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Time effect on the dynamical behavior of a life energy system dynamic model

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Abstract

This article is concerned with a life energy system dynamic model with two different delays. A set of sufficient criteria which ensures the local stability and the existence of Hopf bifurcation for the model is derived. Some explicit formulas which determine the nature of Hopf bifurcations are obtained by means of the normal form theory and center manifold theorem. Our analytical findings are supported by numerical experiments. Finally, a brief conclusion is included.

Keywords: Life energy system model (LESM), delay, stability, Hopf bifurcation.
2010 MSC: 34K20, 34C25.

1. Introduction

Since Lindeman's classical studies [17, 18], many researchers are widely aware of the importance of ecological energetics. More advanced theory on energy [9–20] was put forward to endow the energy with profound connotation in a chemical and thermodynamic foundation. Over the past decades, energy system dynamic theory has attracted a great deal of attention and many excellent and interesting results have been reported. For example, Kooijman [10] developed the dynamic energy budget theory, Kooijman et al. [11] illustrated the application of the dynamic energy budget theory by analyzing some predator-prey systems, Kooijman et al. [12] discussed the application of mass and energy conservation laws in physiologically structured population models of heterotrophic organisms, and Kooijman and Troost [13] explained how the dynamic energy budget theory's various modules. For more related works on this aspect, one can refer to [11–19].

A life system usually consists of two or more components which are linked by way of energy exchange. There are many forms of energy exchange, and each exists under specialized condition. Based on this viewpoint and in order to depict an energy dynamic process of an ecosystem, Huang and Zu [6–8]

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proposed the following life energy system dynamical model (LESM)

$$\begin{aligned} \dot{w}_i(t) &= -\frac{\lambda_{ii}}{N_{ii}} w_i^2 + \lambda_{ii} w_i + \sum_{i \neq j}^{n} \left(\lambda_{ij} - \frac{\lambda_{ij}}{N_{ij}} w_i \right) w_j - \phi_i \\ &= -A_i w_i^2 + B_i w_i + \sum_{i \neq j}^{n} (C_{ij} - D_{ij} w_i) w_j - \phi_i, \ i, j = 1, 2, \cdots, n, \end{aligned} \tag{1.1}$$

where A_i, B_i, C_{ij} and D_{ij} are the eigenparameters determined by the energy capacity and energy-capturing ability of component i. The term $(-A_i w_i^2 + B_i w_i)$ indicates the energy flow which the component i exchanges with the environment outside the life energy system. The term $(C_{ij} - D_{ij} w_i) w_j$ represents the energy flows which the component i exchanges with the other $(n - 1)$ components within the life energy system. The more detail biological meaning of the coefficients of system (1.1) is listed in Table 1, one can also see [6–8].

Table 1: The state variables and parameters for the LESM (1.1)

n	Number of components in the life system
w_i	Energy of component i
ϕ_i	Energy consumption ratio of component i
λ_{ii}	Energy intake ability rate of component i from the outside of the system
λ_{ij}	Energy intake ability rate of component i from component j in the system
N_{ii}	The maximum energy capacity rate of component i
N_{ij}	Energy flow exchange rate of component i with component j
A_i	Eigenparameter
B_i	Eigenparameter
C_i	Eigenparameter
D_i	Eigenparameter

Huang [6] pointed out that analogue processes exist in less restricted ecological systems at the population level, such as the prey-predator relation of a herbivore and an omnivore. The herbivore captures energy from plants, and the omnivore from both preys and plants. Such a system exists under specific conditions, depending on the vegetation, abundance of the prey species, and other indirect condition, such as seasonality. Using subscript 1 to label a prey population and 2 to label a predator population, Huang [6] considered the following simplified LESM of system

$$\begin{cases} \dot{w}_1(t) = -A_1 w_1^2 + B_1 w_1 + [C_{12} - D_{12} w_1] w_2 - \phi_1, \\ \dot{w}_2(t) = -A_2 w_2^2 + B_2 w_2 + [C_{21} - D_{21} w_2] w_1 - \phi_2. \end{cases} \tag{1.2}$$

Considering that there is time delay during the course of the exchanges between the two energy components, Zhang et al. [27] incorporated a discrete delay into the model (1.2), then system (1.2) takes the form

$$\begin{cases} \dot{w}_1(t) = -A_1 w_1^2 + B_1 w_1 + [C_{12} - D_{12} w_1] w_2(t - \tau_1) - \phi_1, \\ \dot{w}_2(t) = -A_2 w_2^2 + B_2 w_2 + [C_{21} - D_{21} w_2] w_1(t - \tau_2) - \phi_2. \end{cases} \tag{1.3}$$

Let (w_1^*, w_2^*) be an equilibrium solution of (1.3). Making the variable changes $x_1 = w_1 - w_1^*, x_2 = w_2 - w_2^*$, Zhang et al. [27] rewrote system (1.3) as the following equivalent form

$$\begin{cases} \dot{x}_1(t) = -a_1 x_1^2 + b_1 x_1 + [c_{12} - d_{12} x_1] x_2(t - \tau_1), \\ \dot{x}_2(t) = -a_2 x_2^2 + b_2 x_2 + [c_{21} - d_{21} x_2] x_1(t - \tau_2), \end{cases} \tag{1.4}$$

where

$$\begin{cases} a_1 = A_1,\ a_2 = A_2,\ d_{12} = D_{12},\ d_{21} = D_{21}, \\ b_1 = B_1 - 2A_1 w_1^* - D_{12} w_2^*,\ b_2 = B_2 - 2A_2 w_2^* - D_{21} w_1^*, \\ c_{12} = C_{12} - D_{12} w_1^*,\ c_{21} = C_{21} - D_{21} w_2^*. \end{cases} \tag{1.5}$$

Taking the sum of the two delays as bifurcation parameter, they made a detailed discussion on the existence of Hopf bifurcation and Hopf bifurcation properties of system (1.4). Different from paper [27], Xiao et al. [22] discussed the bifurcation behavior of the following system

$$\begin{cases} \dot{x}_1(t) = -a_1 x_1^2 + b_1 x_1 + [c_{12} - d_{12} x_1] x_2(t - \tau_1), \\ \dot{x}_2(t) = -a_2 x_2^2 + b_2 x_2 + [c_{21} - d_{21} x_2] x_1(t - \tau_2), \end{cases} \tag{1.6}$$

which is under the assumption $b_1 = b_2 = b$ and $\tau_1 + \tau_2 = 2\tau$ in system (1.6). Regarding the combined interaction parameter $\eta = \sqrt{c_{12}c_{21}}$ as the bifurcation parameter, Xiao et al. [22] showed that a Hopf bifurcation occurs when the parameter η passes through a sequence of critical values for system (1.6).

A natural question is what the two different delays τ_1 and τ_2 have effect on the dynamical behavior of system (1.3). In this paper, we will focus on this topic. Up to now, there is no paper that deals with the topic on what the two different time delays have effect Hopf bifurcation nature for (1.3). The derived results of this article complement the known studies in [8, 22, 27].

2. Stability of the equilibrium and local Hopf bifurcations

In this section, we shall consider the stability of the equilibrium and the existence of local Hopf bifurcations.

