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Convergence theorems for two finite families of some generalized nonexpansive mappings in hyperbolic spaces

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Abstract

We propose and analyze a one-step explicit iterative algorithm for two finite families of mappings satisfying condition (C) in hyperbolic spaces. Our results are new and generalize several recent results in uniformly convex Banach spaces and CAT(0) spaces, simultaneously.

Keywords: Hyperbolic space, one-step iterative algorithm, nonexpansive mapping, condition (C), common fixed point, strong convergence, $\triangle$-convergence.
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1. Introduction and preliminaries

In Banach spaces, iterative construction of fixed points of nonexpansive mappings and their generalizations depend on the linear structure of the space. A nonlinear framework for the iterative construction of fixed points of certain classes of nonlinear mappings is a metric space embedded with a convex structure. Different notions of convexity in metric spaces are available (see, for example Kohlenbach [11], Menger [13], Reich and Shafrir [15], Penot [14] and Takahashi [17]).

A metric space (X, d) is called a convex metric space [13] if

(i) there exists a family $\mathcal{F}$ of metric segments such that any two points x, y in X are endpoints of a unique metric segment $[x, y] \in \mathcal{F}$ ($[x, y]$ is an isometric image of $[0, d(x, y)]$);

(ii) the unique point $z = \alpha x \oplus (1-\alpha)y$ of $[x, y]$ satisfies

$$d(x, z) = (1-\alpha)d(x, y), \quad \text{and} \quad d(z, y) = \alpha d(x, y), \quad \text{for } \alpha \in I = [0, 1].$$

Above definition provides that $0x \oplus 1y = y, 1x \oplus 0y = x$ and $\alpha x \oplus (1-\alpha)x = x$.

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A convex metric space X is hyperbolic if

$$d\left(\alpha x \oplus (1-\alpha)y, \alpha z \oplus (1-\alpha)w\right) \leqslant \alpha d\left(x,z\right) + (1-\alpha)\, d\left(y,w\right),$$

for all $x, y, z, w \in X$ and $\alpha \in I$ (see also [1]).

For $z = w$, the hyperbolic inequality reduces to:

$$d\left(\alpha x \oplus (1-\alpha)y, z\right) \leqslant \alpha d\left(x,z\right) + (1-\alpha)\, d\left(y,z\right), \tag{1.1}$$

which is the convex structure inequality due to Takahashi [17].

A subset K of X is convex, if $\alpha x \oplus (1-\alpha)y \in K$ for all $x, y \in K$ and $\alpha \in I$. Normed spaces and their subsets are linear hyperbolic spaces while CAT(0) spaces qualify for the criteria of nonlinear hyperbolic spaces. For a fixed $a \in X$, $r > 0$ and $\varepsilon > 0$, set

$$\delta(r,\varepsilon) = \inf_{\substack{d(a,x)\leqslant r, d(a,y)\leqslant r,\\ d(x,y)\geqslant r\varepsilon}} \left(1 - \frac{1}{r} d\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right)\right),$$

for any $x, y \in X$. Then the hyperbolic space X is uniformly convex if $\delta(r,\varepsilon) > 0$.

From now onwards we assume that X is a uniformly convex hyperbolic space with the property that for every $s \geqslant 0$, $\varepsilon > 0$, there exists $\eta(s,\varepsilon) > 0$ depending on s and ε such that $\delta(r,\varepsilon) > \eta(s,\varepsilon) > 0$ for any $r > s$.

Let $T : K \to K$ be a mapping. A point $x \in K$ is a fixed point of T, if $Tx = x$. Denote by $F(T)$, the set of all fixed points of T. The mapping T is

(i) nonexpansive, if $d(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in K$;

(ii) quasi-nonexpansive, if $d(Tx, y) \leqslant d(x, y)$ for all $x \in K, y \in F(T)$;

(iii) said to satisfy condition (C), if $\frac{1}{2}d(x, Tx) \leqslant d(x, y)$ implies $d(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in K$.

It has been shown in [16] that condition (C) is weaker than nonexpansiveness but stronger than quasi-nonexpansiveness. Moreover, if a mapping T satisfies condition (C), it may or may not be continuous.

We present the following example (see also [16]).

