x_m}), \text{ and } \lim_{n,m\to\infty} (M_{s x_n, x_n, x_m} - m_{s x_n, x_n, x_m})$$
exist and are finite.
3. An M_s-metric space is said to be complete if every M_s-Cauchy sequence $\{x_n\}$ converges to a point x such that
$$\lim_{n\to\infty} (m_s(x_n, x_n, x) - m_{s x_n, x_n, x}) = 0 \text{ and } \lim_{n\to\infty} (M_{s x_n, x_n, x} - m_{s x_n, x_n, x}) = 0.$$

A ball in the M_s-metric (X, m_s) space with center $x \in X$ and radius $\eta > 0$ is defined by
$$B_s[x, \eta] = \{y \in X \mid m_s(x,x,y) - m_{s x,x,y} \leqslant \eta\}.$$

The topology of (X, M_s) is generated by means of the basis $\beta = \{B_s[x, \eta] : \eta > 0\}$.

Lemma 1.5. *Assume $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M_s-metric space (X, m_s). Then,*

$$\lim_{n\to\infty}(m_s(x_n, x_n, y_n) - m_{s x_n,x_n,y_n}) = m_s(x, x, y) - m_{s x,x,y}.$$

Proof. The proof follows by the inequality (4) in Definition 1.2. Indeed, we have

$$|(m_s(x_n, x_n, y_n) - m_{s x_n,x_n,y_n}) - (m_s(x, x, y) - m_{s x,x,y})| \leqslant 2[(m_s(x_n, x_n, x) - m_{s x_n,x_n,x}) + (m_s(y_n, y_n, y) - m_{s y_n,y_n,y})].$$

□

2. Fixed point theorems

In this section, we consider some results about the existence and the uniqueness of fixed point for self-mappings on an M_s-metric space, under different contraction principles.

Theorem 2.1. *Let (X, m_s) be a complete M_s-metric space and T be a self-mapping on X satisfying the following condition:*

$$m_s(Tx, Tx, Ty) \leqslant k m_s(x, x, y), \tag{2.1}$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point u. Moreover, $m_s(u, u, u) = 0$.

Proof. Since $k \in [0, 1)$, we can choose a natural number n_0 such that for a given $0 < \epsilon < 1$, we have $k^{n_0} < \frac{\epsilon}{8}$. Let $T^{n_0} \equiv F$ and $F^i x_0 = x_i$ for all natural numbers i, where x_0 is arbitrary. Hence, for all $x, y \in X$, we have

$$m_s(Fx, Fx, Fy) = m_s(T^{n_0}x, T^{n_0}x, T^{n_0}y) \leqslant k^{n_0} m_s(x, x, y).$$

For any i, we have

$$\begin{aligned} m_s(x_{i+1}, x_{i+1}, x_i) &= m_s(Fx_i, Fx_i, Fx_{i-1}) \\ &\leqslant k^{n_0} m_s(x_i, x_i, x_{i-1}) \\ &\leqslant k^{n_0+i} m_s(x_1, x_1, x_0) \to 0 \text{ as } i \to \infty. \end{aligned}$$

Similarly, by (2.1) we have $m_s(x_i, x_i, x_i) \to 0$ as $i \to \infty$. Thus, we choose l such that

$$m_s(x_{l+1}, x_{l+1}, x_l) < \frac{\epsilon}{8} \text{ and } m_s(x_l, x_l, x_l) < \frac{\epsilon}{4}.$$

Now, let $\eta = \frac{\epsilon}{2} + m_s(x_l, x_l, x_l)$. Define the set

$$B_s[x_l, \eta] = \{y \in X \mid m_s(x_l, x_l, y) - m_{s x_l,x_l,y} \leqslant \eta\}.$$

