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Convergence analysis of a novel iteration algorithm for solving split feasibility problems

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Abstract

In this paper, our aim is to construct a convergence theorem in Banach spaces via the following Ishikawa recursive algorithm

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n x_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings. Moreover, we also apply these results to solve a split feasibility problem.

Keywords: Split feasibility problem, fixed point, nonexpansive mapping, weak convergence.
2010 MSC: 47J25, 47H45.

1. Introduction and preliminaries

Throughout this paper, we always assume that E is a real Banach space and H is a real Hilbert space, respectively. Let C and Q (C_n and Q_n, $n = 0, 1, 2, \cdots$) denote the nonempty closed convex subsets of the Hilbert spaces H_1 and H_2. Let T be a self-mapping of C. Recall that T is said to be a nonexpansive mapping, if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

Here $F(T)$ denotes the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We use $\rightharpoonup$ ($\to$) to denote weak (strong) convergence, $\omega_w(x_n) = \{x : \exists x_{n_k} \rightharpoonup x\}$ to denote the w-limit set of $\{x_n\}$.

On the fixed point problems of the nonexpansive mappings which is an important class of nonlinear mappings, there are many interesting convergence results during the past decades, see [7, 21, 31] and the references therein.

Krasnosel'skii [15] and Mann [17] used the following algorithm which is now called the K-M algorithm

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \tag{1.1}$$

where $\alpha_n \subset [0, 1]$ and the initial point $x_0 \in C$ have no restrictions.

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In 1979, Reich [24] proved that the sequence defined by (1.1) converges weakly to $q \in F(T)$, if E is a uniformly convex Banach space with a Fréchet differentiable norm, $T : C \to C$ is a nonexpansive self-mapping with $F(T) \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$.

In 2011, Zhang et al. [29] proposed modified Halpern and Ishikawa iteration algorithms for solving the fixed points of nonexpansive mappings in Banach spaces. For the convergence of modified Halpern and Ishikawa iterative algorithms, we refer authors to [8, 9, 20, 28] for more details. In 2016, Hieu et al. [11] introduced three parallel hybrid extragradient methods and obtained the set of fixed points of nonexpansive mappings in a real Hilbert space.

The split feasibility problem (SFP) is to find a point

$$x \in C, \text{ such that } Ax \in Q, \tag{1.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator. Censor and Elfving [5] first introduced the SFP in a Hilbert space. Recently, SFP which attracts attentions of many researchers, has been widely used in many applications such as signal processing and other fields, see [3, 4, 10, 14] and the references therein.

It has been proved that if the SFP (1.2) has a solution, it is not hard to find a solution x^* to (1.2) is equivalent to a fixed point equation

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*. \tag{1.3}$$

In order to solve the SFP (1.2), Byren [3] proposed the popular CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad n \geqslant 0, \tag{1.4}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

As we know, the CQ algorithm (1.4) is a special case of the $K-M$ algorithm (1.1) (see [27]). Due to the fixed point formulation (1.3) of the SFP (1.2), we can apply the $K-M$ algorithm (1.1) to the operator

$$P_C(I - \gamma A^*(I - P_Q)A),$$

to produce a sequence $\{x_n\}$ given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad n \geqslant 0, \tag{1.5}$$

where $\gamma \in (0, 2/\lambda)$ and again λ being the special radium of the operator A^*A. Then we can see that as long as $\{\alpha_n\}$ satisfies condition $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, we have weak convergence of the sequence $\{x_n\}$ generated by the algorithm (1.5). If possible errors are taken into consideration, then we should study perturbations of the closed convex sets C and Q. For example, Zhao and Yang [30] considered the following perturbed algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_{C_n}(x_n - \gamma A^*(I - P_{Q_n})Ax_n),$$

where $\{C_n\}$ and $\{Q_n\}$ are sequences of closed convex subsets of H_1 and H_2, respectively, which converges to C and Q, respectively, in the sense of Mosco [1]. This is a motivation for the authors to study the following more general algorithm which generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n, \tag{1.6}$$

where $\{T_n\}$ is a sequence of nonexpansive mapping in Hilbert space H.

In 2005, under certain conditions, Zhao and Yang [30] studied the convergence of (1.6) in a finite-dimensional Hilbert space.

