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Hybrid steepest-descent methods for systems of variational inequalities with constraints of variational inclusions and convex minimization problems

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Abstract

Two hybrid steepest-descent schemes (implicit and explicit) for finding a solution of the general system of variational inequalities (in short, GSVI) with the constraints of finitely many variational inclusions for maximal monotone and inverse-strongly monotone mappings and a minimization problem for a convex and continuously Fréchet differentiable functional (in short, CMP) have been presented in a real Hilbert space. We establish the strong convergence of these two hybrid steepest-descent schemes to the same solution of the GSVI, which is also a common solution of these finitely many variational inclusions and the CMP. Our results extend, improve, complement and develop the corresponding ones given by some authors recently in this area.

Keywords: Hybrid steepest-descent method, system of variational inequalities, variational inclusion, monotone mapping.
2010 MSC: 49J30, 47H09, 47J20, 49M05.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $T : C \to C$ be a self-mapping on C. We denote by $\text{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers. A mapping $A : H \to H$ is called $\bar{\gamma}$-strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geqslant \bar{\gamma} \|x\|^2, \quad \forall x \in H.$$

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A mapping $F : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geqslant 0$ such that

$$\|Fx - Fy\| \leqslant L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then F is called a nonexpansive mapping; if $L \in [0, 1)$ then F is called a contraction.

Let $\mathcal{A} : C \to H$ be a nonlinear mapping on C. The variational inequality problem (VIP) associated with the set C and the mapping $\mathcal{A}$ is stated as follows: find $x^* \in C$ such that

$$\langle \mathcal{A}x^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, \mathcal{A})$.

The VIP (1.1) was first discussed by Lions [16] and now is well-known. Variational inequalities have extensively been investigated, see [1, 2, 15, 17, 18, 20, 22, 24, 26–28] for more details. In 1976, Korpelevich [14] proposed an iterative algorithm for solving VIP (1.1) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_n = P_C(x_n - \tau\mathcal{A}x_n), \\ x_{n+1} = P_C(x_n - \tau\mathcal{A}y_n), \quad \forall n \geqslant 0, \end{cases}$$

with $\tau > 0$ a given number, which is known as the extragradient method. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see e.g., [4, 5, 8, 10, 11] and references therein, to name but a few.

In 2001, Yamada [23] introduced the following hybrid steepest-descent method for solving the VIP (1.1) with $C = \mathrm{Fix}(S)$

$$x_{n+1} = (I - \lambda_n \mu\mathcal{A})Sx_n, \quad \forall n \geqslant 0,$$

where $S : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$, $\mathcal{A} : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0, 0 < \mu < \frac{2\eta}{\kappa^2}$, and then proved that under appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of VIP (1.1) with $C = \mathrm{Fix}(S)$.

Furthermore, let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C

$$\text{minimize } \{f(x) : x \in C\} \tag{1.2}$$

(assuming the existence of minimizers). We denote by Γ the set of minimizers of CMP (1.2). It is well-known that the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ determined by the gradient ∇f and the metric projection P_C:

$$x_{n+1} := P_C(x_n - \lambda\nabla f(x_n)), \quad \forall n \geqslant 0, \tag{1.3}$$

or more generally,

$$x_{n+1} := P_C(x_n - \lambda_n\nabla f(x_n)), \quad \forall n \geqslant 0, \tag{1.4}$$

where, in both (1.3) and (1.4), the initial guess x_0 is taken from C arbitrarily, the parameters λ or λ_n are positive real numbers. The convergence of algorithms (1.3) and (1.4) depends on the behavior of the gradient ∇f.

In order to find a solution of the minimization problem (1.2) for a convex and continuously Fréchet differentiable functional $f : C \to \mathbf{R}$, Ceng et al. [6] proposed the following iterative method

$$x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)Sx_n], \quad \forall n \geqslant 0.$$

In [7], Ceng et al. introduced one general composite implicit scheme that generates a net $\{x_t\}$ in an implicit way

$$x_t = (I - \theta_t A)Tx_t + \theta_t[Tx_t - t(\mu FTx_t - \gamma f(x_t))].$$

Very recently, inspired by Ceng et al. [7], Jung [13] introduced one general composite implicit scheme that generates a net $\{x_t\}$ in an implicit way

$$x_t = (I - \theta_t A)T_t x_t + \theta_t[t\gamma V x_t + (I - t\mu F)T_t x_t],$$

and also proposed another general composite explicit scheme that generates a sequence $\{x_n\}$ in an explicit way