Note that, $x_l \in B_s[x_l, \eta]$. Therefore $B_s[x_l, \eta] \neq \emptyset$. Let $z \in B_s[x_l, \eta]$ be arbitrary. Hence,

$$\begin{aligned} m_s(Fz, Fz, Fx_l) &\leqslant k^{n_0} m_s(z, z, x_l) \\ &\leqslant \frac{\epsilon}{8}[\frac{\epsilon}{2} + m_{s z,z,x_l} + m_s(x_l, x_l, x_l)] \\ &< \frac{\epsilon}{8}[1 + 2m_s(x_l, x_l, x_l)]. \end{aligned}$$

Also, we know that $m_s(Fx_l, Fx_l, x_l) = m_s(x_{l+1}, x_{l+1}, x_l) < \frac{\epsilon}{8}$. Therefore,

$$m_s(Fz, Fz, x_l) - m_{s Fz,Fz,x_l} \leqslant 2[(m_s(Fz, Fz, Fx_l) - m_{s Fz,Fz,Fx_l})] + (m_s(Fx_l, Fx_l, x_l) - m_{s Fx_l,Fx_l,x_l})$$

$$\begin{aligned}
&\leqslant 2m_s(Fz, Fz, Fx_l) + m_s(Fx_l, Fx_l, x_l)] \\
&\leqslant 2\frac{\epsilon}{8}(1 + 2m_s(x_l, x_l, x_l)) + \frac{\epsilon}{8} \\
&= \frac{\epsilon}{4} + \frac{\epsilon}{8} + \frac{\epsilon}{2}m_s(x_l, x_l, x_l) \\
&< \frac{\epsilon}{2} + m_s(x_l, x_l, x_l).
\end{aligned}$$

Thus, $Fz \in B_b[x_l, \eta]$ which implies that F maps $B_b[x_l, \eta]$ into itself. Thus, by repeating this process we deduce that for all $n \geqslant 1$ we have $F^n x_l \in B_b[x_l, \eta]$ and that is $x_m \in B_b[x_l, \eta]$ for all $m \geqslant l$. Therefore, for all $m > n \geqslant l$ where $n = l + i$ for some i

$$\begin{aligned}
m_s(x_n, x_n, x_m) &= m_s(Fx_{n-1}, Fx_{n-1}, Fx_{m-1}) \\
&\leqslant k^{n_0} m_s(x_{n-1}, x_{n-1}, x_{m-1}) \\
&\leqslant k^{2n_0} m_s(x_{n-2}, x_{n-2}, x_{m-2}) \\
&\;\;\vdots \\
&\leqslant k^{in_0} m_s(x_l, x_l, x_{m-i}) \\
&\leqslant m_s(x_l, x_l, x_{m-i}) \\
&\leqslant \frac{\epsilon}{2} + m_{s x_l, x_l, x_{m-i}} + m_s(x_l, x_l, x_l) \\
&\leqslant \frac{\epsilon}{2} + 2m_s(x_l, x_l, x_l).
\end{aligned}$$

Also, we have $m_s(x_l, x_l, x_l) < \frac{\epsilon}{4}$, which implies that $m_s(x_n, x_n, x_m) < \epsilon$ for all $m > n > l$, and thus $m_s(x_n, x_n, x_m) - m_{s x_n, x_n, x_m} < \epsilon$ for all $m > n > l$. By the contraction condition (2.1) we see that the sequence $\{m_s(x_n, x_n, x_n)\}$ is decreasing and hence, for all $m > n > l$, we have

$$\begin{aligned}
M_{s x_n, x_n, x_m} - m_{s x_n, x_n, x_m} &\leqslant M_{s x_n, x_n, x_m} \\
&= m_s(x_n, x_n, x_n) \\
&\leqslant k m_s(x_{n-1}, x_{n-1}, x_{n-1}) \\
&\;\;\vdots \\
&\leqslant k^n m_s(x_0, x_0, x_0) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, we deduce that

$$\lim_{n,m\to\infty}(m_s(x_n, x_n, x_m) - m_{s x_n, x_n, x_m}) = 0, \text{ and } \lim_{n\to\infty}(M_{s x_n, x_n, x_m} - m_{s x_n, x_n, x_m}) = 0.$$