Denote $\bar{\tau} = \max\{\tau_1, \tau_2\}$ and let $X = C([-\bar{\tau}, 0]; R)$ be the Banach space of continuous mapping from $[-\bar{\tau}, 0]$ to R equipped with the sup-norm. The initial conditions of system (1.6) are given as follows:

$$x_i(\theta) \geqslant 0,\quad x_i(0) > 0,\quad i = 1, 2, \theta \in [-\bar{\tau}, 0]. \tag{2.1}$$

It follows from [3, 14] that system (1.6) with initial conditions (2.1) has a unique solution $(x_1(t), x_2(t))$.

Lemma 2.1. *Let* $(x_1(t), x_2(t))^T$ *be the solution of* (1.6) *satisfying conditions* (2.1). *If* $c_{12} > 0, c_{21} > 0$, *then* $x_1(t)$ *and* $x_2(t)$ *are positive.*

Proof. First, we prove that $x_1(t)$ and $x_2(t)$ are positive. Assume the contrary, then let $t_0 > 0$ be the first time such that $x_1(t_0) = 0$. In view of the first equation of (1.6), we have $\dot{x}_1(t_0) = c_{12} x_2(t_0 - \tau_1) > 0$ which implies that $x_1(t) < 0$ for $t \in (t_0 - \varepsilon, t_0)$, where ε is an arbitrarily small positive constant. In a similar way, we can prove that $x_2(t)$ is positive. This completes the proof. □

Let (w_1^*, w_2^*) be an equilibrium solution of (1.3). Here (w_1^*, w_2^*) satisfies the following equations:

$$\begin{cases} A_1 (w_1^*)^2 - B_1 w_1^* - [C_{12} - D_{12} w_1^*] w_2^* + \phi_1 = 0, \\ A_2 (w_2^*)^2 - B_2 w_2^* - [C_{21} - D_{21} w_2] w_1^* + \phi_2 = 0. \end{cases}$$

Making the variable changes $\bar{w}_1 = w_1 - w_1^*, \bar{w}_2 = w_2 - w_2^*$ and still denote $\bar{w}_i(t)$ $(i = 1, 2)$ by $w_i(t)$ $(i = 1, 2)$, respectively, then system (1.3) takes the following equivalent form

$$\begin{cases} \dot{w}_1(t) = -a_1 w_1^2 + b_1 w_1 + [c_{12} - d_{12} w_1] w_2(t - \tau_1), \\ \dot{w}_2(t) = -a_2 w_2^2 + b_2 w_2 + [c_{21} - d_{21} w_2] w_1(t - \tau_2), \end{cases} \tag{2.2}$$

where a_i, b_i $(i = 1, 2), c_{12}, c_{21}, d_{12}$ and d_{21} are defined by (1.5). The linear system of Eq. (2.2) around the trivial solution takes the form

$$\begin{cases} \dot{w}_1(t) = b_1 w_1(t) + c_{12} w_2(t - \tau_1), \\ \dot{w}_2(t) = b_2 w_2(t) + c_{21} w_1(t - \tau_2). \end{cases} \tag{2.3}$$

The characteristic equation of Eq. (2.3) reads as

$$\det\begin{bmatrix} \lambda - b_1 & -c_{12}e^{-\lambda\tau_1} \\ -c_{21}e^{-\lambda\tau_2} & \lambda - b_2 \end{bmatrix} = 0.$$

Namely,

$$\lambda^2 - (b_1+b_2)\lambda + b_1b_2 - c_{12}c_{21}e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{2.4}$$

Lemma 2.2 ([21]). *For the transcendental equation*

$$P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0,$$

as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ *varies, the sum of orders of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

We will firstly consider two simpler cases, i.e., Case I: $\tau_1 = \tau_2 = 0$ and Case II: $\tau_1 = 0, \tau_2 \neq 0$, then we will turn to the general case, namely, Case III: $\tau_1 > 0, \tau_2 > 0$.

Case I: $\tau_1 = \tau_2 = 0$, (2.4) is written as a second degree polynomial equation

$$\lambda^2 - (b_1+b_2)\lambda + b_1b_2 - c_{12}c_{21} = 0. \tag{2.5}$$

All eigenvalues of (2.5) have negative real parts if

$$b_1 + b_2 < 0, \ b_1b_2 > c_{12}c_{21}. \tag{H1}$$

Then $E_0(u_1^*, u_2^*)$ is locally asymptotically stable if (H1) holds true.

Case II: $\tau_1 = 0, \tau_2 > 0$, (2.4) turns into

$$\lambda^2 - (b_1+b_2)\lambda + b_1b_2 - c_{12}c_{21}e^{-\lambda\tau_2} = 0. \tag{2.6}$$

For $\omega > 0, i\omega$ be a root of (2.6), then it follows that

$$\begin{cases} c_{12}c_{21}\cos\omega\tau_2 = b_1b_2 - \omega^2, \\ c_{12}c_{21}\sin\omega\tau_2 = (b_1+b_2)\omega. \end{cases} \tag{2.7}$$

It follows from (2.7) that

$$\omega^4 + (b_1^2 + b_2^2)\omega^2 + (b_1b_2)^2 - (c_{12}c_{21})^2 = 0. \tag{2.8}$$

It is easy to see that if the condition

$$|b_1b_2| > |c_{12}c_{21}| \tag{H2}$$

holds, then Eq. (2.8) has no positive roots. Hence, all roots of (2.6) have negative real parts when $\tau_2 \in [0, +\infty)$ under the conditions (H1) and (H2).

If (H1) and

$$|b_1b_2| < |c_{12}c_{21}| \tag{H3}$$

hold, then (2.8) has a unique positive root ω_0^2. Substituting ω_0^2 into (2.7), we obtain

$$\tau_{2_n} = \frac{1}{\omega_0}\left\{\arccos\left[\frac{b_1b_2 - \omega^2}{c_{12}c_{21}}\right] + 2n\pi\right\}, n = 0, 1, 2, \cdots.$$

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be a root of (2.6) near $\tau_2 = \tau_{2_n}$ and $\alpha(\tau_{2_n}) = 0,\ \omega(\tau_{2_n}) = \omega_0$. In view of (2.6), one has

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = -\frac{[2\lambda - (b_1 + b_2)]e^{\lambda\tau_2}}{c_{12}c_{21}\lambda} - \frac{\tau_2}{\lambda},$$

which leads to

$$\begin{aligned}
\left[\frac{d(\mathrm{Re}\lambda(\tau))}{d\tau_2}\right]^{-1}_{\tau_2=\tau_{2_n}} &= -\mathrm{Re}\left\{\frac{[2\lambda - (b_1 + b_2)]e^{\lambda\tau_2}}{c_{12}c_{21}\lambda}\right\}_{\tau_2=\tau_{2_n}}\\
&= \mathrm{Re}\left\{\frac{(b_1 + b_2)\cos\omega_0\tau_{2_n} + 2\omega_0\sin\omega_0\tau_{2_n} + i[(b_1 + b_2)\sin\omega_0\tau_{2_n} - 2\omega_0\cos\omega_0\tau_{2_n}]}{c_{12}c_{21}\omega_0 i}\right\}\\
&= \frac{[(b_1 + b_2)\sin\omega_0\tau_{2_n} - 2\omega_0\cos\omega_0\tau_{2_n}]c_{12}c_{21}\omega_0}{(c_{12}c_{21}\omega_0)^2}\\
&= \frac{b_1^2 + b_2^2 + 2\omega_0^2}{(c_{12}c_{21})^2} > 0.
\end{aligned}$$

Noting that

$$\mathrm{sign}\left\{\frac{d(\mathrm{Re}\lambda)}{d\tau_2}\right\}_{\tau_2=\tau_{2_n}} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau_2}\right]\right\}_{\tau_2=\tau_{2_n}} = 1,$$

we have

$$\left[\frac{d(\mathrm{Re}\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2_n}} > 0.$$

It follows from the analysis and the Corollary 2.4 in Ruan and Wei [21] that the following results hold true.