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F)T_n x_n, \\ x_{n+1} = (I - \beta_n A)T_n x_n + \beta_n y_n, \quad \forall n \geqslant 0. \end{cases}$$

On the other hand, let $F_1, F_2 : C \to H$ be two mappings. Consider the following general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^* \rangle \geqslant 0, \quad \forall x \in C, \\ \langle \nu_2 F_2 x^* + y^* - x^*, x - y^* \rangle \geqslant 0, \quad \forall x \in C, \end{cases} \tag{1.5}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. The solution set of GSVI (1.5) is denoted by $\mathrm{GSVI}(C, F_1, F_2)$. Recently, many authors have been devoting the study of the GSVI (1.5); see e.g., [9, 11, 25] and the references therein. In 2008, Ceng et al. [10] transformed the GSVI (1.5) into the fixed point problem of the mapping $G = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$, that is, $Gx^* = x^*$, where $y^* = P_C(I - \nu_2 F_2)x^*$. Throughout this paper, the fixed point set of the mapping G is denoted by Ξ.

Let B be a single-valued mapping of C into H and R be a multivalued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{1.6}$$

We denote by $I(B, R)$ the solution set of the variational inclusion (1.6). In particular, if $B = R = 0$, then $I(B, R) = C$. If $B = 0$, then problem (1.6) becomes the inclusion problem introduced by Rockafellar [19]. Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H,$$

where λ is a positive number.

In this paper, we introduce one hybrid implicit steepest-descent scheme and another hybrid explicit steepest-descent scheme for finding a solution of the GSVI (1.5) with the constraints of finitely many variational inclusions for maximal monotone and inverse-strongly monotone mappings and the minimization problem (1.2) for a convex and continuously Fréchet differentiable functional in a real Hilbert space. We establish the strong convergence of these two hybrid steepest-descent schemes to the same solution of the GSVI (1.5), which is also a common solution of these finitely many variational inclusions and the CMP (1.2). In particular, we make use of weaker control conditions than previous ones for the sake of proving strong convergence. Our results extend, improve, complement and develop the corresponding ones announced by some authors recently in this area.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1 ([24, 27]). *Given any $x \in H$ and $z \in C$, one has*

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \leqslant 0,\ \forall y \in C$;

(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \leqslant \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geqslant \|P_C x - P_C y\|^2,\ \forall y \in H$, *which hence implies that P_C is nonexpansive and monotone.*

Definition 2.2. A mapping $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geqslant \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = \frac{1}{2}(I + S)$, where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.3. A mapping $F : C \to H$ is said to be

(i) monotone, if
$$\langle Fx - Fy, x - y\rangle \geqslant 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone, if there exists a constant $\eta > 0$ such that
$$\langle Fx - Fy, x - y\rangle \geqslant \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) α-inverse-strongly monotone, if there exists a constant $\alpha > 0$ such that
$$\langle Fx - Fy, x - y\rangle \geqslant \alpha\|Fx - Fy\|^2, \quad \forall x, y \in C.$$

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that the projection P_C is 1-ism. Inverse-strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

On the other hand, it is obvious that if $F : C \to H$ is α-inverse-strongly monotone, then F is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda F)u - (I - \lambda F)v\|^2 \leqslant \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Fu - Fv\|^2. \tag{2.1}$$

So, if $\lambda \leqslant 2\alpha$, then $I - \lambda F$ is a nonexpansive mapping from C to H.

Proposition 2.4 ([10]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the GSVI (1.5) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x, \quad \forall x \in C,$$

where $y^* = P_C(I - \nu_2 F_2)x^*$.

In particular, if the mapping $F_j : C \to H$ is ζ_j-inverse-strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by Ξ the fixed point set of the mapping G.

Definition 2.5. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1-\alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged mappings.

Proposition 2.6 ([3]). *Let $T : H \to H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*

(ii) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 2.7 ([3]). *Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

The notation $\mathrm{Fix}(T)$ denotes the set of all fixed points of the mapping T, that is, $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.8. *Let H be a real Hilbert space. Then the following hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all $x, y \in H$;*

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$;*

(c) *if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

Lemma 2.9 ([12]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C with $\mathrm{Fix}(S) \neq \emptyset$. Then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Lemma 2.10 ([25]). *Let $F : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection implies*

$$u \in \mathrm{VI}(C, F) \quad \Leftrightarrow \quad u = P_C(u - \lambda F u), \quad \lambda > 0.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C,$$

where $F : C \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leqslant \kappa\|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y\rangle \geqslant \eta\|x - y\|^2,$$

for all $x, y \in C$.