Hence, the sequence $\{x_n\}$ is an M_s-Cauchy. Since X is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} m_s(x_n, x_n, u) - m_{s x_n, x_n, u} = 0, \quad \lim_{n\to\infty} M_{s x_n, x_n, u} - m_{s x_n, x_n, u} = 0.$$

The contraction condition (2.1) implies that $m_s(x_n, x_n, x_n) \to 0$ as $n \to \infty$. Moreover, notice that

$$\lim_{n\to\infty} M_{s x_n, x_n, u} - m_{s x_n, x_n, u} = \lim_{n\to\infty} |m_s(x_n, x_n, x_n) - m_s(u, u, u)| = 0,$$

and hence $m_s(u, u, u) = 0$. Since $x_n \to u$, $m_s(u, u, u) = 0$ and $m_s(x_n, x_n, x_n) \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} m_s(x_n, x_n, u) = \lim_{n\to\infty} m_{s x_n, x_n, u} = 0$. Since $m_s(Tx_n, Tx_n, Tu) \leqslant k m_s(x_n, x_n, u) \to 0$ as $n \to \infty$, then $Tx_n \to Tu$.

Now, we show that $Tu = u$. By Lemma 1.5 and that $Tx_n \to Tu$ and $x_{n+1} = Tx_n \to u$, we have

$$\lim_{n\to\infty} m_s(x_n, x_n, u) - m_{s x_n, x_n, u} = 0 = \lim_{n\to\infty} m_s(x_{n+1}, x_{n+1}, u) - m_{s x_{n+1}, x_{n+1}, u}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} m_s(Tx_n, Tx_n, u) - m_{sTx_n,Tx_n,u} \\
&= m_s(u,u,u) - m_{sTu,Tu,u} \\
&= m_s(Tu,Tu,u) - m_{sTu,Tu,u}.
\end{aligned}
$$

Hence, $m_s(Tu,Tu,u) = m_{sTu,Tu,u} = m_s(u,u,u)$, but also by the contraction condition (2.1) we see that $m_{sTu,Tu,u} = m_s(Tu,Tu,Tu)$. Therefore, (2) in Definition 1.2 implies that $Tu = u$.

To prove the uniqueness of the fixed point u, assume that T has two fixed points $u, v \in X$; that is, $Tu = u$ and $Tv = v$. Thus,

$$
\begin{aligned}
m_s(u,u,v) = m_s(Tu,Tu,Tv) \leqslant km_s(u,u,v) < m_s(u,u,v), \\
m_s(u,u,u) = m_s(Tu,Tu,Tu) \leqslant km_s(u,u,u) < m_s(u,u,u),
\end{aligned}
$$

and

$$
m_s(v,v,v) = m_s(Tv,Tv,Tv) \leqslant km_s(v,v,v) < m_s(v,v,v),
$$

which implies that $m_s(u,u,v) = 0 = m_s(u,u,u) = m_s(v,v,v)$, and hence $u = v$ as desired. Finally, assume that u is a fixed point of T. Then applying the contraction condition (2.1) with $k \in [0,1)$, implies that

$$
\begin{aligned}
m_s(u,u,u) &= m_s(Tu,Tu,Tu) \\
&\leqslant km_s(u,u,u) \\
&\;\;\vdots \\
&\leqslant k^n m_s(u,u,u).
\end{aligned}
$$

Taking the limit as n tends to infinity, implies that $m_s(u,u,u) = 0$. □

In the following result, we prove the existence and uniqueness of a fixed point for a self-mapping in M_s-metric space, but under a more general contraction.

Theorem 2.2. *Let (X, m_s) be a complete M_s-metric space and T be a self-mapping on X satisfying the following condition:*

$$
m_s(Tx,Tx,Ty) \leqslant \lambda[m_s(x,x,Tx) + m_s(y,y,Ty)], \tag{2.2}
$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point u, where $m_s(u,u,u) = 0$.