Lemma 2.3. *For $\tau_1 = 0$, assume that (H1) holds true. Then the following conclusions hold:*

(i) *If (H2) holds, then the equilibrium $E_0(u_1^*, u_2^*)$ of system (1.3) is locally asymptotically stable for all $\tau_2 \geqslant 0$.*

(ii) *If (H3) holds, then the equilibrium $E_0(w_1^*, w_2^*)$ of system (1.3) is locally asymptotically stable for $\tau_2 < \tau_{2_0}$ and unstable for $\tau_2 < \tau_{2_0}$. Furthermore, system (1.3) undergoes a Hopf bifurcation at the equilibrium $E_0(w_1^*, w_2^*)$ when $\tau_2 = \tau_{2_0}$.*

Case III: $\tau_1 > 0, \tau_2 > 0$. Fix τ_2 in its stable interval and take τ_1 as a parameter. We consider (2.2) under the assumptions (H1) and (H3). Let $i\omega^*(\omega^* > 0)$ be a root of (2.4), then we can obtain

$$\begin{cases} c_{12}c_{21}\cos\omega^*\tau_2\cos\omega\tau_1 - c_{12}c_{21}\sin\omega^*\tau_2\sin\omega^*\tau_1 = -(\omega^*)^2,\\ c_{12}c_{21}\sin\omega^*\tau_2\cos\omega\tau_1 - c_{12}c_{21}\cos\omega^*\tau_2\sin\omega^*\tau_1 = -(b_1 + b_2)\omega^*. \end{cases} \tag{2.9}$$

Then

$$(\omega^*)^4 + (b_1 + b_2)^2(\omega^*)^2 - 2(c_{12}c_{21})^2 = 0. \tag{2.10}$$

It is easy to check that (2.10) has a positive roots

$$\omega^* = \left[\frac{-(b_1 + b_2)^2 + \sqrt{(b_1 + b_2)^4 + 8(c_{12}c_{21})^2}}{2}\right]^{\frac{1}{2}}.$$

For the fixed ω^*, there exists a sequence $\{\tau_1^j | j = 1, 2, 3, \cdots\}$, such that (2.9) holds. Let

$$\tau_{1_0} = \min\{\tau_1^j | j = 1, 2, \cdots\}.$$

When $\tau_1 = \tau_{1_0}$, Eq. (2.4) has a pair of purely imaginary roots $\pm i\omega^*$ for $\tau_2 \in [0, \tau_{2_0})$.

In the following, we are going to verify transversality condition.

$$\left[\frac{d(\mathrm{Re}\lambda)}{d\tau_1}\right]_{\lambda=i\omega^*} = \frac{\theta_1\theta_3 - \theta_2\theta_4}{\theta_1^2+\theta_2^2},$$

where

$$\theta_1 = b_1 + b_2 + c_{12}c_{21}(\tau_{1_0}+\tau_2)\cos[\omega^*(\tau_{1_0}+\tau_2)], \tag{2.11}$$
$$\theta_2 = 2\omega^* + c_{12}c_{21}(\tau_{1_0}+\tau_2)\sin[\omega^*(\tau_{1_0}+\tau_2)], \tag{2.12}$$
$$\theta_3 = c_{12}c_{21}\omega^*\sin[\omega^*(\tau_{1_0}+\tau_2)], \tag{2.13}$$
$$\theta_4 = -c_{12}c_{21}\omega^*\cos[\omega^*(\tau_{1_0}+\tau_2)]. \tag{2.14}$$

We assume that

$$\theta_1\theta_3 \neq \theta_2\theta_4. \tag{H4}$$

According to Hopf bifurcation theory [3] and the ideas of [1, 24, 25], we have the following theorem.

Theorem 2.4. *Let $\theta_1, \theta_2, \theta_3$, and θ_4 be defined by (2.11), (2.12), (2.13), and (2.14), respectively. For system (1.3), assume that (H1), (H3) and (H4) hold true, and $\tau_2 \in [0, \tau_{2_0})$, then the equilibrium $E_0(u_1^*, u_2^*)$ is locally asymptotically stable when $\tau_1 \in (0, \tau_{1_0})$, and system (1.3) undergoes a Hopf bifurcation at the equilibrium $E_0(u_1^*, u_2^*)$ when $\tau_1 = \tau_{1_0}$.*

3. Natures of Hopf bifurcation

Throughout this section, we suppose that model (2.2) undergoes Hopf bifurcation at the equilibrium $E_0(u_1^*, u_2^*)$ for $\tau_1 = \tau_{1_0}$, and then $\pm i\omega^*$ is corresponding purely imaginary roots of the characteristic equation at the equilibrium $E_0(u_1^*, u_2^*)$.

Without loss of generality, we assume that $\tau_2^* < \tau_{1_0}$, where $\tau_2^* \in (0, \tau_{2_0})$. Let $\bar{u}_i(t) = u_i(\tau t)$ $(i = 1, 2)$ and $\tau_1 = \tau_{1_0} + \mu$, where τ_{1_0} is defined by (2.10) and $\mu \in R$, still denote $\bar{u}_i$ by u_i, then (2.2) takes the form

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{3.1}$$

where $u(t) = (u_1(t), u_2(t))^T \in C$ and $u_t(\theta) = u(t+\theta) = (u_1(t+\theta), u_2(t+\theta))^T \in C$, and $L_\mu : C \to R, F : R \times C \to R$ are given by

$$L_\mu\phi = (\tau_{1_0}+\mu)B\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\end{pmatrix} + (\tau_{1_0}+\mu)C\begin{pmatrix}\phi_1\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)\\ \phi_2\left(-\frac{\tau_2^*}{\tau_{1_0}}\right)\end{pmatrix} + (\tau_{1_0}+\mu)D\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\end{pmatrix}$$

and

$$F(\mu, \phi) = (\tau_{1_0}+\mu)(f_1, f_2)^T,$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C$,

$$B = \begin{pmatrix} b_1 & 0\\ 0 & b_2\end{pmatrix}, \; C = \begin{pmatrix} 0 & 0\\ c_{21} & 0\end{pmatrix}, \; D = \begin{pmatrix} 0 & c_{12}\\ 0 & 0\end{pmatrix}$$

and

$$f_1 = -a_1\phi_1^2(0) - d_{12}\phi_1(0)\phi_2(-1),$$
$$f_2 = -a_2\phi_2^2(0) - d_{21}\phi_2(0)\phi_1\left(-\frac{\tau_2^*}{\tau_{1_0}}\right).$$

By Section 2, it is easy to see that if $\mu = 0$, then (3.1) undergoes a Hopf bifurcation at the equilibrium $E_0(u_1^*, u_2^*)$ and the associated characteristic equation of (3.1) has a pair of simple imaginary roots $\pm\omega^*\tau_{1_0}$.