Theorem 1.1. *Let T and T_n be nonexpansive operators in Hilbert space H for $k = 0, 1, 2, \cdots$, $T_n \to T$ and $\{\alpha_n\} \subset (0, 1)$ satisfying*

(i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n D_\rho(T_n, T) < \infty$ *for any given* $\rho > 0$, *where* $D_\rho(T_n, T) = \sup\{\|T_n x - Tx\| : \|x\| \leqslant \rho\}$.

Then the sequence $\{x_n\}$ *defined by* (1.6) *converges weakly to a fixed point of* T.

Remark 1.2. In [30, page 1794], the $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ weakly converges to a fixed point z of T do not imply that $\liminf_{j\to\infty} \|x_{n_j} - z\| = 0$, unless the space is finite dimensional.

In 2006, Xu [26] extended Zhao and Yang [30] from finite dimensional Hilbert spaces to infinite dimensional Banach spaces and they obtained the following result.

Theorem 1.3. *Assume that* X *is a uniformly convex Banach space which has a Fréchet differentiable norm. Let* T *be a nonexpansive operator in the Banach space* X, $F(T)$ *is the set of fixed points and* $F(T)$ *is nonempty. Let* $\{T_n\}$ *be a sequence of nonexpansive mappings on* C. *If assumptions* (i) *and* (ii) *of Theorem* 1.1 *are satisfied, then the sequence* $\{x_n\}$ *generated by the algorithm* (1.6) *converges weakly to a fixed point of* T.

Recently, Qu et al. [22] and Moudafi [18] studied the split feasibility problem by the relaxed alternating CQ-algorithm and CQ-like algorithms. In 2014, [6] present weak and strong convergence theorems of solutions to a split feasibility problem for a family of nonspreading-type mapping in Hilbert spaces.

For each $x_0 \in C$, the iteration sequence $\{x_n\}$ is called an Ishikawa iteration scheme, if

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n. \end{cases}$$

The Ishikawa iteration scheme was introduced by Ishikawa [12] and he proved that the sequence generated by this algorithm must converge to a fixed point of a Lipschitzian pseudo-contractive mapping in a convex compact subset of Hilbert spaces. After that, lots of authors studied the Ishikawa (two-step) iteration algorithm for solving the zero points of nonlinear operators, the equilibrium problems, the variational inequalities problems in Hilbert spaces and Banach spaces, see [13, 16, 19, 23] and the references therein.

In this paper, motivated by Zhao and Yang [30], Xu [26] and the above works, we proposed the following Ishikawa iteration algorithm, given $x_0 \in C$

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n x_n, \end{cases} \tag{1.7}$$

where $0 \leqslant \alpha_n, \beta_n \leqslant 1$ and $\{T_n\}$ is a sequence of nonexpansive mappings. We show that the sequence $\{x_n\}$ weakly converges to a fixed point of T. We also apply this result to solve the SFP (1.2) via the following iteration algorithm

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_{C_n}(y_n - \gamma A^*(I - P_{Q_n})Ay_n), \\ y_n = (1-\beta_n)x_n + \beta_n P_{C_n}(x_n - \gamma A^*(I - P_{Q_n})Ax_n). \end{cases} \tag{1.8}$$

We show that $\{x_n\}$ weakly converges to a solution of the SFP (1.2).

The aim of this paper is to present the above Ishikawa algorithm to solve the SFP, these results mainly improve the exited results in Zhao et al. [30], Xu [26] and Qu et al. [22]. Specifically, we list the following highlights:

- The results extend and improve the corresponding results from finite dimensional Hilbert spaces to infinite dimensional Banach spaces.
- The conditions in this paper are much mild. Indeed, we remove the assumptions in [30, Theorem 2.2] that the sequence $\{C_n\}$ and $\{Q_n\}$ Mosco converge to C and Q, respectively.

- Our results extend the K-M algorithm to the Ishikawa algorithm.
- Our algorithm is efficient for solving the SFP.

In order to get our main results, we need the following preliminaries.

Definition 1.4. An operator $S : H \to H$ is called an averaged operator, if it can be shown as the following combining form:

$$S = (1-\alpha)I + \alpha T,$$

where I is the identity operator and $T : H \to H$ is a nonexpansive operator and $\alpha \in (0, 1)$.

As an special case, if $\alpha = 1/2$, the projections are averaged operators.