Example 1.1. Take $X = \mathbb{R}, K = [0,3]\,, T_1, T_2 : K \to K$ are given by

$$T_1\left(x\right) = \begin{cases} 0 & \text{if } x \neq 3, \\ \frac{2}{3} & \text{if } x = 3, \end{cases}$$

and

$$T_2\left(x\right) = \begin{cases} 0 & \text{if } x \neq 3, \\ \frac{3}{2} & \text{if } x = 3. \end{cases}$$

Then we observe that

(i) T_1 satisfies condition (C) and T_1 is not nonexpansive as it is discontinuous at $x = 3$;

(ii) $F\left(T_2\right) = \{0\}$ and T_2 is quasi-nonexpansive but T_2 does not satisfy condition (C).

Denote by $N = \{1, 2, 3, \cdots, r\}$, the indexing set. To reduce computational cost of a two-step iterative algorithm for two finite families $\{S_n : n \in N\}$ and $\{T_n : n \in N\}$ of nonexpansive mappings on a convex subset K of a Banach space, Khan et al. [9] introduced the following one-step implicit iterative algorithm (see also [12]):

$$x_0 \in K, \quad x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n S_n x_n,$$

where $S_n = S_{n(\mathrm{mod}\, r)}$ and $T_n = T_{n(\mathrm{mod}\, r)}$, $0 \leqslant \alpha_n,\ \beta_n, \gamma_n \leqslant 1$ and satisfy $\alpha_n + \beta_n + \gamma_n = 1$.

Keeping in mind that explicit iterative algorithm is simpler than an implicit iterative algorithm and has less computational cost, Gunduz and Akbulut [6] constructed a one-step explicit iterative algorithm for two finite families $\{S_n : n \in N\}$ and $\{T_n : n \in N\}$ of nonexpansive mappings in a hyperbolic space as under

$$x_1 \in K, \quad x_{n+1} = \alpha_n T_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} S_n x_n \oplus \left(1 - \frac{\beta_n}{1-\alpha_n}\right) x_n\right), \tag{1.2}$$

where $S_n = S_{n(\mathrm{mod}\, r)}$, $T_n = T_{n(\mathrm{mod}\, r)}$, $0 < b \leqslant \alpha_n,\ \beta_n \leqslant c < 1$, $\alpha_n + \beta_n < 1$.

Nonexpansive mappings are always continuous but mappings satisfying condition (C) may or may not be continuous. We study (1.2) for discontinuous mappings, namely mappings satisfying condition (C). For more details about convergence analysis of different iterative algorithms for different classes of mappings, we refer the reader to [7, 8].

Let $\{x_n\}$ be a bounded sequence in a metric space X. We define a functional $r(., \{x_n\}) : X \to \mathbb{R}^+$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n),$$

for all $x \in X$. The asymptotic radius of $\{x_n\}$ with respect to $K \subseteq X$ is defined as

$$r(\{x_n\}) = \inf_{x\in K} r\left(x, \{x_n\}\right).$$

A point $y \in K$ is called the asymptotic center of $\{x_n\}$ with respect to $K \subseteq X$ if

$$r(y, \{x_n\}) \leqslant r(x, \{x_n\}),$$

for all $x \in K$. The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

A sequence $\{x_n\}$ in (X, d) :

(iv) is Fejér monotone with respect to a subset K of X, if $d(x_{n+1}, x) \leqslant d(x_n, x)$ for all $x \in K$;

(v) $\triangle$-converges to $x \in X$, if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [10]. In this case, we write x as $\triangle$-limit of $\{x_n\}$, i.e., $\triangle$-$\lim_n x_n = x$.

To develop our main section, we need the following lemmas.

Lemma 1.2 ([16]). *Let* T *be a mapping on a subset* K *of a metric space* X. *Assume that* T *satisfies condition (C) and has a fixed point. Then* T *is quasi-nonexpansive.*

Lemma 1.3 ([16]). *Let* T *be a mapping on a subset* K *of a metric space* X. *If* T *satisfies condition (C), then*

$$d\left(x, Ty\right) \leqslant 3d\left(Tx, x\right) + d\left(x, y\right),$$

for all $x, y \in K$.