By the representation theorem, there exists a matrix function with bounded variation components $\eta(\theta,\mu), \theta \in [-1,0]$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \qquad \text{for } \phi \in C.$$

In fact, we can choose

$$\eta(\theta,\mu) = \begin{cases} (\tau_{1_0}+\mu)(B+C+D), & \theta = 0, \\ (\tau_{1_0}+\mu)(C+D), & \theta \in \left[-\frac{\tau_2^*}{\tau_{1_0}}, 0\right), \\ (\tau_{1_0}+\mu)D, & \theta \in \left(-1, -\frac{\tau_2^*}{\tau_{1_0}}\right), \\ 0, & \theta = -1. \end{cases}$$

For $\phi \in C([-1,0], R^2)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leqslant \theta < 0, \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0 \end{cases}$$

and

$$R\phi = \begin{cases} 0, & -1 \leqslant \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Then (3.1) is equivalent to the following form

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{3.2}$$

where $u_t(\theta) = u(t+\theta), \theta \in [-1,0]$.

For $\psi \in C([0,1], (R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases}$$

For $\phi \in C([-1,0], R^2)$ and $\psi \in C([0,1], (R^2)^*)$, define the bilinear form

$$< \psi, \phi >= \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta,0)$, the $A = A(0)$ and A^* are adjoint operators. By Section 2, we know that the eigenvalues of $A(0)$ and A^* are $\pm i\omega^*\tau_{1_0}$. It is easy to obtain

$$q(\theta) = (1,\alpha)^T e^{i\omega^*\tau_{1_0}\theta}, \quad q^*(s) = M(1,\alpha^*)e^{i\omega^*\tau_{1_0}s}, \quad M = \frac{1}{K},$$

where

$$\alpha = \frac{i\omega^* - b_1}{c_{12}e^{-i\omega^*}}, \quad \alpha^* = -\frac{i\omega^* + b_1}{c_{12}e^{-i\omega^*\frac{\tau_2^*}{\tau_{1_0}}}},$$

$$K = 1 + \bar{\alpha}\alpha^* + \tau_{1_0}\left[\bar{\alpha}c_{12}e^{i\omega^*} + \frac{\tau_2^*}{\tau_{1_0}}c_{21}\alpha^* e^{i\omega^*\frac{\tau_2^*}{\tau_{1_0}}}\right].$$

Furthermore, $< q^*(s), q(\theta) >= 1$ and $< q^*(s), \bar{q}(\theta) >= 0$.

In view of the method of Hassard et al. [4], we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of Eq. (3.1) when $\mu = 0$.

Define

$$z(t) =< q^*, u_t >, W(t,\theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} \tag{3.3}$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t),\bar{z}(t),\theta),$$

where

$$W(z(t),\bar{z}(t),\theta) = W(z,\bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots,$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (3.1),

$$\dot{z}(t) = i\omega^*\tau_{1_0}z + \bar{q}^*(\theta)F(0, W(z,\bar{z},\theta) + 2\mathrm{Re}\{zq(\theta)\} \overset{\text{def}}{=} i\omega^*\tau_{1_0}z + \bar{q}^*(0)F_0.$$

That is

$$\dot{z}(t) = i\omega^*\tau_{1_0}z + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots.$$

Hence we obtain the expression of $g(z,\bar{z})$ as follows

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)F_0(z,\bar{z})\\
&= F(0,u_t)\\
&= \bar{M}\tau_{1_0}\left[-a_1 - d_{12}\alpha + \bar{\alpha}^*(-a_2\alpha^2 - d_{21}\alpha)\right] z^2\\
&\quad + \bar{M}\tau_{1_0}\left[-2a_1 - 2d_{12}\mathrm{Re}\{\alpha\} + \bar{\alpha}^*(-2a_2\mathrm{Re}\{\alpha\} - 2d_{21}\mathrm{Re}\{\alpha\})\right] z\bar{z}\\
&\quad + \bar{M}\tau_{1_0}\left[-a_1 - d_{12}\bar{\alpha} + \bar{\alpha}^*(-a_2\alpha^2 - d_{21}\bar{\alpha})\right] \bar{z}^2\\
&\quad + \bar{M}\tau_{1_0}\Bigg\{ - a_1\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) - d_{21}\Bigg(W_{11}^{(2)}(-1) + \alpha W_{11}^{(1)}(0)\\
&\quad + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\bar{\alpha}\Bigg) + \bar{\alpha}^*\Bigg[- a_2\left(2\alpha W_{11}^{(2)}(0) + W_{20}^{(2)}(0)\bar{\alpha}\right)\\
&\quad - d_{12}\left((W_{11}^{(1)}(-\frac{\tau_2^*}{\tau_{1_0}})\alpha + \frac{1}{2}\bar{\alpha}W_{20}^{(1)}(-\frac{\tau_2^*}{\tau_{1_0}}) + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\right)\Bigg]\Bigg\} z^2\bar{z} + \text{h.o.t.},
\end{aligned}$$

then it is easy to obtain

$$\begin{aligned}
g_{20} &= 2\bar{M}\tau_{1_0}\left[-a_1 - d_{12}\alpha + \bar{\alpha}^*(-a_2\alpha^2 - d_{21}\alpha)\right],\\
g_{11} &= 2\bar{M}\tau_{1_0}\left[-a_1 - d_{12}\mathrm{Re}\{\alpha\} + \bar{\alpha}^*(-a_2\mathrm{Re}\{\alpha\} - d_{21}\mathrm{Re}\{\alpha\})\right],\\
g_{02} &= 2\bar{M}\tau_{1_0}\left[-a_1 - d_{12}\bar{\alpha} + \bar{\alpha}^*(-a_2\alpha^2 - d_{21}\bar{\alpha})\right],\\
g_{21} &= 2\bar{M}\tau_{1_0}\Bigg\{ - a_1\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) - d_{21}\Bigg(W_{11}^{(2)}(-1) + \alpha W_{11}^{(1)}(0)\\
&\quad + \frac{1}{2}W_{20}^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\bar{\alpha}\Bigg) + \bar{\alpha}^*\Bigg[- a_2\left(2\alpha W_{11}^{(2)}(0) + W_{20}^{(2)}(0)\bar{\alpha}\right)\\
&\quad - d_{12}\left((W_{11}^{(1)}(-\frac{\tau_2^*}{\tau_{1_0}})\alpha + \frac{1}{2}\bar{\alpha}W_{20}^{(1)}(-\frac{\tau_2^*}{\tau_{1_0}}) + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\right)\Bigg]\Bigg\}.
\end{aligned}$$