Definition 1.5. If T is an operator with domain $D(T)$ and range $R(T)$ in H.

(i) T is called monotone, if

$$\langle x - y, Tx - Ty\rangle \geqslant 0, \quad \forall x, y \in D(T).$$

(ii) For a real number $\nu > 0$, T is called to be ν-inverse strongly monotone (ν-ism) (or co-coercive), if it satisfies the following inequality

$$\langle x - y, Tx - Ty\rangle \geqslant \nu\|Tx - Ty\|^2.$$

So, we can easily get the following conclusions.

(i) If T is nonexpansive, then $I - T$ is monotone and a projection P_K is 1-ism.

(ii) T is averaged $\Leftrightarrow$ the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$.

The following lemma is trivial.

Lemma 1.6. *Let $\{\mu_n\}$ and $\{\nu_n\}$ be nonnegative sequences satisfying $\sum_{n=0}^{\infty} \mu_n < \infty$ and $\nu_{n+1} \leqslant \nu_n + \mu_n$, $n = 0, 1, \cdots$. Then $\{\nu_n\}$ is a convergent sequence.*

2. Main results

Theorem 2.1. *Let E be a real uniformly convex Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be nonexpansive mapping and $\{T_n\}$ be a sequence of nonexpansive mappings on C. Let $\{x_n\}$ be defined in (1.7), where $0 \leqslant \alpha_n, \beta_n \leqslant 1$ satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\quad \sum_{n=0}^{\infty} \alpha_n\beta_n < \infty$, $\quad \lim_{n\to\infty} \beta_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n D_\rho(T_n, T) < \infty$, for every $\rho > 0$, where

$$D_\rho(T_n, T) = \sup\{\|T_n x - Tx\| : \|x\| \leqslant \rho\}.$$

Then $\{x_n\}$ converges weakly to a fixed point P of T

Proof. First we show that $\{x_n\}$ is bounded.
Take $z \in F(T)$, it follows that

$$\begin{aligned} \|x_{n+1} - z\| &\leqslant (1-\alpha_n)\|x_n - z\| + \alpha_n\|T_n y_n - T_n z\| + \alpha_n\|T_n z - Tz\| \\ &\leqslant (1-\alpha_n)\|x_n - z\| + \alpha_n\|y_n - z\| + \alpha_n\|T_n z - Tz\|. \end{aligned} \tag{2.1}$$

Similarly, we have

$$\begin{aligned} \|y_n - z\| &\leqslant (1-\beta_n)\|x_n - z\| + \beta_n\|T_n x_n - T_n z\| + \beta_n\|T_n z - Tz\| \\ &\leqslant \|x_n - z\| + \beta_n\|T_n z - Tz\|. \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}\|x_{n+1}-z\| &\leqslant \|x_n-z\|+\alpha_n(1+\beta_n)\|T_n z-Tz\|\\ &\leqslant \|x_n-z\|+2\alpha_n D_{\|z\|}(T_n,T).\end{aligned}$$

By condition (ii), we see that $\lim\limits_{n\to\infty}\|x_n-z\|$ exists. Hence, $\{x_n\}$ is bounded, so $\{T_n x_n\}$ and $\{Tx_n\}$ are bounded, too.

Next, we claim that, $\|x_n-Tx_n\|\to 0$ as $n\to\infty$.

Let $\rho=\sup\{\|x_n\|,\|T_n x_n\|: n\geqslant 0\}<\infty$ and let $r=\rho+\|z\|+2\sup\{\alpha_n D_\rho(T_n,T)\}$.

Now since E is uniformly convex, there exists a continuous strictly convex function φ with $\varphi(0)=0$ by [25]. Hence, we have

$$\|\lambda x+(1-\lambda)y\|^2\leqslant \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\varphi(\|x-y\|), \tag{2.3}$$

for all $x,y\in E$ such that $\|x\|\leqslant r$ and $\|y\|\leqslant r$ and for all $\lambda\in[0,1]$. In particular, setting $e_n=T_n y_n-Ty_n$ (note that $\|e_n\|\leqslant D_\rho(T_n,T)$) and a constant M_1 so that, $M_1\geqslant\sup\{2\|x_n-z\|+\alpha_n\|e_n\|: n\geqslant 0\}$. By (2.3) and condition (i), we have.

