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Approximate controllability of impulsive Hilfer fractional differential inclusions

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Abstract

In this paper, firstly by utilizing the theory of operators semigroup, probability density functions via impulsive conditions, we establish a new $PC_{1-\nu}$-mild solution for impulsive Hilfer fractional differential inclusions. Secondly we prove the existence of mild solutions for the impulsive Hilfer fractional differential inclusions by using fractional calculus, multi-valued analysis and the fixed-point technique. Then under some assumptions, the approximate controllability of associated system are formulated and proved. An example is provided to illustrate the application of the obtained theory. ©2017 All rights reserved.

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1. Introduction

The theory of fractional differential equations has emerged as an important area of investigation since it describes the property of memory and heredity of various materials and processes in comparison with corresponding theory of classical differential equations. It has been found that the differential equations involving fractional time derivatives are more realistic to describe many phenomena in practical cases than those of integer order time derivatives. In recent years, more and more mathematicians, physicists, and engineers are attracted to this area and notable contributions have been made to both theory and applications of fractional differential equations. For example, for fractional derivative operators with non-locality, one can see the monographs of Baleanu et al. [4], Diethelm [9], Kilbas et al. [16], Lakshmikantham et al. [17], Miller and Ross [22], Podlubny [24] and Tarasov [28]. For local fractional derivative operators that describe non-differentiable problems from fractal physical phenomena, we can see the monograph of Yang et al. [31]. Another new operator called conformable fractional derivative has some properties that are distinct from those usual in other formulations [2]. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have many results (see for example [3, 7, 19, 26, 32–35]). On the other hand, Hilfer [13] proposed a generalized Riemann-Liouville fractional derivative, for short, Hilfer fractional derivative, which includes Riemann-Liouville fractional

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derivative and Caputo fractional derivative. This operator appeared in the theoretical simulation of dielectric relaxation in glass forming materials. It seems that Hilfer et al. [14] have initially proposed linear differential equations with the new fractional operator: Hilfer fractional derivative and applied operational calculus to solve such simple fractional differential equtions. Thereafter, Furati et al. [11], Gu and Trujillo [12] extended the study nonlinear problems and presented the existence, nonexistence and stability results for initial value problems of nonlinear fractional differential equations with Hilfer fractional derivative in a suitable weighted space of continuous functions.

The theory of impulsive differential equations and inclusions of integer order has found its extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. During the last ten years, impulsive differential equations and inclusions with different conditions have been intensely studied by many mathematicians, see [1, 10, 20]. At present, the foundations of the general theory are already laid, and many of them are formulated in detail in Benchohra et al. [6]. However, impulsive differential equations and inclusions of fractional order have not been much investigated and many aspects of them are yet to be explored. For some recent works on impulsive fractional differential equations and inclusions, see [18, 23, 27, 29, 30] and the references therein.

Control theory is an interdisciplinary branch of engineering and mathematics that deals with influence behavior of dynamical systems. Controllability is one of the fundamental concepts in mathematical control theory. This is a qualitative property of dynamical control systems and it is of particular importance in control theory. Many fundamental problems of control theory such as pole assignment, stabilization and optimal control may be solved under the assumption that the system is controllable. From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. Exact controllability enables to steer the system to arbitrary final state (see for example [1, 5, 18, 20]) while approximate controllability means that system can be steered an arbitrary small neighborhood of final state. Approximately controllability systems are more prevalent and very often approximate controllability is completely adequate in applications. Therefore, it is important, in fact necessary to study the weaker concept of controllability, namely approximate controllability for nonlinear systems. In recent years, there are some papers on the approximate controllability of the nonlinear evolution systems under different conditions [7, 19, 21, 25, 26, 32]. The conditions are established with the help of semigroup theory and fixed point theorem under the assumption that the associated linear system is approximately controllable. However, it should be emphasized that to the best of our knowledge, the approximate controllability of impulsive Hilfer fractional differential inclusions in Banach spaces has not been investigated yet and it is also the motivation of this paper. In order to fill this gap, in this paper, we study the approximate controllability of Hilfer fractional differential inclusions with impulsive using fixed point theorem, fractional calculus and the assumption that the associated linear system is approximately controllable. At last, an example is given to illustrate the abstract results.

The objective of this paper is to investigate the approximate controllability of the following impulsive fractional differential inclusions involving Hilfer fractional derivative:

$$\begin{cases} D_{0^+}^{q,p}x(t) \in Ax(t) + F(t, x(t)) + Bu(t), & t \in (0, b], \quad t \neq t_k, \\ \Delta I_{0^+}^{1-\nu}x(t) \mid_{t=t_k} = G_k(t_k, x(t_k^-)), & k = 1, 2, \cdots, m, \\ I_{0^+}^{1-\nu}x(t) \mid_{t=0} = x_0 \in X, \end{cases} \tag{1.1}$$

where $D_{0^+}^{q,p}$ is the Hilfer derivative of order q and type p which will be given in next section, $0 \leqslant p \leqslant 1$, $\frac{1}{2} < q \leqslant 1$, and $\nu = p + q - pq$; $x(\cdot)$ takes values in Banach space X with norm $\|\cdot\|$; $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{S(t), t \geqslant 0\}$ on X. Let $J = [0, b]$, U is a Banach space, the control function u takes its values in $L^2(J, U)$; B is a linear bounded operator from U to X; $F : J \times X \to \mathcal{P}(X) := 2^X \backslash \{\emptyset\}$ is a multivalued map satisfying some assumptions and $x_0 \in X$; $G_k : J \times X \to X$ are given functions that will be specified later. $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\Delta I_{0^+}^{1-\nu}x(t_k) = I_{0^+}^{1-\nu}x(t_k^+) - I_{0^+}^{1-\nu}x(t_k^-) = \Gamma(\nu)\big[\lim_{t\to t_k^+}(t-t_k)^{1-\nu}x(t) - \lim_{t\to t_k^-}(t-t_k)^{1-\nu}x(t)\big]$ ([16], Lemma 3.2, Chapter 3). Furthermore, since $x(t_k^-)$ is bounded (see the definition of $PC_{1-\nu}(J,X)$ in Section 2), we can also have $\Delta I_{0^+}^{1-\nu}x(t_k) = \Gamma(\nu)\lim_{t\to t_k^+}(t-t_k)^{1-\nu}x(t)$. $I_{0^+}^{1-\nu}x(t_k^+)$ and $I_{0^+}^{1-\nu}x(t_k^-)$ denote the right and the left limits of $I_{0^+}^{1-\nu}x(t)$ at $t=t_k, k=1,2,\cdots,m$, respectively.

2. Preliminaries

Let $C(J,X)$ denote the Banach space of all X-valued continuous functions from $J=[0,b]$ to X with the norm $\|x\|_C = \sup_{t\in J}\|x(t)\|_X$. Let $J'=(0,b]$, $C_{1-\nu}(J,X)=\{x\in C(J',X): t^{1-\nu}x(t)\in C(J,X)\}$ with the norm $\|x\|_{C_{1-\nu}} = \sup\{t^{1-\nu}\|x(t)\|_X, t\in J'\}$. Obviously, the space $C_{1-\nu}(J,X)$ is a Banach space.

In order to define the mild solutions of problem (1.1), we also consider the Banach space $PC_{1-\nu}(J,X) = \{x: (t-t_k)^{1-\nu}x(t)\in C((t_k,t_{k+1}],X)$ and $\lim_{t\to t_k^+}(t-t_k)^{1-\nu}x(t)$ exists, $k=1,2,\cdots,m\}$ with the norm

$$\|x\|_{PC_{1-\nu}} = \max\Big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\nu}\|x(t)\|_X : k=1,2,\cdots,m\Big\}.$$

We need some basic definitions and properties about fractional calculus, essential principles of multi-valued analysis, primary facts in semigroup theory and some lemmas.

The following definitions concerning with the fractional calculus can be found in the books [9, 16, 22, 24].

Definition 2.1. The fractional integral for a function g from lower limit 0 and order α can be defined as

$$I_{0^+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(s)}{(t-s)^{1-\alpha}}ds, \quad \alpha>0, \quad t>0,$$

where Γ is the gamma function, and right hand side of upper equality is defined point-wise on R^+.