Now we compute $W_{20}^{(i)}(\theta), W_{11}^{(i)}(\theta)$ $(i = 1, 2)$ in g_{21}. From (3.2) and (3.3), we get

$$\dot{W} = \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)F_0 q(\theta)\}, & -1 \leqslant \theta < 0, \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)F_0 q(\theta)\} + F_0, & \theta = 0, \end{cases} \stackrel{\text{def}}{=} AW + H(z, \bar{z}, \theta),$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.4}$$

Comparing the coefficients, we obtain

$$(A - 2i\tau_{1_0}\omega^*)W_{20} = -H_{20}(\theta), \tag{3.5}$$

$$AW_{11}(\theta) = -H_{11}(\theta) \tag{3.6}$$

and we know that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \tag{3.7}$$

Comparing the coefficients of (3.7) with (3.4) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{3.8}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.9}$$

From (3.5), (3.8), and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\omega^*\tau_{1_0}W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta).$$

Noting that $q(\theta) = q(0)e^{i\omega^*\tau_{1_0}\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^*\tau_{1_0}}q(0)e^{i\omega^*\tau_{1_0}\theta} + \frac{i\bar{g}_{02}}{3\omega^*\tau_{1_0}}\bar{q}(0)e^{-i\omega^*\tau_{1_0}\theta} + E_1 e^{2i\omega^*\tau_{1_0}\theta}, \tag{3.10}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T \in R^2$ is a constant vector.

Similarly, from (3.6), (3.9), and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta), \tag{3.11}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^*\tau_{1_0}}q(0)e^{i\omega^*\tau_{1_0}\theta} + \frac{i\bar{g}_{11}}{\omega^*\tau_{1_0}}\bar{q}(0)e^{-i\omega^*\tau_{1_0}\theta} + E_2, \tag{3.12}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^T \in R^2$ is a constant vector. In what follows, we shall seek appropriate E_1,E_2 in (3.10) and (3.12), respectively. It follows from the definition of A, (3.8), and (3.9) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega^*\tau_{1_0}W_{20}(0) - H_{20}(0) \tag{3.13}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.14}$$

where $\eta(\theta) = \eta(0, \theta)$.

From (3.5), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2\tau_{1_0}(H_1, H_2)^T, \tag{3.15}$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + 2\tau_{1_0}(P_1, P_2)^T, \tag{3.16}$$

where

$$H_1 = -a_1 - a_{12}\alpha, \qquad H_2 = -a_2\alpha^2 - d_{21}\alpha,$$
$$P_1 = -a_1 + d_{12}\mathrm{Re}\{\alpha\}, \qquad P_2 = a_2\mathrm{Re}\{\alpha\} - d_{21}\mathrm{Re}\{\alpha\}.$$

Noting that

$$\left(i\omega^*\tau_{1_0}I - \int_{-1}^{0} e^{i\omega^*\tau_{1_0}\theta}d\eta(\theta)\right)q(0) = 0,$$
$$\left(-i\omega^*\tau_{1_0}I - \int_{-1}^{0} e^{-i\omega^*\tau_{1_0}\theta}d\eta(\theta)\right)\bar{q}(0) = 0,$$

and substituting (3.10) and (3.15) into (3.13), we have

$$\left(2i\omega^*\tau_{1_0}I - \int_{-1}^{0} e^{2i\omega^*\tau_{1_0}\theta}d\eta(\theta)\right)E_1 = 2\tau_{1_0}(H_1, H_2)^T.$$

That is

$$\begin{pmatrix} 2i\omega^* - b_1 & -c_{12}e^{2i\omega^*\tau_{1_0}} \\ -c_{21}e^{-2i\omega^*\tau_2^*} & 2i\omega^* - b_2 \end{pmatrix} E_1 = 2(H_1, H_2)^T.$$

It follows that

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, \quad E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},$$

where

$$\Delta_1 = (2i\omega^* - b_1)(2i\omega^* - b_2) - c_{12}c_{21}e^{2i\omega^*\tau_{1_0}+\tau_2^*},$$
$$\Delta_{11} = -2(a_1 + a_{12}\alpha)(2i\omega^* - b_1) - 2(a_2\alpha^2 + d_{21}\alpha)c_{12}e^{2i\omega^*\tau_{1_0}},$$
$$\Delta_{12} = -2(a_2\alpha^2 + d_{21}\alpha)(2i\omega^* - b_1) - 2(a_1 + a_{12}\alpha)c_{21}e^{-2i\omega^*\tau_2^*}.$$

In a similar way, substituting (3.11) and (3.16) into (3.14), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right)E_2 = 2\tau_{1_0}(P_1, P_2)^T.$$

That is

$$\begin{pmatrix} b_1 & c_{12} \\ c_{21} & b_2 \end{pmatrix} E_2 = 2(-P_1, -P_2)^T.$$

It follows that

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, \quad E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \tag{3.17}$$

where

$$\Delta_2 = b_1b_2 - c_{12}c_{21},$$
$$\Delta_{21} = 2(a_2\mathrm{Re}\{\alpha\} - d_{21}\mathrm{Re}\{\alpha\})c_{12} - 2(-a_1 + d_{12}\mathrm{Re}\{\alpha\})b_2,$$
$$\Delta_{22} = 2(-a_1 + d_{12}\mathrm{Re}\{\alpha\})c_{21} - 2(a_2\mathrm{Re}\{\alpha\} - d_{21}\mathrm{Re}\{\alpha\})b_1.$$

From (3.10), (3.12), (3.8), and (3.17), we can calculate g_{21} and derive the following coefficients:

$$c_1(0) = \frac{i}{2\omega^*\tau_{1_0}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_{1_0})\}}, \qquad \beta_2 = 2\mathrm{Re}(c_1(0)), \qquad T_2 = -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tau_{1_0})\}}{\omega^*\tau_{1_0}}.$$

Summarizing the above analysis, we get the following Theorem 3.1.

Theorem 3.1. *The periodic solution is forward (backward) if* $\mu_2 > 0$ $(\mu_2 < 0)$*; the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if* $\beta_2 < 0$ $(\beta_2 > 0)$*; and the periods of the bifurcating periodic solutions increase (decrease) if* $T_2 > 0$ $(T_2 < 0)$.

Remark 3.2. The method of this paper can be applied to deal with various delayed differential models with two different time delays and we can show that the different delay has important effect on the stability and Hopf bifurcation natures.

Remark 3.3. In many cases, we can consider the stability of the delayed model by constructing a suitable Lyapunov functional. In this paper, we investigated the stability by analyzing the characteristic equation roots of the characteristic equation. This method is generally more easier to handle the stability of involved model than Lyapunov method.