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|(1-\alpha_n)(x_n-z+\alpha_n e_n)+\alpha_n(Ty_n-z+\alpha_n e_n)\|^2\\ &\leqslant (1-\alpha_n)\|x_n-z+\alpha_n e_n\|^2+\alpha_n\|Ty_n-z+\alpha_n e_n\|^2\\ &\quad -\alpha_n(1-\alpha_n)\varphi(\|x_n-Ty_n\|)\\ &\leqslant (1-\alpha_n)(\|x_n-z\|^2+2\alpha_n\|x_n-z\|\|e_n\|+\alpha_n^2\|e_n\|^2)\\ &\quad +\alpha_n(\|Ty_n-z\|^2+2\alpha_n\|e_n\|\|Ty_n-z\|+\alpha_n^2\|e_n\|^2)\\ &\quad -\alpha_n(1-\alpha_n)\varphi(\|x_n-Ty_n\|)\\ &\leqslant \|x_n-z\|^2+M_1\alpha_n D_\rho(T_n,T)-\alpha_n(1-\alpha_n)\varphi(\|x_n-Ty_n\|).\end{aligned}$$

It follows that

$$\alpha_n(1-\alpha_n)\varphi(\|x_n-Ty_n\|)\leqslant \|x_n-z\|^2-\|x_{n+1}-z\|^2+M_1\alpha_n D_\rho(T_n,T). \tag{2.4}$$

Since $\lim\limits_{n\to\infty}\|x_n-z\|$ exists, condition (i) and (2.4) imply that

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\varphi(\|x_n-Ty_n\|)<\infty,$$

which further implies that $\liminf\limits_{n\to\infty}\varphi(\|x_n-Ty_n\|)=0$. Hence,

$$\liminf_{n\to\infty}\|x_n-Ty_n\|)=0.$$

It follows from (1.7) that

$$\|x_n-y_n\|=\beta_n\|x_n-T_n x_n\|\to 0.$$

Then

$$\liminf_{n\to\infty}\|x_n-Tx_n\|=0. \tag{2.5}$$

Since $\{x_n\}$ and $\{T_n x_n\}$ are bounded, there exists a constant M_2 satisfying $2\|T_n x_n-x_n\|\leqslant M_2$. Hence, we have

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &= \|(1-\alpha_n)x_n+\alpha_n T_n y_n-Tx_{n+1}\|\\ &= \|(1-\alpha_n)x_n+\alpha_n T_n y_n-Tx_n+Tx_n-Tx_{n+1}\|\\ &\leqslant (1-\alpha_n)\|x_n-Tx_n\|+\alpha_n\|T_n y_n-Tx_n\|\end{aligned}$$

$$
\begin{aligned}
&+\|x_{n+1}-x_n\| \\
&=(1-\alpha_n)\|x_n-Tx_n\|+\alpha_n\|T_ny_n-Tx_n\| \\
&\quad+\alpha_n\|x_n-T_ny_n\| \\
&\leqslant (1-\alpha_n)\|x_n-Tx_n\|+\alpha_n\|T_ny_n-Tx_n\| \\
&\quad+\alpha_n\|x_n-Tx_n\|+\alpha_n\|Tx_n-T_ny_n\| \\
&=\|x_n-Tx_n\|+2\alpha_n\|T_ny_n-Tx_n\| \\
&\leqslant \|x_n-Tx_n\|+2\alpha_n\|T_ny_n-Ty_n\|+2\alpha_n\|y_n-x_n\| \\
&\leqslant \|x_n-Tx_n\|+2\alpha_nD_\rho(T_n,T)+2\alpha_n\beta_n\|T_nx_n-x_n\| \\
&\leqslant \|x_n-Tx_n\|+2\alpha_nD_\rho(T_n,T)+\alpha_n\beta_nM_2.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}\alpha_nD_\rho(T,T_n)<\infty$, $\sum_{n=1}^{\infty}\alpha_n\beta_nM_2<\infty$, by Lemma 1.6, we obtain

$$\lim_{n\to\infty}\|x_n-Tx_n\| \text{ exists.}$$

This together with (2.5) implies that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

The demiclosedness principle for nonexpansive mappings (see [2]) implies that

$$\omega_w(x_n)\subset F(T).$$

To prove that $\{x_n\}$ is weakly convergent to a fixed point of T, it now suffices to prove that $\omega_w(x_n)$ consists of exactly one point.