Lemma 1.4 ([2]). *Let* K *be a nonempty closed subset of a complete metric space* (X, d) *and* $\{x_n\}$ *be Fejér monotone with respect to* K. *Then* $\{x_n\}$ *converges to* $p \in K$, *if and only if* $\lim_{n\to\infty} d(x_n, K) = 0$.

Lemma 1.5 ([3]). *Let* K *be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space* X. *Then every bounded sequence* $\{x_n\}$ *in* X *has a unique asymptotic center with respect to* K.

Lemma 1.6 ([3]). *Let* K *be a nonempty closed and convex subset of a uniformly convex hyperbolic space and* $\{x_n\}$ *a bounded sequence in* K *such that* $A(\{x_n\}) = \{y\}$ *and* $r(\{x_n\}) = \rho$. *If* $\{y_m\}$ *is another sequence in* K *such that* $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, *then* $\lim_{m\to\infty} y_m = y$.

Lemma 1.7 ([5]). *Let X be a uniformly convex hyperbolic space. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in X such that*

$$\limsup_{n \longrightarrow \infty} d(u_n, x) \leqslant r, \quad \limsup_{n \longrightarrow \infty} d(v_n, x) \leqslant r,$$

and

$$\lim_{n \longrightarrow \infty} d(a_n u_n \oplus (1 - a_n) v_n, x) = r,$$

for some $r \geqslant 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.

From now onwards, we set $F = \cap_{i \in N} (F(T_i) \cap F(S_i)) \neq \phi$ for two finite families $\{T_n : n \in N\}$ and $\{S_n : n \in N\}$ of mappings on K.

2. Main results

We start with the following lemma.

Lemma 2.1. *Let K be a closed and convex subset of a hyperbolic space X and the two finite families $\{S_n : n \in N\}$ and $\{T_n : n \in N\}$ of mappings on K, satisfy condition (C). If $F \neq \phi$, then for the sequence $\{x_n\}$ defined in (1.2), we have the followings:*

(a) *$\{x_n\}$ is Fejér monotone with respect to F;*

(b) *$\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$;*

(c) *$\lim_{n\to\infty} d(x_n, F)$ exists.*

Proof. Let $p \in F$. In the light of Lemma 1.2, we apply inequality (1.1) to the iterative algorithm (1.2) and get that

$$\begin{aligned} d(x_{n+1}, p) &= d\left(\alpha_n T_n x_n \oplus (1 - \alpha_n)\left(\frac{\beta_n}{1 - \alpha_n} S_n x_n \oplus \left(1 - \frac{\beta_n}{1 - \alpha_n}\right) x_n\right), p\right) \\ &\leqslant \alpha_n d(T_n x_n, p) + \beta_n d(S_n x_n, p) + (1 - \alpha_n - \beta_n) d(x_n, p) \\ &\leqslant \alpha_n d(x_n, p) + \beta_n d(x_n, p) + (1 - \alpha_n - \beta_n) d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

The above inequality provides that

(a) $\{x_n\}$ is Fejér monotone with respect to F; and

(b) $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.

Also the inequality $\inf_{p \in F} d(x_{n+1}, p) \leqslant \inf_{p \in F} d(x_n, p)$ gives that

(c) $\lim_{n\to\infty} d(x_n, F)$ exists.

□

Lemma 2.2. *Let K be a closed and convex subset of a complete uniformly convex hyperbolic space X and let $\{S_n : n \in N\}$ and $\{T_n : n \in N\}$ be two finite families of mappings on K satisfying condition (C). If $\{x_n\}$ is any bounded sequence in K such that*

$$\lim_{n\to\infty} d(x_n, S_l x_n) = 0 = \lim_{n\to\infty} d(x_n, T_l x_n),$$

for all $l \in N$, then $F \neq \phi$.