Proof. Let $x_0 \in X$ be arbitrary. Consider the sequence $\{x_n\}$ is defined by $x_n = T^n x_0$ and $m_{s_n} = m_s(x_n, x_n, x_{n+1})$. Note that if there exists a natural number n such that $m_{s_n} = 0$, then x_n is a fixed point of T and we are done. So, we may assume that $m_{s_n} > 0$ for $n \geqslant 0$. By (2.2), we obtain for any $n \geqslant 0$,

$$
\begin{aligned}
m_{s_n} = m_s(x_n, x_n, x_{n+1}) &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_n) \\
&\leqslant \lambda[m_s(x_{n-1}, x_{n-1}, Tx_{n-1}) + m_s(x_n, x_n, Tx_n)] \\
&= \lambda[m_s(x_{n-1}, x_{n-1}, x_n) + m_s(x_n, x_n, x_{n+1})] \\
&= \lambda[m_{s_{n-1}} + m_{s_n}].
\end{aligned}
$$

Hence, $m_{s_n} \leqslant \lambda m_{s_{n-1}} + \lambda m_{s_n}$, which implies $m_{s_n} \leqslant \mu m_{s_{n-1}}$, where $\mu = \frac{\lambda}{1-\lambda} < 1$ as $\lambda \in [0, \frac{1}{2})$. By repeating this process we get

$$
m_{s_n} \leqslant \mu^n m_{s_0}.
$$

Thus, $\lim_{n\to\infty} m_{s_n} = 0$. By (2.2), for all natural numbers n, m, we have

$$
\begin{aligned}
m_s(x_n, x_n, x_m) = m_s(T^n x_0, T^n x_0, T^m x_0) &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_{m-1}) \\
&\leqslant \lambda[m_s(x_{n-1}, x_{n-1}, Tx_{n-1}) + m_s(x_{m-1}, x_{m-1}, Tx_{m-1})]
\end{aligned}
$$

$$= \lambda[m_s(x_{n-1}, x_{n-1}, x_n) + m_s(x_{m-1}, x_{m-1}, x_m)]$$
$$= \lambda[m_{s_{n-1}} + m_{s_{m-1}}].$$

Since $\lim_{n\to\infty} m_{s_n} = 0$, for every $\epsilon > 0$, we can find a natural number n_0 such that $m_{s_n} < \frac{\epsilon}{2}$ and $m_{s_m} < \frac{\epsilon}{2}$ for all $m, n > n_0$. Therefore, it follows that

$$m_s(x_n, x_n, x_m) \leqslant \lambda[m_{s_{n-1}} + m_{s_{m-1}}] < \lambda[\frac{\epsilon}{2} + \frac{\epsilon}{2}] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } n, m > n_0.$$

This implies that

$$m_s(x_n, x_n, x_m) - m_{sx_n,x_n,x_m} < \epsilon \text{ for all } n, m > n_0.$$

Now, for all natural numbers n, m we have

$$\begin{aligned} M_{sx_n,x_n,x_m} &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-1}) \\ &\leqslant \lambda[m_s(x_{n-1}, x_{n-1}, Tx_{n-1}) + m_s(x_{n-1}, x_{n-1}, Tx_{n-1})] \\ &= \lambda[m_s(x_{n-1}, x_{n-1}, x_n) + m_s(x_{n-1}, x_{n-1}, x_n)] \\ &= \lambda[m_{s_{n-1}} + m_{s_{n-1}}] \\ &= 2\lambda m_{s_{n-1}}. \end{aligned}$$

As $\lim_{n\to\infty} m_{s_{n-1}} = 0$, for every $\epsilon > 0$ we can find a natural number n_0 such that $m_{s_n} < \frac{\epsilon}{2}$ and for all $m, n > n_0$. Therefore, it follows that

$$M_{sx_n,x_n,x_m} \leqslant \lambda[m_{s_{n-1}} + m_{s_{n-1}}] < \lambda[\frac{\epsilon}{2} + \frac{\epsilon}{2}] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } n, m > n_0,$$

which implies that

$$M_{sx_n,x_n,x_m} - m_{sx_n,x_n,x_m} < \epsilon \text{ for all } n, m > n_0.$$

Thus, $\{x_n\}$ is an M_s-Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} m_s(x_n, x_n, u) - m_{sx_n,x_n,u} = 0.$$