Indeed, there are $\bar{x},\tilde{x}\in\omega_w(x_n)$ $(x_{n_i}\rightharpoonup\bar{x},\ \ x_{m_j}\rightharpoonup\tilde{x})$. Note that $\lim_{n\to\infty}\|x_n-\bar{x}\|$ and $\lim_{n\to\infty}\|x_n-\tilde{x}\|$ exist. If $\tilde{x}\neq\bar{x}$, then

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n-\tilde{x}\|^2&=\lim_{j\to\infty}\|(x_{m_j}-\bar{x})+(\bar{x}-\tilde{x})\|^2\\
&=\lim_{j\to\infty}\|x_{m_j}-\bar{x}\|^2+\|\bar{x}-\tilde{x}\|^2\\
&>\lim_{i\to\infty}\|x_{n_i}-\bar{x}\|^2\\
&=\lim_{i\to\infty}\|(x_{n_i}-\tilde{x})+(\tilde{x}-\bar{x})\|^2\\
&=\lim_{i\to\infty}\|x_{n_i}-\tilde{x}\|^2+\|\tilde{x}-\bar{x}\|^2\\
&>\lim_{i\to\infty}\|x_{n_i}-\tilde{x}\|^2\\
&=\lim_{n}\|x_n-\tilde{x}\|^2.
\end{aligned}
$$

This is a contradiction. The proof is completed. □

Corollary 2.2. *Let C be a closed convex subset of a Hilbert space H. Assume that $T:C\to C$ is a nonexpansive mapping such that $F(T)\neq\emptyset$. Assume also that $\{T_n\}$ is a sequence of nonexpansive mappings on C. Let the sequence $\{x_n\}$ be defined by (1.7), where $0\leqslant\alpha_n,\beta_n\leqslant 1$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)=\infty$, $\sum_{n=0}^{\infty}\alpha_n\beta_n<\infty$, $\lim_{n\to\infty}\beta_n=0$;

(ii) $\sum_{n=0}^{\infty}\alpha_nD_\rho(T_n,T)<\infty$, *for every $\rho>0$, where*

$$D_\rho(T_n,T)=\sup\{\|T_nx-Tx\|:\|x\|\leqslant\rho\}.$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

Remark 2.3. Theorem 2.1 extends Theorem 2.1 of Zhao and Yang [30] from finite dimensional Hilbert spaces to infinite dimensional Banach spaces. Corollary 2.2 extends Theorem 2.1 of Zhao and Yang [30] from the K-M algorithm to the Ishikawa algorithm.

Below we show the applications of algorithm (1.7) to the split feasibility problem.

We now apply Corollary 2.2 to the SFP (1.2).

Recall that ρ-distance between two closed convex subsets E_1 and E_2 of a Hilbert space H is defined by

$$d_\rho(E_1, E_2) = \sup_{\|x\| \leqslant \rho} \|P_{E_1}x - P_{E_2}x\|.$$

Theorem 2.4. *Assume that the sequence $\{x_n\}$ is generated by the perturbed averaging CQ algorithm (1.8), the sequences $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$ satisfy the conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty, \quad \sum_{n=0}^{\infty} \alpha_n\beta_n < \infty, \quad \lim_{n\to\infty} \beta_n = 0;$

(ii) $\sum_{n=0}^{\infty} \alpha_n d_\rho(C_n, C) < \infty$ *and* $\sum_{n=0}^{\infty} \alpha_n d_\rho(Q_n, Q) < \infty$, $\forall \rho > 0$.

Then $\{x_n\}$ converges weakly to a solution of the SFP (1.2).