Proof. Since $\{x_n\}$ is bounded, therefore $\{x_n\}$ has a unique asymptotic centre, that is, $A(\{x_n\}) = \{x\}$. We show that $x \in F(S_l)$. By Lemma 1.3, we have

$$d(x_n, S_l x) \leqslant 3d(x_n, S_l x_n) + d(x_n, x),$$

which further implies

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, S_l x) &\leqslant 3\limsup_{n\to\infty} d(x_n, S_l x_n) + \limsup_{n\to\infty} d(x_n, x)\\ &= \limsup_{n\to\infty} d(x_n, x).\end{aligned}$$

By the uniqueness of asymptotic centers, we have that $S_l x = x$. Similarly, we can prove that $T_l x = x$. Hence $F \neq \phi$. □

Lemma 2.3. *Let K be a closed and convex subset of a hyperbolic space X and let $\{S_n : n \in N\}$ and $\{T_n : n \in N\}$ be two finite families of mappings on K satisfying condition (C) and $\{x_n\}$ be given in (1.2). Then $\{x_n\}$ is bounded and*

$$\lim_{n\to\infty} d(x_n, S_l x_n) = 0 = \lim_{n\to\infty} d(x_n, T_l x_n),$$

for all $l \in N$.

Proof. Let $p \in F$. Then by Lemma 2.1, $\lim_{n\to\infty} d(x_n, p)$ exists and therefore $\{x_n\}$ is bounded. Next, we show that

$$\lim_{n\to\infty} d(x_n, S_l x_n) = 0 = \lim_{n\to\infty} d(x_n, T_l x_n),$$

for all $l \in N$. Assume that $\lim_{n\to\infty} d(x_n, p) = c$. The result is trivial, if $c = 0$. If $c > 0$, then $\lim_{n\to\infty} d(x_n, p) = c$ can be written as

$$\lim_{n\to\infty} d\left(\alpha_n T_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} S_n x_n \oplus \left(1 - \frac{\beta_n}{1-\alpha_n}\right) x_n\right), p\right) = c. \tag{2.1}$$

Since T_n satisfies condition (C) and has a fixed point p, therefore $d(T_n x_n, p) \leqslant d(x_n, p)$ which further implies that $\limsup_{n\to\infty} d(T_n x_n, p) \leqslant c$.

Also the inequality

$$\begin{aligned} d\left(\frac{\beta_n}{1-\alpha_n} S_n x_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right) x_n, p\right) &\leqslant \frac{\beta_n}{1-\alpha_n} d(S_n x_n, p) + \left(1-\frac{\beta_n}{1-\alpha_n}\right) d(x_n, p)\\ &\leqslant \frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1-\frac{\beta_n}{1-\alpha_n}\right) d(x_n, p)\\ &= d(x_n, p), \end{aligned}$$

provides that

$$\limsup_{n\to\infty} d\left(\frac{\beta_n}{1-\alpha_n} S_n x_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right) x_n, p\right) \leqslant c. \tag{2.2}$$

Setting $x = p$, $r = c$, $a_n = \alpha_n$, $w_n = T_n x_n$, $z_n = \frac{\beta_n}{1-\alpha_n} S_n x_n \oplus \left(1 - \frac{\beta_n}{1-\alpha_n}\right) x_n$ in Lemma 1.7 together with (2.1) and (2.2), we get

$$\lim_{n\to\infty} d\left(T_n x_n, \frac{\beta_n}{1-\alpha_n} S_n x_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right) x_n\right) = 0. \tag{2.3}$$

Observe that

$$d(x_{n+1}, T_n x_n) = d\left(\alpha_n T_n x_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} S_n x_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right) x_n\right), T_n x_n\right)$$

$$\leqslant (1-\alpha_n)d\left(\frac{\beta_n}{1-\alpha_n}S_nx_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right)x_n, T_nx_n\right)$$
$$\leqslant (1-a)d\left(\frac{\beta_n}{1-\alpha_n}S_nx_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right)x_n, T_nx_n\right).$$

Taking $\limsup_{n\to\infty}$ on both sides in the above inequality and using (2.3), we have

$$\lim_{n\to\infty} d(x_{n+1}, T_nx_n) = 0. \tag{2.4}$$

Moreover by triangle inequality,

$$d(x_{n+1}, p) \leqslant d(x_{n+1}, T_nx_n) + d\left(T_nx_n, \frac{\beta_n}{1-\alpha_n}S_nx_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right)x_n\right)$$
$$+ d\left(\frac{\beta_n}{1-\alpha_n}S_nx_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right)x_n, p\right).$$