Definition 2.2. Riemann-Liouville derivative of order α with the lower limit 0 for a function $f:[0,\infty)\to R$ can be defined as

$$^{R-L}D_{0^+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}ds, \quad t>0, \quad 0\leqslant n-1<\alpha<n.$$

Definition 2.3. The Caputo derivative of order α for a function $f:[0,\infty)\to R$ can be denoted by

$$^{C}D_{0^+}^{\alpha}f(t) = {}^{R-L}D^{\alpha}\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t>0, \quad 0\leqslant n-1<\alpha<n.$$

Definition 2.4. The left Hilfer derivative of order $0<q\leqslant 1$ and type $0\leqslant p\leqslant 1$ of function $f(t)$ is defined by

$$D_{0^+}^{q,p}f(t) = (I_{0^+}^{p(1-q)}D(I_{0^+}^{(1-p)(1-q)}f))(t),$$

where $D := \frac{d}{dt}$.

Remark 2.5.

(i) When $p=0$ and $0<q<1$, the Hilfer derivative corresponds to the Riemann-Liouville fractional derivative:

$$D_{0^+}^{q,0}f(t) = \frac{d}{dt}\left(I_{0^+}^{1-q}f\right)(t) = {}^{R-L}D_{0^+}^{q}f(t).$$

(ii) When $p=1$ and $0<q<1$, the Hilfer derivative corresponds to the classical Caputo fractional derivative:

$$D_{0^+}^{q,1}f(t) = I_{0^+}^{1-q}\frac{d}{dt}f(t) = {}^{C}D_{0^+}^{q}f(t).$$

Let $\mathcal{P}(X)$ be the set of all nonempty subsets of X. We will use the following notations:

$\mathcal{P}_{cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is closed}\}$, $\mathcal{P}_{b}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is bounded}\}$, $\mathcal{P}_{cv}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is convex}\}$, $\mathcal{P}_{cp}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is compact}\}$.

Proposition 2.6 ([8]).

(i) *A measurable function* $u : J \to X$ *is Bochner integrable, if and only if* $\|u\|$ *is Lebesgue integrable.*

(ii) *A multi-valued map* $F : X \to 2^X$ *is said to be convex-valued (closed-valued), if* $F(u)$ *is convex (closed) for all* $u \in X$*, is said to be bounded on bounded sets, if* $F(B) = \cup_{u\in B}$ *is bounded in* X *for all* $B \in \mathcal{P}_b(X)$.

(iii) *A map* F *is said to be upper semi-continuous (u.s.c.) on* X*, if for each* $u_0 \in X$*, the set* $F(u_0)$ *is a nonempty closed subset of* X *and if for each open subset* Ω *of* X *containing* $F(u_0)$*, there exists an open neighborhood* ∇ *of* u_0 *such that* $F(\nabla) \subseteq \Omega$.

(iv) *A map* F *is said to be completely continuous, if* $F(B)$ *is relatively compact for every* $B \in \mathcal{P}_b(X)$*. If the multi-valued map* F *is completely continuous with nonempty compact values, then* F *is u.s.c. if and only if* F *has a closed graph, i.e.,* $u_n \to u$, $y_n \to y$, $y_n \in F(u_0)$ *imply* $y \in F(u)$*. We say that* F *has a fixed point if there is* $u \in X$ *such that* $u \in F(u)$.

(v) *A multi-valued map* $F : J \to \mathcal{P}_{cl}(X)$ *is said to be measurable, if for each* $u \in X$ *the function* $y : J \to R$ *defined by* $y(t) = d(u, F(t)) = \inf\{\|u - z\|, z \in F(t)\}$ *is measurable.*

(vi) *A multi-valued map* $F : X \to 2^X$ *is said to be condensing, if for any bounded subset* $B \subset X$ *with* $\alpha(B) \neq 0$ *we have* $\alpha(F(B)) < \alpha(B)$*, where* $\alpha(\cdot)$ *denotes the Kuratowski measure of non-compactness defined as follows:* $\alpha(B) := \inf\{d > 0 : B \text{ can be covered by a finite number of balls of radius } d\}$.

Lemma 2.7 ([29]). *For* $\sigma \in (0, 1]$ *and* $0 < a < b$*, we have* $|b^\sigma - a^\sigma| \leqslant (b - a)^\sigma$.

Lemma 2.8 (Lasota and Opial [26]). *Let* J *be a compact real interval and let* X *be a Banach space. The multivalued map* $F : J \times X \to \mathcal{P}_{b,cl,cv}(X)$ *is measurable to* t *for each fixed* $x \in X$*, u.s.c. to* x *for each* $t \in J$ *and for each* $x \in C(J, X)$ *the set* $S_{F,x} = \{f \in L^1(J, X) : f(t) \in F(t, x(t)), \text{for a.e. } t \in J\}$ *is nonempty. Let* Γ *be a linear continuous mapping from* $L^1(J, X)$ *to* $C(J, X)$*, then the operator*

$$\Gamma \circ S_F : C(J, X) \to \mathcal{P}_{b,cl,cv}(C(J, X)),$$

$$x \mapsto (\Gamma \circ S_F)(x) = \Gamma(S_{F,x}),$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.9 ([8]). *Let* D *be a nonempty subset of* X *which is bounded, closed and convex. Suppose* $G : D \to 2^X\backslash\{\emptyset\}$ *is u.s.c. with closed, convex values such that* $G(D) \subset D$ *and* $G(D)$ *is compact. Then* G *has a fixed point.*

Lemma 2.10 ([29]). *Let* $0 < \nu \leqslant 1$ *and let* $x_{1-\nu}(t) = I_{0^+}^{1-\nu}x(t)$ *be the fractional integral of order* $1 - \nu$*. If* $x(t) \in PC_{1-\nu}(J, X)$ *and* $x_{1-\nu}(t) \in PC(J, X)$*, then one has the following equality:*

$$I_{0^+}^{\nu} D_{0^+}^{\nu} x(t) = \begin{cases} x(t) - x_{1-\nu}(t)\mid_{t=0} \frac{t^{\nu-1}}{\Gamma(\nu)}, & t \in (0, t_1], \\ x(t) - \sum_{i=1}^{k} \frac{\Delta x_{1-\nu}(t_i)}{\Gamma(\nu)}(t - t_i)^{\nu-1} & \\ \quad - x_{1-\nu}(t)\mid_{t=0} \frac{t^{\nu-1}}{\Gamma(\nu)}, & t \in (t_k, t_{k+1}], \end{cases}$$

where $\Delta x_{1-\nu}(t_k) = x_{1-\nu}(t_k^+) - x_{1-\nu}(t_k^-), k = 1, 2, \cdots, m$.

We first consider a nonhomogeneous impulsive linear fractional system of the form

$$\begin{cases} D_{0^+}^{q,p} x(t) = Ax(t) + h(t), & t \in J', \;\; t \neq t_k \\ \Delta I_{0^+}^{1-\nu} x(t)\mid_{t=t_k} = y_k, & k = 1, 2, \cdots, m, \\ I_{0^+}^{1-\nu} x(t)\mid_{t=0} = x_0 \in X, & \end{cases} \tag{2.1}$$

where $h \in PC(J, X)$. We suppose that $x(\cdot) = v(\cdot) + w(\cdot)$, where v is the continuous mild solution for

$$\begin{cases} D_{0^+}^{q,p} v(t) = Av(t) + h(t), & t \in J', \\ I_{0^+}^{1-\nu} v(t)\,|_{t=0} = x_0 \in X, \end{cases} \tag{2.2}$$

on J, and w is the PC-mild solution for

$$\begin{cases} D_{0^+}^{q,p} w(t) = Aw(t), & t \in J', t \neq t_k, \\ \Delta I_{0^+}^{1-\nu} w(t)\,|_{t=t_k} = y_k, & k = 1, 2, \cdots, m, \\ I_{0^+}^{1-\nu} w(t)\,|_{t=0} = 0 \in X. \end{cases} \tag{2.3}$$

Indeed, by adding together (2.2) and (2.3), it follows (2.1). Note v is continuous, so $v(t_k^+) = v(t_k^-)$, $k = 1, 2, \cdots, m$. On the other hand, any solution of (2.1) can be decomposed to (2.2) and (2.3).