Proof. Set $T = P_C(I-\gamma A^*(I-P_Q)A)$ and $T_n = P_{C_n}(I-\gamma A^*(I-P_{Q_n})A)$. Then T and T_n are nonexpansive with $\gamma < \frac{2}{\|A\|^2}$. Indeed, write

$$U = A^*(I-P_Q)A,$$

and

$$S = P_C(I-\gamma U.$$

Since P_Q and $I-P_Q$ are 1-ism, we calculate

$$\begin{aligned}
\langle x-y, Ux-Uy\rangle &= \langle x-y, A^*(I-P_Q)Ax - A^*(I-P_Q)Ay\rangle \\
&= \langle Ax-Ay, (I-P_Q)Ax-(I-P_Q)Ay\rangle \\
&\geqslant \|(I-P_Q)Ax-(I-P_Q)Ay\|^2 \\
&\geqslant \frac{1}{\|A\|^2}\|Ux-Uy\|^2.
\end{aligned}$$

Hence, U is $\frac{1}{\|A\|^2}$-inverse strongly monotone, which implies that γU is $\frac{1}{\gamma\|A\|^2}$-ism, which in turn implies that $I-\gamma U$ is averaged for $\|A\|^2\gamma < 2$, i.e. $\gamma < \frac{2}{\|A\|^2}$.

Hence, we get that $S = P_C(I-\gamma U)$ is averaged. Then $S = P_C(I-\gamma U)$ is nonexpansive, so are $S_n = P_{C_n}(I-\gamma U_n)$. Since the SFP (1.2) is consistent, $F(T)$ is nonempty. Note that $F(T)$ is the solution set of the SFP (1.2). Also the perturbed averaging CQ algorithm (1.8) can be written as

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n x_n. \end{cases}$$

Given $\rho > 0$. Letting

$$\widetilde{\rho} = \sup\{\max\{\|Ax\|, \|x-\gamma A^*(I-P_Q)Ax\|\} : \|x\| \leqslant \rho\} < \infty,$$

we compute for $x \in H$, such that $\|x\| \leqslant \rho$,

$$\begin{aligned}
\|T_nx - Tx\| &\leqslant \|P_{C_n}(x-\gamma A^*(I-PQ_n)Ax) - P_{C_n}(x-\gamma A^*(I-P_Q)Ax)\| \\
&\quad + P_{C_n}(x-\gamma A^*(I-P_Q)Ax) - P_C(x-\gamma A^*(I-P_Q)Ax)\| \\
&\leqslant \|P_{C_n}(x-\gamma A^*(I-P_Q)Ax) - P_C(x-\gamma A^*(I-P_Q)Ax)\| \\
&\quad + \gamma\|A^*(P_QAx - P_{Q_n}Ax)\| \\
&\leqslant d_{\widetilde{\rho}}(C_n, C) + \gamma\|A\|d_{\widetilde{\rho}}(Q_n, Q).
\end{aligned}$$

This shows that
$$D_\rho(T_n, T) \leqslant d_{\tilde{\rho}}(C_n, C) + \gamma\|A\| d_{\tilde{\rho}}(Q_n, Q).$$
It follows from condition (ii) that
$$\sum_{n=0}^{\infty} \alpha_n D_\rho(T_n, T) \leqslant \sum_{n=0}^{\infty} \alpha_n d_{\tilde{\rho}}(C_n, C) + \gamma\|A\| \sum_{n=0}^{\infty} \alpha_n d_{\tilde{\rho}}(Q_n, Q) < \infty.$$

Now we can apply Corollary 2.2 to conclude that the sequence $\{x_n\}$ defined by the perturbed averaging CQ algorithm (1.8) converges weakly to a solution of SFP (1.2) □

Remark 2.5. Theorem 2.4 extends [30, Theorem 2.2] from the K-M algorithm to the Ishikawa iteration algorithm and removes the assumption in [30, Theorem 2.2] that the sequences $\{C_n\}$ and $\{Q_n\}$ Mosco converge to C and Q, respectively.