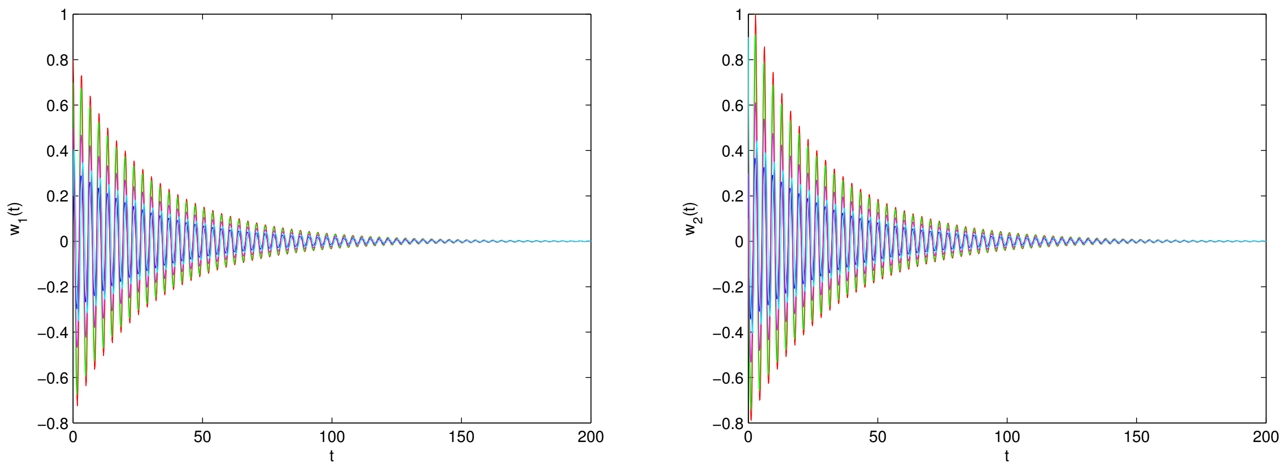
4. Numerical examples

Consider the following system:

$$\begin{cases} \dot{w}_1(t) = -0.1w_1^2 + 1.5w_1 + [2 - 0.3w_1]w_2(t - \tau_1), \\ \dot{w}_2(t) = -0.2w_2^2 + 1.5w_2 + [-2.75 - w_2]w_1(t - \tau_2). \end{cases} \tag{4.1}$$

Obviously, system (4.1) has a zero equilibrium $E_0(0,0)$. When $\tau_1 = 0$, we can easily obtain that (H1) and (H3) are fulfilled. Let $n = 0$ and by means of Matlab software, we get $\omega_0 \approx 0.4489, \tau_{2_0} \approx 0.75$. From Lemma 2.3, we know that the transversal condition is satisfied. Figure 1 shows that the zero equilibrium $E_0(0,0)$ is asymptotically stable for $\tau_2 < \tau_{2_0} \approx 0.75$. In Figure 2 we can see that the zero equilibrium $E_0(0,0)$ is unstable for $\tau_2 > \tau_{2_0} \approx 0.75$ and when $\tau_2 = \tau_{2_0} \approx 0.75$, Eq. (4.1) undergoes a Hopf bifurcation at the zero equilibrium $E_0(0,0)$, i.e., a small amplitude periodic solution occurs around $E_0(0,0)$ when $\tau_1 = 0$ and τ_2 is close to $\tau_{2_0} = 0.75$.

Keep $\tau_2 = 0.6 \in (0, 0.75)$ and take τ_1 as a parameter. We have $\tau_{1_0} \approx 0.16$. Then the zero equilibrium is asymptotically when $\tau_1 \in [0, \tau_{1_0})$. The Hopf bifurcation value of Eq. (4.1) is $\tau_{1_0} \approx 0.16$. By the algorithm derived in Section 3, we can obtain $\lambda'(\tau_{1_0}) \approx 0.3215 - 0.1428i, c_1(0) \approx -1.0823 - 2.1130i, \mu_2 \approx 3.3664, \beta_2 \approx -2.1646, T_2 \approx 101.1$. It follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus, the zero equilibrium $E_0(0,0)$ is stable when $\tau_1 < \tau_{1_0}$ which is presented by the computer simulations (see Figure 3). When τ_1 passes through the critical value τ_{1_0}, the zero equilibrium $E_0(0,0)$ loses its stability and a Hopf bifurcation occurs. Due to $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau_1 > \tau_{1_0}$, and these bifurcating periodic solutions from $E_0(0,0)$ at τ_{1_0} are stable, which are depicted in Figure 4.



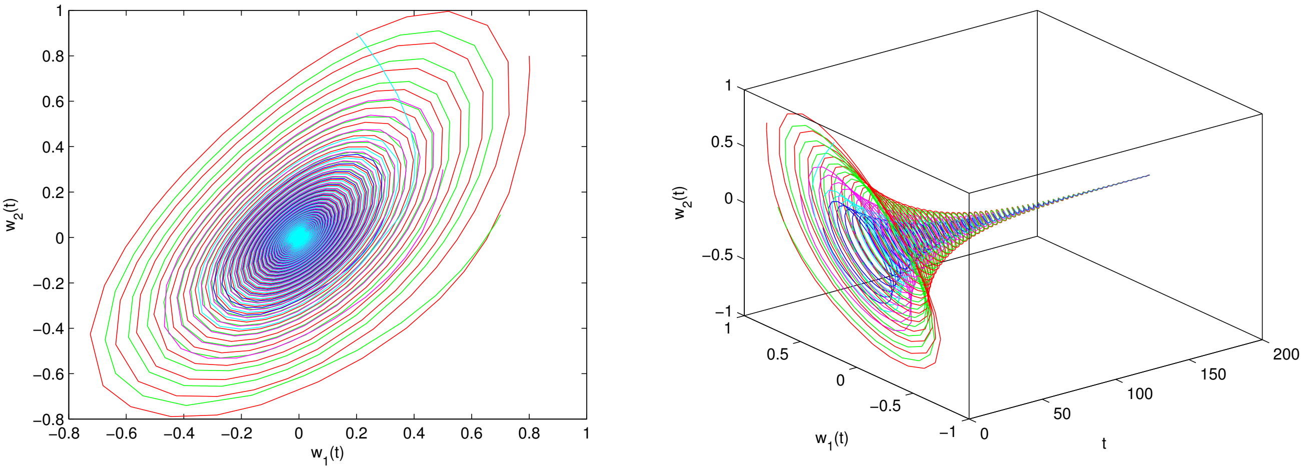


Figure 1: Numerical results for system (4.1) when $\tau_1 = 0, \tau_2 = 0.72 < \tau_{2_0} \approx 0.75$ and the initial value $(w_1(0), w_2(0)) = (0.8, 0.8)$. The zero equilibrium $E_0(0,0)$ is asymptotically stable. The initial values are (0.8, 0.8), (0.3,0.1), (0.7,0.1),(0.5,0.3),(0.2,0.9), respectively.