Firstly, from [12] we know that a mild solution of (2.2) is given by

$$v(t) = S_{p,q}(t)x_0 + \int_0^t K_q(t-s)h(s)ds, \quad t \in J',$$

where $S_{p,q}(t) = I_{0^+}^{p(1-q)} K_q(t)$, $K_q(t) = t^{q-1}P_q(t)$, $P_q(t) = \int_0^\infty q\theta M_q(\theta)S(t^q\theta)d\theta$ and $M_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\rho_q(\theta^{-\frac{1}{q}})$, $\rho_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q)$.

Now we can obtain the PC-mild solution of system (2.3) using Lemma 2.10:

$$w(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Aw(s)ds, & t \in (0, t_1], \\ \sum_{i=1}^k \frac{\Delta w_{1-\nu}(t_i)}{\Gamma(\nu)}(t-t_i)^{\nu-1} & \\ \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Aw(s)ds, & t \in (t_k, t_{k+1}],\ k = 1, 2, \cdots, m. \end{cases} \tag{2.4}$$

Obviously, equation (2.4) can be written as

$$w(t) = \sum_{i=1}^k \frac{y_i}{\Gamma(\nu)}(t-t_i)^{\nu-1}\chi_i(t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Aw(s)ds, \quad t \in J, \tag{2.5}$$

where

$$\chi_i(t) = \begin{cases} 0, & t \in (0, t_i], \\ 1, & t \in (t_i, b]. \end{cases}$$

Let $\lambda > 0$. Taking the Laplace transformation to the equation (2.5), we obtain

$$W(\lambda) = \sum_{i=1}^k \frac{y_i e^{-\lambda t_i}}{\lambda^\nu} + \frac{1}{\lambda^q}AW(\lambda),$$

i.e.,

$$W(\lambda) = \sum_{i=1}^k (\lambda^q I - A)^{-1} y_i e^{-\lambda t_i}.$$

Thus, one can obtain the PC-mild solution of (2.3) as

$$w(t) = \sum_{i=1}^k \chi_i(t)S_{p,q}(t-t_i)y_i.$$

By the above arguments, the PC-mild solution of (2.1) is given by

$$x(t) = S_{p,q}(t)x_0 + \sum_{i=1}^k \chi_i(t)S_{p,q}(t-t_i)y_i + \int_0^t K_q(t-s)h(s)ds.$$

According to the above result, we can introduce the following definition of the PC-mild solution for system (1.1).

Definition 2.11. A function $x \in PC_{1-\nu}(J, X)$ is called a mild solution of problem (1.1), if it satisfies the following fractional integral equations:

(i) $I_{0+}^{1-\nu}x(t)\mid_{t=0}= x_0 \in X$;

(ii) there exists $f \in S_{F,x}$ such that $f(t) \in F(t, x(t))$ and

$$x(t) = \begin{cases} S_{p,q}(t)x_0 + \int_0^t K_q(t-s)[Bu(s) + f(s)]ds, & t \in (0, t_1], \\ S_{p,q}(t)x_0 + \int_0^t K_q(t-s)[Bu(s) + f(s)]ds & \\ \quad + \sum_{i=1}^{k} S_{p,q}(t-t_i)G_i(t_i, x(t_i^-)), & t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m. \end{cases}$$

We need the following assumption:

(H_0) $S(t)$ is continuous in the uniform operator topology for $t \geqslant 0$ and $\{S(t), t \geqslant 0\}$ is uniformly bounded, i.e., there exists $M > 1$ such that $\sup_{t\in[0,\infty)} |S(t)| < M$.

The following essential propositions can be found in the paper [12, 33].

Proposition 2.12. *Under assumption (H_0), $P_q(t)$ is continuous in the uniform operator topology for $t > 0$.*

Proposition 2.13. *Under assumption (H_0) for any fixed $t > 0$, $\{K_q(t), t > 0\}$ and $\{S_{p,q}(t), t > 0\}$ are linear operators and for any $x \in X$,*

$$\|K_q(t)x\| \leqslant \frac{Mt^{q-1}}{\Gamma(q)}\|x\|, \quad \|S_{p,q}(t)x\| \leqslant \frac{Mt^{\nu-1}}{\Gamma(\nu))}\|x\|.$$

Proposition 2.14. *Under assumption (H_0), $\{K_q(t), t > 0\}$ and $\{S_{p,q}(t), t > 0\}$ are strongly continuous, which means that for any $x \in X$ and $0 < t' < t'' \leqslant b$, we have*

$$\|K_q(t')x - K_q(t'')x\| \to 0, \quad \|S_{p,q}(t')x - S_{p,q}(t'')x\| \to 0, \quad \text{as } t'' \to t'.$$

Definition 2.15. Let $x(\cdot, u)$ be a mild solution of system (1.1) corresponding to the control $u \in L^2(J, U)$ and the initial value $x_0 \in X$. The set $\mathfrak{R}(b, x_0) = \{x(b; u) : u \in L^2(J, U), x(0; u) = x_0\}$ is reachable set of system (1.1) at terminal time b. If $\overline{\mathfrak{R}(b, x_0)} = X$, then system (1.1) is said to be approximately controllable on the interval J.

It is convenient at this point to introduce two relevant operators:

$$\Gamma_0^b = \int_0^b K_q(b-s)BB^*K_q^*(b-s)ds, \quad \frac{1}{2} < q \leqslant 1,$$

and

$$R(a, \Gamma_0^b) = (aI + \Gamma_0^b)^{-1}, \quad a > 0.$$

In order to study the approximate controllability for the nonlinear impulsive system (1.1), we first consider the approximate controllability of its linear part:

$$\begin{cases} D_{0+}^{p,q}x(t) \in Ax(t) + (Bv)(t), & t \in J', \quad t \neq t_k, \quad \frac{1}{2} < q \leqslant 1, \\ \Delta I_{0+}^{1-\nu}x(t)\mid_{t=t_k}= y_k, & k = 1, 2, \cdots, m, \\ I_{0+}^{1-\nu}x(t)\mid_{t=0}= x_0 \in X, & \end{cases} \tag{2.6}$$

where $B : U \to X$ is a linear bounded operator, $v \in L^2(J, U)$.

Lemma 2.16 ([5]). *The linear fractional differential system (2.6) is approximately controllable on J, if and only if $aR(a, \Gamma_0^b) \to 0$ as $a \to 0^+$ in the strong operator topology.*

3. Main results

In this part, we establish the existence of PC-mild solution and some sufficient conditions of the approximately controllability for system (1.1).

For convenient, let us introduce some notations:

$$M_B = \|B\|, \quad b_1 = \Big(\frac{1-\beta}{q-\beta}\Big)^{1-\beta}, \quad \Theta = b_1 b^{q-\beta}.$$

Firstly, we introduce the following hypotheses:

(H_1) Semigroup $S(t)$ is compact for each $t > 0$ and $\|aR(a, \Gamma_0^b)\| \leqslant 1, \forall a > 0$.

(H_2) The multivalued map $F : J \times X \to \mathcal{P}_{b,cl,cv}(X)$ satisfies the following:

- (2a) $F(t, \cdot) : X \to X$ is u.s.c. for each $t \in J$ and for each $x \in X$, the function $F(t, \cdot) : J \to X$ is strongly measurable to t, and for each $x \in X$, the set $S_{F,x} = \{f \in L^1(J, X) : f(t) \in F(t, x(t)), \text{ for a.e. } t \in J\}$ is nonempty;
- (2b) there exist a function $H(t) \in L^{\frac{1}{\beta}}(J), \beta \in (0, q)$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$, such that for any $(t, x) \in J \times X$, we have $\|F(t, x(t))\|_X = \sup\{\|f(t)\|_X : f(t) \in F(t, x(t))\} \leqslant H(t) \times \psi(\|x\|_{PC_{1-\nu}}), \lim_{r\to\infty} \inf \frac{\psi(r)}{r} = \Lambda < \infty$.

(H_3) There exist positive constants $d_k (k = 1, 2, \cdots, m)$, satisfy: $M \sum_{k=1}^{m} d_k (t_k - t_{k-1})^{\nu-1} < \Gamma(p(1 - q) + q)$ such that

$$\|G_k(t_k, x) - G_k(t_k, y)\| \leqslant d_k \|x - y\|_X, \quad \forall\, x, y \in X.$$

(H_4) $\frac{Mb^{1-\nu}\Theta}{\Gamma(q)} \|P\|_{L^{\frac{1}{\beta}}} \Lambda \Big[1 + \frac{M^2 M_B^2 b^{2q-1}}{a\Gamma^2(q)(2q-1)}\Big] < 1$.