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References

[1] H. Attouchi, *Variational convergence for functions and operators*, Applicable Mathematics Series, Pitman (Advanced Publishing Program), Boston, MA, (1984). 1
[2] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math., **32** (1979), 107–116. 2
[3] C. Byren, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems, **18** (2002), 441–453. 1, 1
[4] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems, **20** (2014), 103–120. 1
[5] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, **8** (1994), 221–239. 1
[6] S.-S. Chang, J. K. Kim, Y. J. Cho, J. Y. Sim, *Weak- and strong-convergence theorems of solutions to split feasibility problem for nonspreading type mapping in Hilbert spaces*, Fixed Point Theory Appl., **2014** (2014), 12 pages. 1
[7] S. Y. Cho, B. A. Bin Dehaish, X.-L. Qin, *Weak convergence of a splitting algorithm in Hilbert spaces*, J. Appl. Anal. Comput., **7** (2017), 427–438. 1
[8] S. Y. Cho, W.-L. Li, S. M. Kang, *Convergence analysis of an iterative algorithm for monotone operators*, J. Inequal. Appl., **2013** (2013), 14 pages. 1
[9] S. Y. Cho, X.-L. Qin, L. Wang, *Strong convergence of a splitting algorithm for treating monotone operators*, Fixed Point Theory Appl., **2014** (2014), 15 pages. 1
[10] N.-N. Fang, Y.-P. Gong, *Viscosity iterative methods for split variational inclusion problems and fixed point problems of a nonexpansive mapping*, Commun. Optim. Theory, **2016** (2016), 15 pages. 1
[11] D. V. Hieu, L. D. Muu, P. K. Anh, *Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings*, Numer. Algorithms, **73** (2016), 197–217. 1
[12] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44** (1974), 147–150. 1
[13] J. K. Kim, S. Y. Cho, X.-L. Qin, *Some results on generalized equilibrium problems involving strictly pseudocontractive mappings*, Acta Math. Sci. Ser. B Engl. Ed., **31** (2011), 2041–2057. 1
[14] J. K. Kim, Salahuddin, *A system of nonconvex variational inequalities in Banach spaces*, Commun. Optim. Theory, **2016** (2016), 19 pages. 1
[15] M. A. Krasnosel'skii, *Two remarks on the method of successive approximations*, (Russian) Uspehi Mat. Nauk (N.S.), **10** (1955), 123–127. 1
[16] S.-T. Lv, *Some results on a two-step iterative algorithm in Hilbert spaces*, J. Nonlinear Funct. Anal., **2015** (2015), 10 pages. 1
[17] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510. 1
[18] A. Moudafi, *A relaxed alternating CQ-algorithm for convex feasibility problems*, Nonlinear Anal., **79** (2013), 117–121. 1

[19] X.-L. Qin, S.-S. Chang, Y. J. Cho, *Iterative methods for generalized equilibrium problems and fixed point problems with applications*, Nonlinear Anal. Real World Appl., **11** (2010), 2963–2972. 1

[20] X.-L. Qin, S. Y. Cho, L. Wang, *A regularization method for treating zero points of the sum of two monotone operators*, Fixed Point Theory Appl., **2014** (2014), 10 pages. 1

[21] X.-L. Qin, J.-C. Yao, *Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators*, J. Inequal. Appl., **2016** (2016), 9 pages. 1

[22] B. Qu, B.-H. Liu, N. Zheng, *On the computation of the step-size for the CQ-like algorithms for the split feasibility problem*, Appl. Math. Comput., **262** (2015), 218–223. 1, 1

[23] S. Rathee, Ritika, *δ-convergence theorems for Mann and Ishikawa iteration procedures with errors in CAT(0) spaces*, Commun. Optim. Theory, **2013** (2013), 11 pages. 1

[24] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67** (1979), 274–276. 1

[25] H.-K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16** (1991), 1127–1138. 2

[26] H.-K. Xu, *A variable Krasnoselskiĭ-Mann algorithm and the multiple-set split feasibility problem*, Inverse Problems, **22** (2006), 2021–2034. 1, 1, 1

[27] Q.-Z. Yang, *The relaxed CQ algorithm solving the split feasibility problem*, Inverse Problems, **20** (2004), 1261–1266. 1

[28] Y.-H. Yao, R.-D. Chen, H.-Y. Zhou, *Iterative process for certain nonlinear mappings in uniformly smooth Banach spaces*, Nonlinear Funct. Anal. Appl., **10** (2005), 651–664. 1

[29] C.-J. Zhang, J.-L. Li, B.-Q. Liu, *Strong convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Comput. Math. Appl., **61** (2011), 262–276. 1

[30] J.-L. Zhao, Q.-Z. Yang, *Several solution methods for the split feasibility problem*, Inverse Problems, **21** (2005), 1791–1799. 1, 1, 1.2, 1, 1, 1, 2.3, 2.5

[31] F. Zhao, L. Yang, *Hybrid projection methods for equilibrium problems and fixed point problems of infinite family of multi-valued asymptotically nonexpansive mappings*, J. Nonlinear Funct. Anal., **2016** (2016), 13 pages. 1