Lemma 2.11 ([10]). *T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|T^\lambda x - T^\lambda y\| \leqslant (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Remark 2.12. Since F is κ-Lipschitzian and η-strongly monotone on C, we get $0 < \eta \leqslant \kappa$. Hence, whenever $0 < \mu < \frac{2\eta}{\kappa^2}$, we have $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 2.13 ([21]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leqslant (1 - \omega_n)a_n + \omega_n\delta_n + r_n, \quad \forall n \geqslant 0,$$

where $\{\omega_n\}$, $\{\delta_n\}$, and $\{r_n\}$ satisfy the following conditions:

(i) $\{\omega_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \omega_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} \delta_n \leqslant 0$ *or* $\sum_{n=0}^{\infty} \omega_n|\delta_n| < \infty$;

(iii) $r_n \geqslant 0$ *for all* $n \geqslant 0$, *and* $\sum_{n=1}^{\infty} r_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14 ([17]). *Assume that A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $0 < \rho \leqslant \|A\|^{-1}$. Then $\|I - \rho A\| \leqslant 1 - \rho\bar{\gamma}$.*

Let LIM be a Banach limit. According to time and circumstances, we use $\mathrm{LIM}_n a_n$ instead of $\mathrm{LIM}a$ for every $a = \{a_n\} \in l^\infty$. The following properties are well-known:

(i) for all $n \geqslant 1$, $a_n \leqslant c_n$ implies $\mathrm{LIM}_n a_n \leqslant \mathrm{LIM}_n c_n$;

(ii) $\mathrm{LIM}_n a_{n+N} = \mathrm{LIM}_n a_n$ for any fixed positive integer N;

(iii) $\liminf_{n\to\infty} a_n \leqslant \mathrm{LIM}_n a_n \leqslant \limsup_{n\to\infty} a_n$ for all $\{a_n\} \in l^\infty$.

Lemma 2.15. *Let $a \in \mathbf{R}$ be a real number and let a sequence $\{a_n\} \in l^\infty$ satisfy the condition $\mathrm{LIM}_n a_n \leqslant a$ for all Banach limit LIM. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leqslant 0$, then $\limsup_{n\to\infty} a_n \leqslant a$.*

Recall that a set-valued mapping $\widetilde{T} : D(\widetilde{T}) \subset H \to 2^H$ is called monotone if for all $x, y \in D(\widetilde{T})$, $f \in \widetilde{T}x$ and $g \in \widetilde{T}y$ imply

$$\langle f - g, x - y\rangle \geqslant 0.$$

A set-valued mapping $\widetilde{T}$ is called maximal monotone if $\widetilde{T}$ is monotone and $(I + \lambda\widetilde{T})D(\widetilde{T}) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(\widetilde{T})$ the graph of $\widetilde{T}$. It is known that a monotone mapping $\widetilde{T}$ is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geqslant 0$ for every $(y, g) \in G(\widetilde{T})$ implies $f \in \widetilde{T}x$. Next we provide an example to illustrate the concept of maximal monotone mapping.

Let $\mathcal{A} : C \to H$ be a monotone and Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{u \in H : \langle v - p, u \rangle \geqslant 0, \ \forall p \in C\}.$$

Define

$$\widetilde{T}v = \begin{cases} \mathcal{A}v + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, it is known in [19] that $\widetilde{T}$ is maximal monotone and $0 \in \widetilde{T}v$ if and only if $v \in VI(C, \mathcal{A})$.

Let $R : D(R) \subset H \to 2^H$ be a maximal monotone mapping. Let $\lambda, \mu > 0$ be two positive numbers.