Now we are in a position to prove the main result of this section.

Theorem 3.1. *Suppose that the hypotheses* (H_0)-(H_4) *are satisfied. Then for each given control function* $u(\cdot) \in L^2(J, U)$, *the initial problem* (1.1) *has a mild solution on* $PC_{1-\nu}(J, X)$.

Proof. Define $B_r = \{x \in PC_{1-\nu}(J, X),\ \|x\|_{PC_{1-\nu}} \leqslant r,\ r > 0\}$, obviously, B_r is a bounded, closed, convex set in $PC_{1-\nu}(J, X)$. For $a > 0$, for all $x(\cdot) \in PC_{1-\nu}, x_1 \in X$, we take the control function as

$$u(t) = B^* K_q^*(b - t) R(a, \Gamma_0^b) P(x(\cdot)),$$

where

$$P(x(\cdot)) = x_1 - S_{p,q}(b)x_0 - \int_0^b K_q(b - s) f(s) ds - \sum_{i=1}^{k} S_{p,q}(b - t_i) G_i(t_i, x(t_i^-)), \quad f \in S_{F,x}.$$

According to this control, we define the operator $\Phi : PC_{1-\nu}(J, X) \to \mathcal{P}(PC_{1-\nu}(J, X))$ as follows:

$$\begin{aligned}\Phi(x) = \Big\{ \omega \in PC_{1-\nu}(J, X) : \omega &= S_{p,q}(t)x_0 + \int_0^t K_q(t - s)[Bu(s) + f(s)] ds,\ f \in S_{F,x},\ t \in (0, t_1], \\ \text{or } \omega &= S_{p,q}(t)x_0 + \int_0^t K_q(t - s)[Bu(s) + f(s)] ds\ + \sum_{i=1}^{k} S_{p,q}(t - t_i) G_i(t_i, x(t_i^-)), \\ &f \in S_{F,x},\ t \in (t_k, t_{k+1}],\ k = 1, 2, \cdots, m \Big\}.\end{aligned}$$

We will show that for all $a > 0$, the operator $\Phi : PC_{1-\nu} \to \mathcal{P}(PC_{1-\nu})$ has a fixed point. Now we divide the proof into five steps.

Step 1. For every $x \in B_r$ the operator Φ is convex.

Let $\omega_1, \omega_2 \in \Phi(x)$, then for each $t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m$, there exist $f_1, f_2 \in S_{F,x}$ such that

$$\begin{aligned}\omega_j(t) &= S_{p,q}(t)x_0 + \int_0^t K_q(t-s)f_j(s)ds + \sum_{i=1}^{k} S_{p,q}(t-t_i)G_i(t_i, x(t_i^-)) \\ &\quad + \int_0^t K_q(t-s)BB^*K_q^*(b-s)R(a, \Gamma_0^b) \\ &\quad \times \Big[x_1 - \int_0^b K_q(b-s)f_j(s)ds - S_{p,q}(b)x_0 - \sum_{i=1}^{k} S_{p,q}(b-t_i)G_i(t_i, x(t_i^-))\Big]ds, \quad j = 1, 2.\end{aligned}$$

Let $\eta \in [0, 1]$, then for each $t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m$, we have

$$\begin{aligned}\eta\omega_1(t) + (1-\eta)\omega_2(t) &= S_{p,q}(t)x_0 + \sum_{i=1}^{k} S_{p,q}(t-t_i)G_i(t_i, x(t_i^-)) + \int_0^t K_q(t-s)[\eta f_1(s) + (1-\eta)f_2(s)]ds \\ &\quad + \int_0^t K_q(t-s)BB^*K_q^*(b-s)R(a, \Gamma_0^b) \cdot \Big\{x_1 - S_{p,q}(b)x_0 - \sum_{i=1}^{k} S_{p,q}(b-t_i) \\ &\quad \times G_i(t_i, x(t_i^-)) - \int_0^b K_q(b-s) \cdot [\eta f_1(s) + (1-\eta)f_2(s)]ds\Big\}ds, \quad j = 1, 2.\end{aligned}$$

Since $S_{F,x}$ is convex, $\eta f_1(s) + (1-\eta)f_2(s) \in S_{F,x}$, thus $\eta\omega_1(t) + (1-\eta)\omega_2(t) \in \Phi(x)$.

Step 2. For every $a > 0$, there is a positive constant $r_0 = r(a)$, such that $\Phi(B_{r_0}) \subset B_{r_0}$.

If this is not true, then for each $r > 0$, there exists $\overline{x} \in B_r$, $\overline{u} \in L^2(J, U)$ corresponding to $\overline{x}$, such that $\Phi(\overline{x}) \nsubseteq B_r$, that is

$$\|\Phi(\overline{x})\|_{PC_{1-\nu}} = \sup\{\|\omega\|_{PC_{1-\nu}} : \omega \in \Phi(\overline{x})\} \geqslant r.$$

By using Holder's inequality and (H_3), we have

$$\begin{aligned}\int_0^t \|K_q(t-s)f(s)\|ds &= \int_0^t (t-s)^{q-1}\|P_q(t-s)f(s)\|ds \\ &\leqslant \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s)\|ds \\ &\leqslant \frac{M\psi(\|x\|_{PC_{1-\nu}})}{\Gamma(q)}\Big(\int_0^t (t-s)^{\frac{q-1}{1-\beta}}ds\Big)^{1-\beta}\Big(\int_0^t H(s)^{\frac{1}{\beta}}ds\Big)^{\beta} \\ &= \frac{M\Theta\|P\|_{L^{\frac{1}{\beta}}}}{\Gamma(q)}\psi(\|x\|_{PC_{1-\nu}}),\end{aligned}$$

$$\begin{aligned}\int_0^t \|K_q(t-s)Bu(s)\|ds &\leqslant \int_0^t (t-s)^{q-1}\|P_q(t-s)Bu(s)\|ds \\ &\leqslant \int_0^t (t-s)^{q-1}(b-s)^{q-1}\|P_q(t-s)BB^*P_q^*(b-s)R(a, \Gamma_0^b)P(x(b))\|ds \\ &\leqslant \frac{M^2M_B^2}{a\Gamma^2(q)}\int_0^t (t-s)^{q-1}(b-s)^{q-1}\|P(x(b))\|ds \\ &\leqslant \frac{M^2M_B^2b^{2q-1}}{a\Gamma(q)(2q-1)}\Big[\|x_1\| + \frac{Mb^{\nu-1}}{\Gamma(p(1-q)+q)}\|x_0\| + \frac{M\|H\|_{L^{\frac{1}{\beta}}}\Theta}{\Gamma(q)} \\ &\quad \times \psi(\|x\|_{PC_{1-\nu}}) + \sum_{i=1}^{k}\frac{M(b-t_i)^{1-\nu}}{\Gamma(p(1-q)+q)}(d_i\|x(t_i^-)\| + \|G_i(t_i, 0)\|)\Big].\end{aligned}$$

If $t \in (0, t_1]$, then we have

$$\begin{aligned}\|t^{1-\nu}\Phi(\bar{x})\| &\leqslant \|t^{1-\nu}S_{p,q}(t)x_0\| + \left\|t^{1-\nu}\int_0^t (t-s)^{q-1}P_q(t-s)[B\overline{u}(s)+f(s)]ds\right\| \\ &\leqslant \frac{M}{\Gamma(p(1-q)+q)}\|x_0\| + \frac{Mb^{1-\nu}\Theta}{\Gamma(q)}\|H\|_{L^{\frac{1}{\beta}}}\psi(\|x\|) + \frac{M^2M_B^2b^{2q-1}}{a\Gamma^2(q)(2q-1)} \\ &\quad \times \Big[b^{1-\nu}\|x_1\| + \frac{M}{\Gamma(p(1-q)+q)}\|x_0\| + \frac{Mb^{1-\nu}\Theta}{\Gamma(q)}\|H\|_{L^{\frac{1}{\beta}}}\psi(\|x\|) \\ &\quad + \sum_{i=1}^{k}\frac{M(b-t_i)^{1-\nu}b^{1-\nu}}{\Gamma(p(1-q)+q)}(d_i\|x(t_i^-)\| + \|G_i(t_i,0)\|)\Big].\end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then we get