Now, we show that u is a fixed point of T in X. For any natural number n we have,

$$\begin{aligned} \lim_{n\to\infty} m_s(x_n, x_n, u) - m_{sx_n,x_n,u} = 0 &= \lim_{n\to\infty} m_s(x_{n+1}, x_{n+1}, u) - m_{sx_{n+1},x_{n+1},u} \\ &= \lim_{n\to\infty} m_s(Tx_n, Tx_n, u) - m_{sTx_n,Tx_n,u} \\ &= m_s(Tu, Tu, u) - m_{sTu,Tu,u}. \end{aligned}$$

This implies that $m_s(Tu, Tu, u) - m_{su,u,Tu} = 0$, and that is $m_s(Tu, Tu, u) = m_{su,u,Tu}$. Now, assume that

$$m_s(Tu, Tu, u) = m_s(Tu, Tu, Tu) \leqslant 2\lambda m_s(u, u, Tu) = 2\lambda m_s(Tu, Tu, u) < m_s(u, u, Tu).$$

Thus,

$$m_s(Tu, Tu, u) = m_s(u, u, u) \leqslant m_s(Tu, Tu, Tu) \leqslant 2\lambda m_s(u, u, Tu) < m_s(u, u, Tu).$$

Therefore, $Tu = u$ and thus u is a fixed point of T.

Next, we show that if u is a fixed point, then $m_s(u, u, u) = 0$. Assume that u is a fixed point of T, then using the contraction (2.2), we have

$$\begin{aligned} m_s(u, u, u) &= m_s(Tu, Tu, Tu) \\ &\leqslant \lambda[m_s(u, u, Tu) + m_s(u, u, Tu)] \\ &= 2\lambda m_s(u, u, Tu) \\ &= 2\lambda m_s(u, u, u) \\ &< m_s(u, u, u) \text{ since } \lambda \in [0, \frac{1}{2}), \end{aligned}$$

that is, $m_s(u, u, u) = 0$.

Finally, to prove the uniqueness, assume that T has two fixed points, say $u, v \in X$. Hence,

$$m_s(u,u,v) = m_s(Tu,Tu,Tv) \leqslant \lambda[m_s(u,u,Tu) + m_s(v,v,Tv)] = \lambda[m_s(u,u,u) + m_s(v,v,v)] = 0,$$

which implies that $m_s(u,u,v) = 0 = m_s(u,u,u) = m_s(v,v,v)$, and hence $u = v$ as required. □

In closing, the authors would like to bring to the reader's attention that in this interesting M_s-metric space it is possible to add some control functions in both contractions of Theorems 2.1 and 2.2.

Theorem 2.3. *Let (X, m_s) be a complete M_s-metric space and T be a self mapping on X satisfying the following condition: for all $x, y, z \in X$*

$$m_s(Tx,Ty,Tz) \leqslant m_s(x,y,z) - \phi(m_s(x,y,z)), \tag{2.3}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function and $\phi^{-1}(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ in X such that $x_n = T^{n-1}x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Note that if there exists an $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then x_n is a fixed point for T. Without loss of generality, assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned} m_s(x_n, x_{n+1}, x_{n+1}) &= m_s(Tx_{n-1}, Tx_n, Tx_n) \\ &\leqslant m_s(x_{n-1}, x_n, x_n) - \phi(m_s(x_{n-1}, x_n, x_n)) \\ &\leqslant m_s(x_{n-1}, x_n, x_n). \end{aligned} \tag{2.4}$$

Similarly, we can prove that $m_s(x_{n-1}, x_n, x_n) \leqslant m_s(x_{n-2}, x_{n-1}, x_{n-1})$. Hence, $m_s(x_n, x_{n+1}, x_{n+1})$ is a monotone decreasing sequence. Hence there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} m_s(x_n, x_{n+1}, x_{n+1}) = r.$$