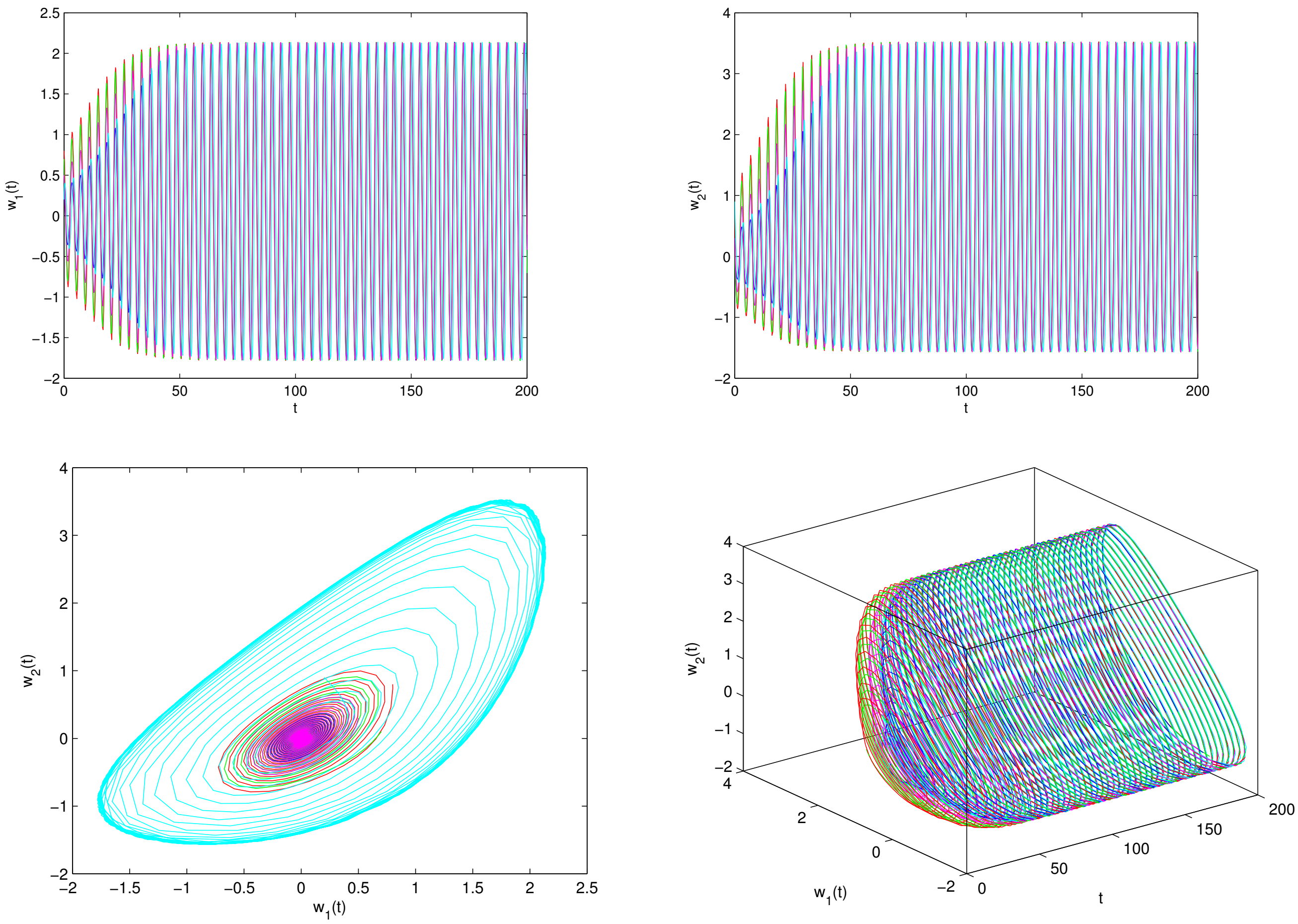


Figure 2: Numerical results for system (4.1) when $\tau_1 = 0, \tau_2 = 0.9 > \tau_{2_0} \approx 0.75$ and the initial value $(w_1(0), w_2(0)) = (0.8, 0.8)$. Hopf bifurcation occurs from the zero equilibrium $E_0(0,0)$. The initial values are (0.8, 0.8), (0.3,0.1), (0.7,0.1),(0.5,0.3),(0.2,0.9), respectively.

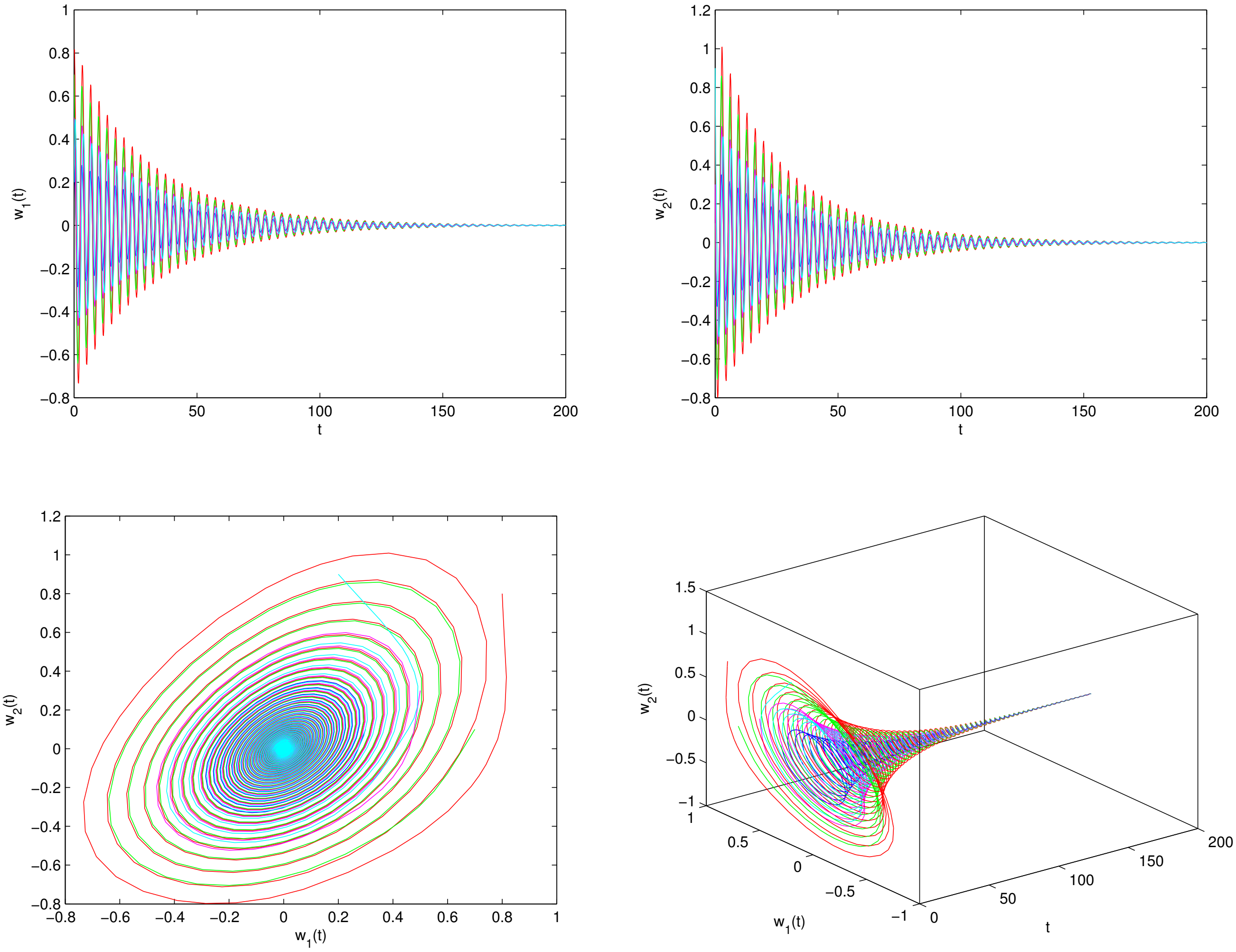
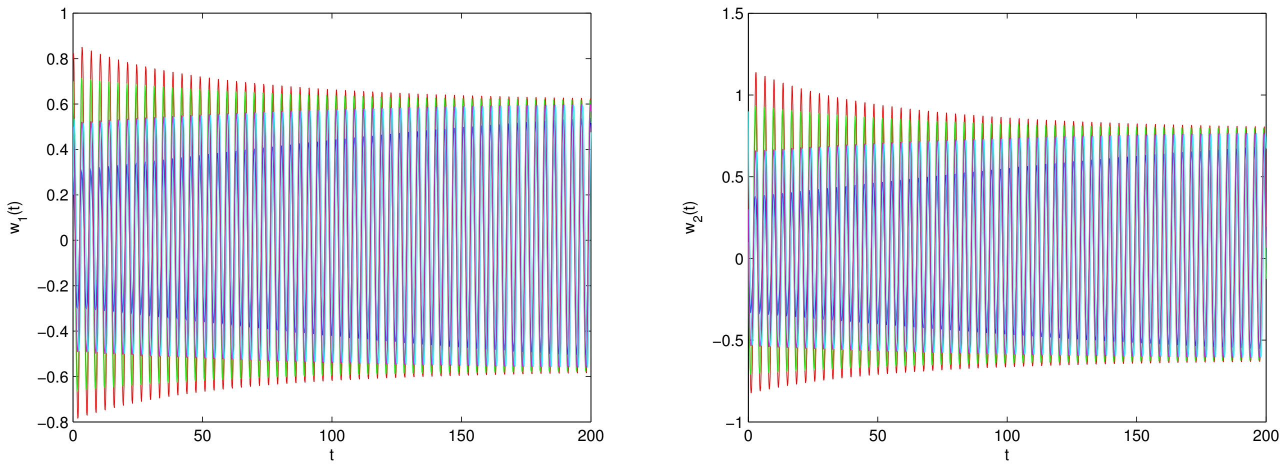