$$\begin{aligned} r \leqslant (t-t_k)^{1-\nu}\|\Phi(\overline{x})\|_X &\leqslant (t-t_k)^{1-\nu}\|S_{p,q}(t)x_0\| + (t-t_k)^{1-\nu}\left\|\int_0^t K_q(t-s)f(s)ds\right\| \\ &\quad + (t-t_k)^{1-\nu}\left\|\int_0^t K_q(t-s)B\overline{u}(s)ds\right\| \\ &\quad + (t-t_k)^{1-\nu}\left\|\sum_{i=1}^{k} S_{p,q}(t-t_i)G_i(t_i, x(t_i^-))\right\| \\ &\leqslant \frac{M\|x_0\|}{\Gamma(p(1-q)+q)} + \frac{M(t-t_k)^{1-\nu}\Theta\|H\|_{L^{\frac{1}{\beta}}}}{\Gamma(q)}\psi(\|\overline{x}\|_{PC_{1-\nu}}) \\ &\quad + \frac{M^2M_B^2b^{2q-1}}{a\Gamma^2(q)(2q-1)}\Big[(t-t_k)^{1-\nu}\|x_1\| + \frac{M\|x_0\|}{\Gamma(p(1-q)+q)} \\ &\quad + \frac{M(t-t_k)^{1-\nu}\Theta}{\Gamma(q)}\|H\|_{L^{\frac{1}{\beta}}}\psi(\|\overline{x}\|_{PC_{1-\nu}}) \\ &\quad + \sum_{i=1}^{k}\frac{M(b-t_i)^{1-\nu}(t-t_k)^{1-\nu}}{\Gamma(p(1-q)+q)}(d_i\|x(t_i^-)\| + \|G_i(t_i,0)\|)\Big] \\ &\quad + \sum_{i=1}^{k}\frac{M(b-t_i)^{1-\nu}(t-t_k)^{1-\nu}}{\Gamma(p(1-q)+q)}\big(d_i\|x(t_i^-)\| + \|G_i(t_i,0)\|\big) \\ &\leqslant \frac{M}{\Gamma(p(1-q)+q)}\|x_0\| + \frac{Mb^{1-\nu}\Theta}{\Gamma(q)}\|H\|_{L^{\frac{1}{\beta}}}\psi(r) + \frac{M^2M_B^2b^{2q-1}}{a\Gamma^2(q)(2q-1)} \\ &\quad \times \Big[b^{1-\nu}\|x_1\| + \frac{M}{\Gamma(p(1-q)+q)}\|x_0\| + \frac{Mb^{1-\nu}\Theta\|H\|_{L^{\frac{1}{\beta}}}}{\Gamma(q)}\psi(r) \\ &\quad + \sum_{i=1}^{k}\frac{Mb^{2(\nu-1)}\big(d_i\|x\| + \|G_i(t_i,0)\|\big)}{\Gamma(p(1-q)+q)}\Big] + \sum_{i=1}^{k}\frac{Mb^{2(\nu-1)}\big(d_i\|x\| + \|G_i(t_i,0)\|\big)}{\Gamma(p(1-q)+q)}.\end{aligned}$$

Dividing both sides by r and taking $r \to \infty$, we obtain

$$\frac{Mb^{1-\nu}\Theta}{\Gamma(q)}\|H\|_{L^{\frac{1}{\beta}}}\Lambda\left[1 + \frac{M^2M_B^2b^{2q-1}}{a\Gamma^2(q)(2q-1)}\right] \geqslant 1,$$

which is a contradiction to (H_4). Thus there exists r_0 such that Φ maps B_{r_0} into itself.

Step 3. $\Phi(x)$ is closed for every $x \in PC_{1-\nu}(J, X)$.

Indeed, for every given $x \in PC_{1-\nu}(J,X)$, let $\{\omega_n\}_{n\geqslant 0} \subset \Phi(x)$ such that as $n \to \infty$, $\omega_n \to \omega \in PC_{1-\nu}(J,X)$. Then there exists $f_n \in S_{F,x}$ such that for each $t \in (t_k, t_{k+1}]$,

$$\begin{aligned}\omega_n(t) = S_{p,q}(t)x_0 + \int_0^t K_q(t-s)f_n(s)ds + \sum_{i=1}^{k} S_{p,q}(t-t_i)G_i(t_i, x(t_i^-)) + \int_0^t K_q(t-s)BB^*K_q^*(b-s) \\ \times R(a,\Gamma_0^b)\Big[x_1 - S_{p,q}(b)x_0 - \int_0^b K_q(b-s)f_n(s)ds - \sum_{i=1}^{k} S_{p,q}(b-t_i)G_i(t_i,x(t_i^-))\Big]ds.\end{aligned}$$

From [8], we know that $S_{F,x}$ is weakly compact in $L^1(J,X)$, which implies that f_n converges weakly to some $f \in S_{F,x}$. Therefore as $n \to \infty$,

$$\begin{aligned}\omega_n(t) \to \omega(t) = S_{p,q}(t)x_0 + \int_0^t K_q(t-s)f(s)ds + \sum_{i=1}^{k} S_{p,q}(t-t_i)G_i(t_i, x(t_i^-)) + \int_0^t K_q(t-s)BB^*K_q^*(b-s) \\ \times R(a,\Gamma_0^b)\Big[x_1 - S_{p,q}(b)x_0 - \int_0^b K_q(b-s)f(s)ds - \sum_{i=1}^{k} S_{p,q}(b-t_i)G_i(t_i,x(t_i^-))\Big]ds.\end{aligned}$$

Thus we prove that $\omega \in \Phi(x)$.

Step 4. $\Phi(x)$ is u.s.c. and condensing.

We decompose $\Phi = \Phi_1 + \Phi_2$, where the operators Φ_1 and Φ_2 are defined by

$$\Phi_1(t) = \begin{cases} 0, & t \in (0,t_1], \\ \sum_{i=1}^{k} S_{p,q}(t-t_i)G_i(t_i,x(t_i^-)), & t \in (t_k,t_{k+1}], \quad k=1,2,\cdots,m, \end{cases}$$

$$\Phi_2 = S_{p,q}(t)x_0 + \int_0^t K_q(t-s)[Bu(s)+f(s)]ds, \quad f \in S_{F,x}, \quad t \in J\setminus\{t_1,t_2,\cdots,t_m\}.$$

According to [15], we if and only if show that Φ_1 is a contraction operator, while Φ_2 is a completely continuous operator.

Let us begin proving that Φ_1 is a contraction operator. For any $x, y \in X$, $t \in (t_k, t_{k+1}]$, we obtain

$$\begin{aligned}(t-t_k)^{1-\nu}\|(\Phi_1 x)(t) - (\Phi_1 y)(t)\| &\leqslant (t-t_k)^{1-\nu}\sum_{i=1}^{k}\|S_{p,q}(t-t_i)\|\cdot\|G_i(t_i,x(t_i^-)) - G_i(t_i,y(t_i^-))\| \\ &\leqslant \frac{M\sum_{i=1}^{k} d_i(t_i-t_{i-1})^{\nu-1}}{\Gamma(p(1-q)+q)}\|x-y\|_{PC_{1-\nu}}.\end{aligned}$$

Thus, Φ_1 is a contraction by assumption (H_3).

Next, we prove that Φ_2 is u.s.c. and completely continuous. We subdivide the proof into three claims.

Claim 1. Φ_2 maps bounded sets into uniformly bounded sets in $PC_{1-\nu}$, i.e. there exists a positive constant r_1 such that $\Phi_2(B_{r_1}) \subset B_{r_1}$.

By employing the technique used in Step 2, one can easily show that there exists $r_1 > 0$ such that $\Phi_2(B_{r_1}) \subset B_{r_1}$.

Claim 2. $\Phi_2(B_r)$ is a family of equicontinuous functions. The equicontinuity of

$$\{S_{p,q}(t)x_0 \mid t \in J/\{t_1,t_2,\cdots,t_m\}\},$$

can be shown using the fact of $S_{p,q}(\cdot)$ is continuous.

Now we only need to check the equicontinuity of the second term in Φ_2.