Now, by taking the limit as $n \to \infty$ in the inequality (2.4), we get $r \leqslant r - \phi(r)$ which leads to a contradiction unless $r = 0$. Therefore,

$$\lim_{n\to\infty} m_s(x_n, x_{n+1}, x_{n+1}) = 0.$$

Suppose that $\{x_n\}$ is not an M_s-Cauchy sequence. Then there exists an $\epsilon > 0$ such that we can find subsequences x_{m_k} and x_{n_k} of $\{x_n\}$ such that

$$m_s(x_{n_k}, x_{m_k}, x_{m_k}) - m_{s x_{n_k}, x_{m_k}, x_{m_k}} \geqslant \epsilon. \tag{2.5}$$

Choose n_k to be the smallest integer with $n_k > m_k$ and satisfies the inequality (2.5).

Hence, $m_s(x_{n_k}, x_{m_{k-1}}, x_{m_{k-1}}) - m_{s x_{n_k}, x_{m_{k-1}}, x_{m_{k-1}}} < \epsilon$. Now,

$$\begin{aligned} \epsilon &\leqslant m_s(x_{m_k}, x_{n_k}, x_{n_k}) - m_{s x_{m_k}, x_{n_k}, x_{n_k}} \\ &\leqslant m_s(x_{m_k}, x_{n_{k-1}}, x_{n_{k-1}}) + 2m_s(x_{n_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) - m_{s x_{m_k}, x_{n_{k-1}}, x_{n_{k-1}}} \\ &\leqslant \epsilon + 2m_s(x_{n_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) \\ &< \epsilon, \end{aligned}$$

as $n \to \infty$. Hence, we have a contradiction. Without loss of generality, assume that $m_{s x_n, x_n, x_m} = m_s(x_n, x_n, x_n)$. Then we have

$$\begin{aligned} 0 \leqslant m_{s x_n, x_n, x_m} - m_{s x_n, x_n, x_m} \leqslant M_{s x_n, x_n, x_m} &= m_s(x_n, x_n, x_n) \\ &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-1}) \end{aligned}$$

$$
\begin{aligned}
&\leqslant m_s(x_{n-1}, x_{n-1}, x_{n-1}) - \phi(m_s(x_{n-1}, x_{n-1}, x_{n-1})) \\
&\leqslant m_s(x_{n-1}, x_{n-1}, x_{n-1}) \\
&\vdots \\
&\leqslant m_s(x_0, x_0, x_0).
\end{aligned}
$$

Hence, $\lim_{n\to\infty} m_{s x_n, x_n, x_m} - m_{s x_n, x_n, x_m}$ exists and finite. Therefore, $\{x_n\}$ is an M_s-Cauchy sequence. Since X is complete, the sequence $\{x_n\}$ converges to an element $x \in X$; that is,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} m_s(x_n, x_n, x) - m_{s x_n, x_n, x} \\
&= \lim_{n\to\infty} m_s(x_{n+1}, x_{n+1}, x) - m_{s x_{n+1}, x_{n+1}, x} \\
&= \lim_{n\to\infty} m_s(Tx_n, Tx_n, x) - m_{s Tx_n, Tx_n, x} \\
&= m_s(Tx, Tx, x) - m_{s Tx, Tx, x}.
\end{aligned}
$$

Similar to the proof of Theorem 2.2, it is not difficult to show that this implies that, $Tx = x$ and so x is a fixed point.

Finally, we show that T has a unique fixed point. Assume that there are two fixed points $u, v \in X$ of T. If we have $m_s(u, u, v) > 0$, then condition (2.3) implies that

$$
m_s(u, u, v) = m_s(Tu, Tu, Tv) \leqslant m_s(u, u, v) - \phi(m_s(u, u, v)) < m_s(u, u, v),
$$

and that is a contradiction. Therefore, $m_s(u, u, v) = 0$ and similarly $m_s(u, u, u) = M_s(v, v, v) = 0$ and thus $u = v$ as desired. □

In closing, is it possible to define the same space but without the symmetry condition, (i.e., $m_s(x, x, y) \neq m_s(y, y, x)$)? If possible, what kind of results can be obtained in such space?

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