Taking $\liminf_{n\to\infty}$ on both sides of the above estimate and then utilizing (2.3) and (2.4), we have

$$c \leqslant \liminf_{n\to\infty} d\left(\frac{\beta_n}{1-\alpha_n}S_nx_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right)x_n, p\right). \tag{2.5}$$

Combining (2.2) and (2.5), we get

$$\lim_{n\to\infty} d\left(\frac{\beta_n}{1-\alpha_n}S_nx_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right)x_n, p\right) = c. \tag{2.6}$$

Choosing $x = p,\ r = c,\ a_n = \frac{\beta_n}{1-\alpha_n},\ w_n = S_nx_n,\ z_n = x_n$ in Lemma 1.7 together with (2.6), we get

$$\lim_{n\to\infty} d(x_n, S_nx_n) = 0. \tag{2.7}$$

From the inequality

$$d(x_{n+1}, x_n) \leqslant d\left(\alpha_nT_nx_n \oplus (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n}S_nx_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right)x_n\right), x_n\right)$$
$$\leqslant \alpha_n d(T_nx_n, x_n) + (1-\alpha_n)\, d\left(\frac{\beta_n}{1-\alpha_n}S_nx_n \oplus \left(1-\frac{\beta_n}{1-\alpha_n}\right)x_n, x_n\right)$$
$$\leqslant \alpha_n d(T_nx_n, x_n) + \beta_n d(S_nx_n, x_n)$$
$$\leqslant \alpha_n \{d(x_{n+1}, T_nx_n) + d(x_{n+1}, x_n)\} + \beta_n d(S_nx_n, x_n),$$

we obtain that

$$d(x_{n+1}, x_n) \leqslant \frac{\alpha_n}{1-\alpha_n} d(x_{n+1}, T_nx_n) + \frac{\beta_n}{1-\alpha_n} d(S_nx_n, x_n)$$
$$\leqslant \frac{c}{1-c} d(x_{n+1}, T_nx_n) + \frac{c}{1-c} d(S_nx_n, x_n).$$

Taking $\limsup_{n\to\infty}$ on both sides in the above inequality and then using (2.4) and (2.7), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.8}$$

For each $l < r$, the inequality

$$d(x_n, x_{n+l}) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+l-1}, x_{n+l}),$$

and (2.8) provide that

$$\lim_{n\to\infty} d(x_n, x_{n+l}) = 0, \quad \text{for each } l < r. \tag{2.9}$$

Since

$$d(x_n, T_n x_n) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, T_n x_n),$$

therefore it follows that

$$\lim_{n\to\infty} d(x_n, T_n x_n) = 0.$$

Using Lemma 1.2, we estimate

$$\begin{aligned} d(x_n, S_{n+l}x_n) &\leqslant d(x_n, x_{n+l}) + d(x_{n+l}, S_{n+l}x_{n+l}) + d(S_{n+l}x_{n+l}, S_{n+l}x_n) \\ &\leqslant d(x_n, x_{n+l}) + 2d(x_{n+l}, S_{n+l}x_{n+l}) + d(x_{n+l}, S_{n+l}x_n) \\ &\leqslant 2d(x_n, x_{n+l}) + 5d(x_{n+l}, S_{n+l}x_{n+l}). \end{aligned}$$

Therefore by $\limsup_{n\to\infty}$ on both sides in the above inequality and then using (2.7) and (2.9), we get that

$$\lim_{n\to\infty} d(x_n, S_{n+l}x_n) = 0, \quad \text{for each } l \in N.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, T_{n+l}x_n) = 0, \quad \text{for each } l \in N.$$

Since for each $l \in N$, the sequence $\{d(x_n, S_l x_n)\}$ is a subsequence of $\cup_{l=1}^{N}\{d(x_n, S_{n+l}x_n)\}$ and

$$\lim_{n\to\infty} d(x_n, S_{n+l}x_n) = 0,$$

for each $l \in N$, therefore

$$\lim_{n\to\infty} d(x_n, S_l x_n) = 0 = \lim_{n\to\infty} d(x_n, T_l x_n), \quad \text{for each } l \in N.$$

□

Now we prove our $\triangle$-convergence result by using the algorithm (1.2) without requiring the continuity of the mappings.