Denote $E = \{y \in PC_{1-\nu}(J,X) : y(t) = t^{1-\nu}\Phi_2(x)(t), y(0) = y(0^+), x \in \overline{B}_r\}$, for $t' = 0$, $0 < t'' \leqslant t_1$, we can easily get $\|y(t'')-y(t')\|\to 0$, as $t''\to 0$. For $0<t'<t''\leqslant t_1$, for each $x\in\overline{B}_r$, there exists $f\in S_{F,x}$ such that

$$\begin{aligned}
\|y(t'')-y(t')\| &\leqslant (t'')^{1-\nu}\Big\|\int_{t'}^{t''}(t''-s)^{q-1}P_q(t''-s)f(s)ds\Big\| \\
&\quad+\Big\|\int_0^{t'}[(t'')^{1-\nu}(t''-s)^{q-1}-(t')^{1-\nu}(t'-s)^{q-1}]P_q(t''-s)f(s)ds\Big\| \\
&\quad+(t')^{1-\nu}\Big\|\int_0^{t'}(t'-s)^{q-1}[P_q(t''-s)-P_q(t'-s)]f(s)ds\Big\| \\
&\quad+(t'')^{1-\nu}\Big\|\int_{t'}^{t''}(t''-s)^{q-1}P_q(t''-s)Bu(s)ds\Big\| \\
&\quad+\Big\|\int_0^{t'}[(t'')^{1-\nu}(t''-s)^{q-1}-(t')^{1-\nu}(t'-s)^{q-1}]P_q(t''-s)Bu(s)ds\Big\| \\
&\quad+(t')^{1-\nu}\Big\|\int_0^{t'}(t'-s)^{q-1}[P_q(t''-s)-P_q(t'-s)]Bu(s)ds\Big\| \\
&\leqslant \frac{M(t'')^{1-\nu}}{\Gamma(q)}\Big\|\int_{t'}^{t''}(t''-s)^{q-1}f(s)ds\Big\| \\
&\quad+\frac{M}{\Gamma(q)}\Big\|\int_0^{t'}[(t'')^{1-\nu}(t''-s)^{q-1}-(t')^{1-\nu}(t'-s)^{q-1}]f(s)ds\Big\| \\
&\quad+(t')^{1-\nu}\Big\|\int_0^{t'}(t'-s)^{q-1}[P_q(t''-s)-P_q(t'-s)]f(s)ds\Big\| \\
&\quad+\frac{M(t'')^{1-\nu}}{\Gamma(q)}\Big\|\int_{t'}^{t''}(t''-s)^{q-1}Bu(s)ds\Big\| \\
&\quad+\frac{M}{\Gamma(q)}\Big\|\int_0^{t'}[(t'')^{1-\nu}(t''-s)^{q-1}-(t')^{1-\nu}(t'-s)^{q-1}]Bu(s)ds\Big\| \\
&\quad+(t')^{1-\nu}\Big\|\int_0^{t'}(t'-s)^{q-1}[P_q(t''-s)-P_q(t'-s)]Bu(s)ds\Big\| \\
&=\sum_{i=1}^{6}I_i,
\end{aligned}$$

where

$$\begin{aligned}
I_1&=\frac{Mb^{1-\nu}b_1(t''-t')^{q-\beta}}{\Gamma(q)}\psi(r)\|H\|_{L^{\frac{1}{\beta}}}, \\
I_2&=\frac{M}{\Gamma(q)}\Big\|\int_0^{t'}[(t'')^{1-\nu}(t''-s)^{q-1}-(t')^{1-\nu}(t'-s)^{q-1}]f(s)ds\Big\|, \\
I_3&=(t')^{1-\nu}\Big\|\int_0^{t'}(t'-s)^{q-1}[P_q(t''-s)-P_q(t'-s)]f(s)ds\Big\|, \\
I_4&=\frac{MM_B(t'')^{1-\nu}}{\Gamma(q)}\Big\|\int_{t'}^{t''}(t''-s)^{q-1}u(s)ds\Big\|, \\
I_5&=\frac{M}{\Gamma(q)}\Big\|\int_0^{t'}[(t'')^{1-\nu}(t''-s)^{q-1}-(t')^{1-\nu}(t'-s)^{q-1}]Bu(s)ds\Big\|, \\
I_6&=(t')^{1-\nu}\Big\|\int_0^{t'}(t'-s)^{q-1}[P_q(t''-s)-P_q(t'-s)]Bu(s)ds\Big\|.
\end{aligned}$$

Since $\beta \in (0, q)$, we have $q-\beta > 0$, $I_1 \to 0$ as $t' \to t''$. Noting that

$$(t')^{1-\nu}(t'-s)^{q-1} - (t'')^{1-\nu}(t''-s)^{q-1} \leqslant (t')^{1-\nu}(t'-s)^{q-1},$$

then by Lebesgue dominated convergence theorem, we derive that $I_2, I_5 \to 0$ as $t''-t' \to 0^+$. From the strong continuity of $\{P_q(t) : t \geqslant 0\}$, there exists a $\delta > 0$ such that $|t''-t'| < \delta$ and $\|P_q(t'') - P_q(t')\| < \tau$, so

$$I_3 \leqslant (t')^{(1-\nu+q-\beta)} b_1 \tau \|\psi(r)\| \cdot \|H\|_{L^{\frac{1}{\beta}}} \to 0, \quad \text{as } \delta \to 0,$$

and

$$I_6 \leqslant (t')^{1-\nu} \tau M_B \int_0^{t'} (t'-s)^{q-1} u(s) ds \to 0, \quad \text{as } \delta \to 0.$$

Note that

$$\begin{aligned} I_4 &\leqslant \frac{MM_B(t'')^{1-\nu}}{\Gamma(q)} \Big\| \int_{t'}^{t''} (t''-s)^{q-1} u(s) ds \Big\| \\ &\leqslant \frac{MM_B(t'')^{1-\nu}}{\Gamma(q)} \Big\| \int_{t'}^{t''} (t''-s)^{q-1} (b-s)^{q-1} B^* P_q^*(b-s) R(a, \Gamma_0^b) P(x(b)) ds \Big\| \\ &\leqslant \frac{M^2 M_B^2 (t'')^{1-\nu}}{a\Gamma^2(q)} \int_{t'}^{t''} (b-s)^{2(q-1)} \|P(x(b))\| ds \\ &\leqslant \frac{M^2 M_B^2 (t'')^{1-\nu}}{a\Gamma^2(q)} \frac{(t''-t')^{2q-1}}{2q-1} \|P(x(b))\| \to 0, \quad \text{as } t''-t' \to 0. \end{aligned}$$

Hence the right-hand side of the above inequality tends to zero independently of $x \in \overline{B}_r$. By recalling the relationship of E and $\Phi_2(\overline{B}_r)$, one can easily deduce that Φ_2 is equicontinuous on $\overline{B}_r$.

Claim 3. $V(t) = \{\omega(t), \omega \in \Phi_2(\overline{B}_r)\}$ is a relatively compact in X.

Let $0 < t \leqslant b$ be fixed, since

$$\begin{aligned} S_{p,q}(t)x_0 &= I_{0^+}^{p(1-q)} K_q(t) x_0 \\ &= \frac{1}{\Gamma(p(1-q))} \int_0^t (t-s)^{p(1-q)-1} s^{q-1} P_q(s) x_0 ds \\ &= \frac{q}{\Gamma(p(1-q))} \int_0^t (t-s)^{p(1-q)-1} s^{q-1} \int_0^\infty \theta M_q(\theta) S(s^q\theta) x_0 d\theta ds, \end{aligned}$$

let $x \in \overline{B}_r$ and $\omega \in \Phi_2(\overline{B}_r)$, then for all $\eta \in (0, t)$ and for all $\delta > 0$, define an operator

$$\begin{aligned} \omega^{\lambda,\delta}(t) = & \frac{q}{\Gamma(p(1-q))} \int_0^{t-\lambda} \int_\delta^\infty \theta M_q(\theta) (t-s)^{p(1-q)-1} s^{q-1} S(s^q\theta) x_0 d\theta ds \\ & + qS(\lambda^q\theta) \int_0^{t-\lambda} \int_\delta^\infty \theta M_q(\theta) (t-s)^{q-1} S((t-s)^q\theta - \lambda^q\theta)[Bu(s) + f(s)] d\theta ds. \end{aligned}$$

From the compactness of $S(\lambda^q\theta)$, $\lambda^q\theta > 0$, we obtain the set $V^{\lambda,\delta}(t) = \{\omega^{\lambda,\delta}(t), \omega^{\lambda,\delta} \in \Phi_2^{\lambda,\delta}(x), x \in \overline{B}_r\}$ is relatively compact in X for all $\lambda \in (0, t)$ and $\delta > 0$.