Figure 3: Numerical results for system (4.1) when $\tau_2 = 0.6, \tau_1 = 0.12 < \tau_{1_0} \approx 0.16$ and the initial value $(w_1(0), w_2(0)) = (0.8, 0.8)$. The zero equilibrium $E_0(0, 0)$ is asymptotically stable. The initial values are (0.8, 0.8), (0.3,0.1), (0.7,0.1),(0.5,0.3),(0.2,0.9), respectively.



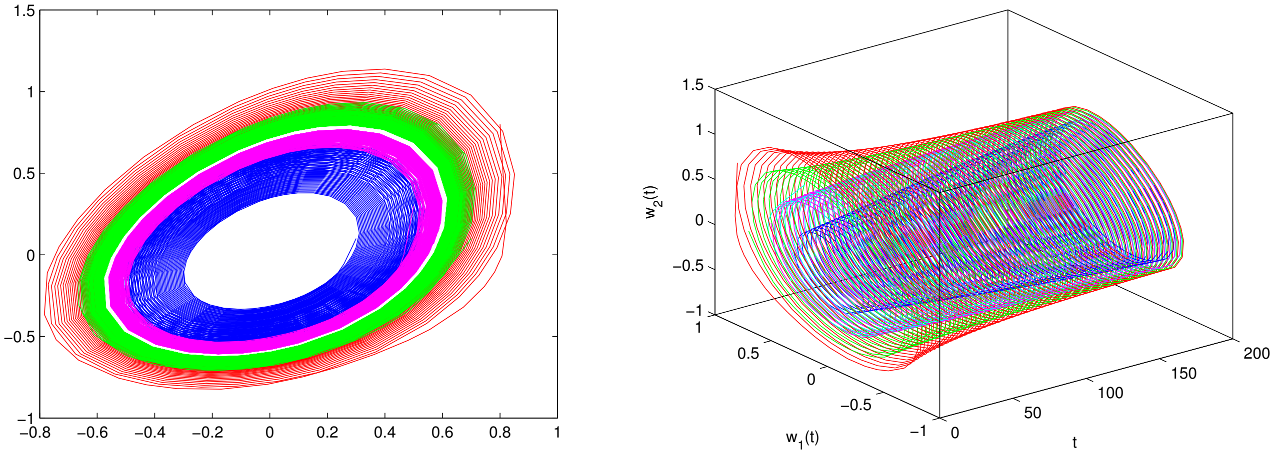


Figure 4: Numerical results for system (4.1) when $\tau_2 = 0.6, \tau_1 = 0.18 > \tau_{1_0} \approx 0.16$ and the initial value $(w_1(0), w_2(0)) = (0.8, 0.8)$. Hopf bifurcation occurs from the zero equilibrium $E_0(0,0)$. The initial values are (0.8, 0.8), (0.3,0.1), (0.7,0.1),(0.5,0.3),(0.2,0.9), respectively.

5. Conclusions

In this paper, we have investigated local stability of the zero equilibrium $E_0(0,0)$ and local Hopf bifurcation of a life energy system dynamic model with two different delays. We have showed that if the conditions (H1), (H3), and (H4) are satisfied and $\tau_2 \in [0, \tau_{2_0})$, then the zero equilibrium $E_0(0,0)$ is asymptotically stable when $\tau_1 \in (0, \tau_{1_0})$, as the delay τ_1 increases, the zero equilibrium $E_0(0,0)$ loses its stability and a sequence of Hopf bifurcations occur at the zero equilibrium $E_0(0,0)$. The direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. To verify some of the mathematical results, computer simulations are carried out. From the viewpoint of energy, we know that if there is no delay when 2-component energy transmits to 1-component energy, and the time delay which 1-component energy transmits to 1-component energy stays in the interval $[0, \tau_{2_0})$, then the two-component energy can keep a stable situation near (w_1*, w_2^*). When the time delay passes a certain critical value, two-component energy will lose stability and two-component energy appears in a periodic phenomenon with limit cycle. When there are time delays when 2-component energy transmits to 1-component energy and 1-component energy transmits to 2-component energy, if we fix the time delay which 2-component energy transmits to 1-component energy in its stable range, then two-component energy will keep in a stable situation near (w_1*, w_2^*). When the time delay which 2-component energy transmits to 1-component energy passes a certain critical value, two-component energy will lose stability and two-component energy appears in a periodic phenomenon with limit cycle. From an ecological viewpoint, the interaction parameter areas of coexistence for two components are desirable. This method is important for understanding the regulatory chemical mechanisms of ecological systems, which provides a control mechanism to ensure a coexistence transition from an equilibrium to a periodic oscillation with desired amplitude and robust period [2, 5, 15, 16, 22, 23, 26]. In addition, we would like to point out that we can also investigate the Hopf bifurcation of system (1.3) by choosing the delay τ_2 as bifurcation parameter. We leave this for future consideration.

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