Theorem 2.4. *Let K be a closed and convex subset of a hyperbolic space X and let $\{S_n : n \in N\}$ and $\{T_n : n \in N\}$ be two finite families of mappings on K satisfying condition (C) and $\{x_n\}$ be given in (1.2). If $F \neq \phi$, then $\{x_n\}$ $\triangle$-converges to an element of F.*

Proof. In the proof of Lemma 2.1, we have that $\{x_n\}$ is bounded. Therefore $\{x_n\}$ has a unique asymptotic centre, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$ and

$$\lim_{n\to\infty} d(u_n, T_l u_n) = 0 = \lim_{n\to\infty} d(u_n, S_l u_n),$$

for each $l \in N$ (by Lemma 2.3).

By Lemma 2.2, we get that $u \in F$. Therefore $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 2.1. If $x \neq u$, then by the uniqueness of asymptotic centres, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leqslant \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction. Hence $x = u$.

Therefore, $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle$-converges to $x \in F$. □

A mapping $T : K \to K$ is semi-compact, if any bounded sequence $\{x_n\}$ in K has a convergent subsequence whenever $d(x_n, Tx_n) \to 0$.

Two finite families $\{T_n : n \in N\}$ and $\{S_n : n \in N\}$ of mappings on K with nonempty common fixed point set F are said to satisfy condition (M), if there exists a nondecreasing function f on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$, for all $t \in (0, \infty)$ such that

$$\max_{i \in N} d\left(x, T_i x\right) \geqslant f(d(x, F)), \quad \text{or} \quad \max_{i \in N} d\left(x, S_i x\right) \geqslant f(d(x, F)),$$

for all $x \in K$.

Using Lemma 1.3 and Lemma 2.3, we obtain the following strong convergence theorems.

Theorem 2.5. *Let K be a closed and convex subset of a hyperbolic space X and let $\{S_n : n \in N\}$ and $\{T_n : n \in N\}$ be two finite families of mappings on K satisfying condition (C) and $\{x_n\}$ be given in (1.2). If at least one $T \in \{T_n : n \in N\}$ or one $S \in \{S_n : n \in N\}$ is semi-compact, then the sequence $\{x_n\}$ converges strongly to an element of F.*

Proof. Let T_1 be semi-compact. Since $\{x_n\}$ is bounded and $d\left(x_n, T_1 x_n\right) \to 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in K$ and

$$\lim_{j \to \infty} d(x_{n_j}, S_l x_{n_j}) = 0 = \lim_{j \to \infty} d(x_{n_j}, T_l x_{n_j}),$$

for each $l \in N$.

By Lemma 1.3, we have

$$d\left(x_{n_j}, S_l q\right) \leqslant 3d\left(x_{n_j}, S_l x_{n_j}\right) + d\left(x_{n_j}, q\right).$$

By $\lim_{j \to \infty}$ on both sides in the above inequality, we have that $S_l q = q$. Similarly $T_l q = q$. That is $q \in F$.

As $\lim_{n \to \infty} d(x_n, p)$ exists (Lemma 2.1), therefore $x_n \to q \in F$. □

Theorem 2.6. *Let K be a closed and convex subset of a complete and uniformly convex hyperbolic space X and let $\{S_n : n \in N\}$ and $\{T_n : n \in N\}$ be two finite families of mappings on K satisfying condition (C) and $\{x_n\}$ be given in (1.2). If $\{T_n : n \in N\}$ or $\{S_n : n \in N\}$ satisfy condition (M), then the sequence $\{x_n\}$ converges strongly to an element of F.*

Proof. Let $z_n \in F\left(T_l\right)$ such that $z_n \to z$. We show that $z \in F\left(T_l\right)$.

Appealing to Lemma 1.3, we have

$$d\left(z_n, T_l z\right) \leqslant 3d\left(z_n, T_l z_n\right) + d\left(z_n, z\right).$$

This gives that $z_n \to T_l z$ and hence $T_l z = z$. Therefore $F\left(T_l\right)$ is closed. Similarly $F\left(S_l\right)$ is closed. Finally we get that F is closed. Using condition (M) and Lemma 2.3, we have that $\lim_{n \to \infty} d(x_n, F) = 0$. Since $\{x_n\}$ is Fejér monotone with respect to the set F, therefore $x_n \to p \in F$. □

The followings are corollaries to our Theorems 2.4–2.6 and yet are new in the literature.