Moreover, for each $x \in \overline{B}_r$, by using Holder's inequality, we have

$$\begin{aligned} \|\omega - \omega^{\lambda,\delta}\|_{PC_{1-\nu}} \leqslant & \sup (t-t_k)^{1-\nu} \frac{Mq}{\Gamma(p(1-q))} \Big[\Big\| \int_0^t (t-s)^{p(1-q)-1} s^{q-1} ds \int_0^\delta \theta M_q(\theta) x_0 d\theta \Big\| \\ & + \Big\| \int_t^{t-\lambda} (t-s)^{p(1-q)-1} s^{q-1} ds \int_\delta^\infty \theta M_q(\theta) x_0 d\theta \Big\| \Big] + q \sup (t-t_k)^{1-\nu} \end{aligned}$$

$$
\begin{aligned}
&\times \Big[\Big\|\int_0^t\int_0^\delta \theta M_q(\theta)(t-s)^{q-1}S((t-s)^q\theta)[f(s)+Bu(s)]d\theta ds\Big\|\\
&+\Big\|\int_t^{t-\lambda}\int_\delta^\infty \theta M_q(\theta)(t-s)^{q-1}S((t-s)^q\theta)[f(s)+Bu(s)]d\theta ds\Big\|\Big]\\
\leqslant\ & \frac{\Gamma(1+q)M\|x_0\|b^{1-\nu}}{\Gamma(p(1-q)+q)}\int_0^\delta \theta M_q(\theta)d\theta+\frac{M\|x_0\|b^{1-\nu}}{\Gamma(p(1-q))\Gamma(q)}\int_t^{t-\lambda}(t-s)^{p(1-q)-1}\\
&\times s^{q-1}ds+qM\Theta b^{1-\nu}\psi(r)\|H\|_{L^{\frac{1}{\beta}}}\int_0^\delta\theta M_q(\theta)d\theta+\frac{Mb^{1-\nu}}{\Gamma(q)}\psi(r)\|H\|_{L^{\frac{1}{\beta}}}\lambda^{q-\beta}\\
&+\frac{\beta M^2M_B^2b^{1-\nu+q}}{(2q-1)\Gamma(q)}\Big[\|x_1\|+\frac{M\|x_0\|b^{\nu-1}}{\Gamma(p(1-q)+q)}+\frac{M\Theta}{\Gamma(q)}\psi(r)\|H\|_{L^{\frac{1}{\beta}}}\\
&+\sum_{i=1}^k\frac{M(b-t_i)^{1-\nu}}{\Gamma(p(1-q)+q)}[d_i\|x(t_i^-)\|+\|G_i(t_i,0)\|]\Big]\int_0^\delta\theta M_q(\theta)d\theta\\
&+\frac{qM^2M_B^2b^{1-\nu}\lambda^{2q-1}}{a(2q-1)\Gamma^2(q)}\Big[\|x_1\|+\frac{M\|x_0\|b^{\nu-1}}{\Gamma(p(1-q)+q)}+\frac{M\Theta}{\Gamma(q)}\psi(r)\|H\|_{L^{\frac{1}{\beta}}}\\
&+\sum_{i=1}^k\frac{M(b-t_i)^{1-\nu}}{\Gamma(p(1-q)+q)}[d_i\|x(t_i^-)\|+\|G_i(t_i,0)\|]\Big]\\
=\ &\sum_{i=1}^6 K_i.
\end{aligned}
$$

Applying the absolute continuity of the Lebesgue integral, we can derive $K_2\to 0$ as $\lambda,\delta\to 0^+$. In addition, we can derive that $K_i\to 0, i=1,3,4,5,6$ as $\lambda,\delta\to 0^+$, where we have used the equality $\int_0^\delta\theta M_q(\theta)d\theta=\frac{1}{\Gamma(1+q)}$. Hence there are relatively compact sets arbitrarily close to the set $V(t)=\{\omega(t),\omega\in\Phi_2(x),x\in\overline{B}_r\}$, which implies $V(t)$ is also relatively compact in X by Arzelá-Ascoli theorem.

From Claims 1–3, we know that Φ_2 is a completely continuous multivalued map.

Step 5. $\Phi_2(x)$ has a closed graph.

Let $x^{(n)}\to x^*(n\to\infty)$, $\omega^{(n)}\to\omega^*(n\to\infty)$. We will prove that $\omega^*\in\Phi_2(x^*)$. Since $\omega^{(n)}\in\Phi_2(x^{(n)})$, there exists $f^{(n)}\in S_{F,x^{(n)}}$, such that for each $t\in J'$,

$$
\begin{aligned}
\omega^{(n)}(t)=S_{p,q}(t)x_0+\int_0^t K_q(t-s)f^{(n)}(s)ds+\int_0^t K_q(t-s)BB^*K_q^*(b-s)R(a,\Gamma_0^b)\\
\times\Big[x_1-S_{p,q}(b)x_0-\int_0^b K_q(b-s)f^{(n)}(s)ds-\sum_{i=1}^k S_{p,q}(b-t_i)G_i(t_i,x(t_i^-))\Big]ds.
\end{aligned}
$$

It remains to prove that the existence of $f^*\in S_{F,x^*}$ such that for each $t\in J'$,

$$
\begin{aligned}
\omega^*(t)=S_{p,q}(t)x_0+\int_0^t K_q(t-s)f^*(s)ds+\int_0^t K_q(t-s)BB^*K_q^*(b-s)\cdot R(a,\Gamma_0^b)\\
\times\Big[x_1-S_{p,q}(b)x_0-\int_0^b K_q(b-s)f^*(s)ds-\sum_{i=1}^k S_{p,q}(b-t_i)G_i(t_i,x(t_i^-))\Big]ds.
\end{aligned}
$$

Consider the linear continuous operator

$$\Gamma: L^{\frac{1}{\beta}}(J,X)\to PC_{1-\nu}(J,X),$$

where

$$(\Gamma f)(t)=\int_0^t K_q(t-s)\Big[f(s)-BB^*K_q^*(b-s)R(a,\Gamma_0^b)\int_0^b K_q(b-\tau)f(\tau)d\tau\Big]ds.$$

Obviously, it follows from Lemma 2.8 that $\Gamma \circ S_F$ is a closed graph operator. Since $\omega^{(n)} \to \omega^*(n \to \infty)$, we can get that as $n \to \infty$,

$$\Big\|\omega^{(n)}(t) - S_{p,q}(t)x_0 - \int_0^t K_q(t-s)BB^*K_q^*(b-s)R(a,\Gamma_0^b)\Big[x_1 - S_{p,q}(b)x_0 - \sum_{i=1}^k S_{p,q}(b-t_i) \cdot G_i(t_i, x(t_i^-))\Big]ds - \Big\{\omega^*(t) - S_{p,q}(t)x_0 - \int_0^t K_q(t-s)BB^*K_q^*(b-s)R(a,\Gamma_0^b) \times \Big[x_1 - S_{p,q}(b)x_0 - \sum_{i=1}^k S_{p,q}(b-t_i)G_i(t_i, x(t_i^-))\Big]ds\Big\}\Big\| \to 0.$$

Moreover we have

$$\omega^{(n)}(t) - S_{p,q}(t)x_0 - \int_0^t K_q(t-s)BB^*K_q^*(b-s)R(a,\Gamma_0^b)\Big[x_1 - S_{p,q}(b)x_0 - \sum_{i=1}^k S_{p,q}(b-t_i)G_i(t_i, x(t_i^-))\Big]ds \in \Gamma(S_{F,x^{(n)}}).$$

Since $x^{(n)} \to x^*$, it follows from Lemma 2.8 that

$$\begin{aligned}\omega^*(t) - S_{p,q}(t)x_0 - \int_0^t K_q(t-s)BB^*K_q^*(b-s)R(a,\Gamma_0^b)\Big[x_1 - S_{p,q}(b)x_0 - \sum_{i=1}^k S_{p,q}(b-t_i)G_i(t_i, x(t_i^-))\Big]ds \\ = \int_0^t K_q(t-s)\Big[f^*(s) - BB^*K_q^*(b-s)R(a,\Gamma_0^b)\int_0^b K_q(b-s)f^*(s)ds\Big]ds,\end{aligned}$$

for some $f^* \in S_{F,x}$, this shows that $\omega^* \in \Phi_2(x^*)$. Hence Φ_2 has a closed graph. Since Φ_2 is a completely continuous multivalued map with compact value, from Proposition 2.6, we get that Φ_2 is u.s.c., on the other hand, Φ_1 is a contraction and hence $\Phi = \Phi_1 + \Phi_2$ is u.s.c. and condensing. Thus from Lemma 2.9, we know operator Φ has a fixed point on B_r, which is a mild solution of system (1.1). This completes the proof. □

The following result concerns the approximate controllability of that problem (1.1). We assume that the following assumption be held.