Corollary 2.7. *Let K be a closed and convex subset of a hyperbolic space X and let $\{T_n : n \in N\}$ be a finite family of mappings on K satisfying condition (C) and $\{x_n\}$ be given by*

$$x_1 \in K, \quad x_{n+1} = \alpha_n T_n x_n \oplus (1 - \alpha_n) x_n,$$

where $T_n = T_{n(\mathrm{mod}\, r)}$, $0 < b \leqslant \alpha_n \leqslant c < 1$. If $\cap_{i \in N} F(T_i) \neq \phi$, then $\{x_n\}$ $\triangle$-converges to an element of $\cap_{i \in N} F(T_i)$.

Proof. Choose $S_i = I$ in Theorem 2.4 for $i \in N$. □

Corollary 2.8. *Let K be a closed and convex subset of a hyperbolic space X and let T be a mapping satisfying condition (C) and $\{x_n\}$ be given by*

$$x_1 \in K, \quad x_{n+1} = \alpha_n T x_n \oplus (1 - \alpha_n) x_n,$$

where $0 < b \leqslant \alpha_n \leqslant c < 1$. If $F(T) \neq \phi$, then $\{x_n\}$ $\triangle$-converges to an element of $F(T)$.

Proof. Choose $S_i = I$ and $T_n = T$ in Theorem 2.4 for $i \in N$. □

Corollary 2.9. *Let K be a closed and convex subset of a hyperbolic space X and let $\{T_n : n \in N\}$ be a finite family of mappings on K satisfying condition (C) and $\{x_n\}$ be given by*

$$x_1 \in K, \quad x_{n+1} = \alpha_n T_n x_n \oplus (1-\alpha_n) x_n,$$

where $T_n = T_{n(\mathrm{mod}\, r)}$, $0 < b \leqslant \alpha_n \leqslant c < 1$. If $\cap_{i \in N} F(T_i) \neq \phi$ and at least one $T \in \{T_n : n \in N\}$ is semi-compact, then the sequence $\{x_n\}$ converges strongly to an element of $\cap_{i \in N} F(T_i)$.

Proof. Choose $S_i = I$ in Theorem 2.5 for $i \in N$. □

Remark 2.10. The CAT (0) spaces and uniformly convex Banach spaces are the special cases of uniformly convex hyperbolic spaces, therefore our results also hold in CAT (0) spaces and uniformly convex Banach spaces, simultaneously.

Remark 2.11. Every nonexpansive mapping is a mapping satisfying condition (C), therefore our theorems generalize the corresponding ones in [1, 4–6, 18, 19], etc.

Remark 2.12. Every nonexpansive mapping is always continuous but a mapping satisfying condition (C) may or may not be continuous. Therefore our results hold for discontinuous mappings also.

Remark 2.13. The iterative algorithm (1.2) is computationally simpler than the following Ishikawa iterative algorithm:

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= \alpha_n T_n y_n \oplus (1-\alpha_n) x_n, \\ y_n &= \beta_n S_n x_n \oplus (1-\beta_n) x_n, \end{aligned}$$

therefore our results are better.

Remark 2.14. The essentials of hypotheses in our results are natural in view of the following observations: $\alpha_n = \frac{n+50}{100n}$, $\beta_n = \frac{n+51}{101n}$, $X = \mathbb{R}$, $K = [0,3]$, $S_i, T_i : C \to C$ are given by

$$T_i(x) = \begin{cases} 0 & \text{if } x \neq 3, \\ \frac{i+1}{i+2} & \text{if } x = 3, \end{cases}$$

and

$$S_i(x) = \begin{cases} 0 & \text{if } x \neq 3, \\ \frac{i+2}{i+4} & \text{if } x = 3. \end{cases}$$

Then

(i) $0 < \alpha_n, \ \beta_n < 1$;

(ii) S_i, T_i satisfy condition (C);

(iii) $\cap_{i \in N} (F(T_i) \cap F(S_i)) = \{0\}$.

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