(H_5) There exists a positive constant L such that $\|F(t, x(t))\| \leqslant L$, for all $(t, x) \in J \times X$.

Theorem 3.2. *Suppose that the hypotheses* (H_0)–(H_5) *are satisfied and the linear system* (2.6) *is approximately controllable on* J. *Then system* (1.1) *is approximately controllable on* J.

Proof. By employing the technique used in Theorem 3.1, we can easily show that for all $0 < a < 1$, the operator Φ has a fixed point in B_r, where $r = r(a)$. Let $x^e(\cdot)$ be a fixed point of Φ in B_r. Any fixed point of Φ is a mild solution of (1.1). This means that there exists $f^e \in S_{F,x^e}$ such that for each $t \in J'$,

$$\begin{aligned}x^e(t) = S_{p,q}(t)x_0 + \int_0^t K_q(t-s)f^e(s)ds + \sum_{i=1}^k S_{p,q}(t-t_i)G_i(t_i, x^e(t_i^-)) + \int_0^t K_q(t-s)BB^*K_q^*(b-s) \\ \times R(a,\Gamma_0^b)\Big[x_1 - S_{p,q}(b)x_0 - \int_0^b K_q(b-s)f^e(s)ds - \sum_{i=1}^k S_{p,q}(b-t_i)G_i(t_i, x^e(t_i^-))\Big]ds.\end{aligned}$$

Define

$$P(x^e) = x_1 - S_{p,q}(b)x_0 - \int_0^b K_q(b-s)f^e(s)ds - \sum_{i=1}^k S_{p,q}(b-t_i)G_i(t_i, x^e(t_i^-)).$$

Noting that $I-\Gamma_0^b R(a,\Gamma_0^b)=aR(a,\Gamma_0^b)$, we get

$$\begin{aligned}
x^e(b) &= S_{p,q}(b)x_0+\int_0^b K_q(b-s)f^e(s)ds+\sum_{i=1}^{k} S_{p,q}(b-t_i)G_i(t_i,x^e(t_i^-))+\int_0^b K_q(b-s)BB^*K_q^*(b-s)\\
&\quad \times R(a,\Gamma_0^b)\Big[x_1-S_{p,q}(b)x_0-\int_0^b K_q(b-s)f^e(s)ds-\sum_{i=1}^{k} S_{p,q}(b-t_i)G_i(t_i,x^e(t_i^-))\Big]ds\\
&= S_{p,q}(b)x_0+\int_0^b K_q(b-s)f^e(s)ds+\sum_{i=1}^{k} S_{p,q}(b-t_i)G_i(t_i,x^e(t_i^-))+\Gamma_0^b R(a,\Gamma_0^b)P(x^e)\\
&= S_{p,q}(b)x_0+\int_0^b K_q(b-s)f^e(s)ds+\sum_{i=1}^{k} S_{p,q}(b-t_i)G_i(t_i,x^e(t_i^-))+P(x^e)-aR(a,\Gamma_0^b)P(x^e)\\
&= x_1-aR(a,\Gamma_0^b)P(x^e).
\end{aligned}$$

In addition, by our assumption (H_5),

$$\int_0^b \|f^e(s)\|^2ds\leqslant L^2b.$$

Consequently, the sequence $\{f^e(s)\}$ is uniformly bounded in $L^{\frac{1}{\beta}}(J,X)$. Thus, there is a subsequence, still denoted by $\{f^e\}$, that converges weakly to, say f, in $L^{\frac{1}{\beta}}(J,X)$.

Denote

$$z=x_1-S_{p,q}(b)x_0-\int_0^b K_q(b-s)f(s)ds-\sum_{i=1}^{k} S_{p,q}(b-t_i)G_i(t_i,x(t_i^-)).$$

We deduce that

$$\|P(x^e)-z\|\leqslant\Big\|\int_0^b K_q(b-s)[f^e(s)-f(s)]ds\Big\|.$$

By the Ascoli-Arzelá theorem, we can show that the linear operator $\Delta\to\int_0^{\cdot} K_\beta(\cdot-s)\Delta(s)ds : L^{\frac{1}{\beta}}(J,X)\to PC_{1-\nu}(J,X)$ is compact, consequently the right-hand side of the upper formula tends to zero as $e\to 0^+$. This implies that as $e\to 0^+$,

$$\begin{aligned}
\|x^e(b)-x_1\| &= \|aR(a,\Gamma_0^b)P(x^e)\|\\
&\leqslant \|aR(a,\Gamma_0^b)(z)\|+\|aR(a,\Gamma_0^b)(P(x^e)-z)\|\\
&\leqslant \|aR(a,\Gamma_0^b)(z)\|+\|P(x^e)-z\|\to 0.
\end{aligned}$$

This proves the approximate controllability of system (1.1). □

4. An example

The partial differential system arises in the mathematical modeling of heat transfer

$$\begin{cases}
D_{0^+}^{q,\frac{2}{3}}x(t,s)\in\frac{\partial^2}{\partial y^2}x(t,s)+\tilde{F}(t,x(t,s))+\eta(t,s), & t\in(0,1]\backslash\{\frac{1}{2}\}, s\in[0,\pi],\\
\Delta I_{0^+}^{\frac{1}{3}(1-q)}x(\frac{1}{2},s)=\frac{|x(s)|}{2\pi+|x(s)|}, & s\in[0,\pi],\\
x(t,0)=x(t,\pi)=0, & t\in[0,1],\\
I_{0^+}^{\frac{1}{3}(1-q)}x(t,s)\mid_{t=0}=x_0(s)\in X, & s\in[0,\pi].
\end{cases}\tag{4.1}$$

To write the above system (4.1) into the abstract system (1.1), we choose the space $X=L^2([0,\pi],\mathbb{R})$ and define the operator $A : D(A) \subset X \to X$ by

$$\begin{aligned} Ay &= y'', \\ D(A) &= \{y \in X : y, y' \text{ are absolutely continuous, } y'' \in X,\ y(0) = y(\pi) = 0\}. \end{aligned}$$

Then, A can be written as

$$Ay = \sum_{k=1}^{\infty} n^2 (y, y_n) y_n, \qquad y \in D(A),$$

where $y_n(y) = \sqrt{\frac{2}{\pi}} \sin ny, (n = 1, 2, \cdots)$ is an orthonormal basis of X. It is well-known that A is the infinitesimal generator of a compact semigroup $S(t)(t \geqslant 0)$ in X given by

$$S(t)y = \sum_{k=1}^{\infty} e^{-n^2 t} (y, y_n) y_n, \ \ y \in X,$$

in particular, $S(\cdot)$ is a uniformly stable semigroup and

$$\|S(t)\| \leqslant e^{-1} < 1 = M.$$

Let $v(t)(s) = x(t, s), t \in J = [0, 1], s \in [0, \pi]$. Now for any $v \in X = L^2([0, \pi], \mathbb{R}), s \in [0, \pi]$, we define function $F : J \times X \to X$ and the bounded linear operator $B : U \to X$ respectively by $F(t, v(t))(s) = \tilde{F}(t, x(t, s)) = \frac{e^{-t}}{1+e^{-t}} \sin(x(t, s))$ and $Bu(t)(s) = \eta(t, s), 0 < s < \pi$, where $\eta : J \times [0, \pi] \to [0, \pi]$ is continuous in t.

Therefore, (4.1) can be reformulated as the abstract system (1.1). Obviously, $\tilde{F}(t, x(t, s))$ is uniformly bounded. On the other hand, assume that the linear fractional differential system corresponding to (4.1) is approximately controllable and other assumptions of Theorem 3.2 hold. Thus, all conditions of Theorem 3.2 are satisfied. Hence the Hilfer fractional differential inclusion with impulsive (4.1) is approximately controllable (but not exactly controllable since the associated C_0-semigroup $S(t)$ is compact) on $[0, 1]$.

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