Lemma 2.16 ([11]). *There holds the resolvent identity*

$$J_{R,\lambda}x = J_{R,\mu}\Big(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_{R,\lambda}x\Big), \quad \forall x \in H.$$

Remark 2.17. For $\lambda, \mu > 0$, there holds the following relation

$$\|J_{R,\lambda}x - J_{R,\mu}y\| \leqslant \|x - y\| + |\lambda - \mu|\Big(\frac{1}{\lambda}\|J_{R,\lambda}x - y\| + \frac{1}{\mu}\|x - J_{R,\mu}y\|\Big), \quad \forall x, y \in H. \tag{2.2}$$

Lemma 2.18 ([26]). *$J_{R,\lambda}$ is single-valued and firmly nonexpansive, i.e.,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x - y \rangle \geqslant \|J_{R,\lambda}x - J_{R,\lambda}y\|^2, \quad \forall x, y \in H.$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 2.19 ([26]). *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem (1.6) if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda Bu).$$

Lemma 2.20 ([29]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous and single-valued mapping. Then for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.21 ([4]). *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this section, we always assume the following:

- $f : C \to \mathbf{R}$ is a convex functional with L-Lipschitz continuous gradient ∇f, $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, and $F_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1, 2$;
- A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $\bar{\gamma} \in (1, 2)$, $V : C \to H$ is an l-Lipschitzian mapping with $l \geqslant 0$, $R_i : C \to 2^H$ is a maximal monotone mapping, and $B_i : C \to H$ is η_i-inverse strongly monotone for each $i = 1, \ldots, N$; $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leqslant \gamma l < \tau$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $G : C \to C$ is a mapping defined by $Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x$ with $0 < \nu_j < 2\zeta_j$ for $j = 1, 2$;

- $P_C(I-\lambda_t\nabla f) = s_t I + (1-s_t)T_t$, where T_t is nonexpansive, $s_t = \frac{2-\lambda_t L}{4} \in (0, \frac{1}{2})$ and $\lambda_t : (0,1) \to (0, \frac{2}{L})$ with $\lim_{t\to 0}\lambda_t = \frac{2}{L}$; $P_C(I-\lambda_n\nabla f) = s_n I + (1-s_n)T_n$ where T_n is nonexpansive, $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ and $\{\lambda_n\} \subset (0, \frac{2}{L})$ with $\lim_{n\to\infty}\lambda_n = \frac{2}{L}$;
- $\Lambda_t^N : C \to C$ is a mapping defined by $\Lambda_t^N x = J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\cdots J_{R_1,\lambda_{1,t}}(I-\lambda_{1,t}B_1)x, t \in (0,1)$, for $\{\lambda_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i = 1, ..., N$; $\Lambda_n^N : C \to C$ is a mapping defined by $\Lambda_n^N x = J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\cdots J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1)x$ with $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\lim_{n\to\infty}\lambda_{i,n} = \lambda_i$, for each $i = 1, ..., N$;
- $\Omega := \bigcap_{i=1}^N I(B_i, R_i) \cap \Xi \cap \Gamma \neq \emptyset$ and P_Ω is the metric projection of H onto Ω;
- $\{\alpha_n\} \subset [0,1]$, $\{s_n\} \subset (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$ and $\{s_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})} \subset (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$.

Next, put

$$\Lambda_t^i = J_{R_i,\lambda_{i,t}}(I-\lambda_{i,t}B_i)J_{R_{i-1},\lambda_{i-1,t}}(I-\lambda_{i-1,t}B_{i-1})\cdots J_{R_1,\lambda_{1,t}}(I-\lambda_{1,t}B_1), \quad \forall t \in (0,1),$$

and

$$\Lambda_n^i = J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}}(I-\lambda_{i-1,n}B_{i-1})\cdots J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1), \quad \forall n \geqslant 0,$$

for all $i \in \{1, ..., N\}$, and $\Lambda_t^0 = \Lambda_n^0 = I$, where I is the identity mapping on H.

Since ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism; see [12] (see also, [1]). By Proposition 2.6 (ii) we know that for $\lambda > 0$, $\lambda\nabla f$ is $\frac{1}{\lambda L}$-ism. So by Proposition 2.6 (iii) we deduce that $I-\lambda\nabla f$ is $\frac{\lambda L}{2}$-averaged. Now since the projection P_C is $\frac{1}{2}$-averaged, it is easy to see from Proposition 2.7 (iv) that the composite $P_C(I-\lambda\nabla f)$ is $\frac{2+\lambda L}{4}$-averaged for $\lambda \in (0, \frac{2}{L})$. Hence we obtain that for each $t \in (0,1)$, $P_C(I-\lambda_t\nabla f)$ is $\frac{2+\lambda_t L}{4}$-averaged for each $\lambda_t \in (0, \frac{2}{L})$. Therefore, we can write

$$P_C(I-\lambda_t\nabla f) = \frac{2-\lambda_t L}{4}I + \frac{2+\lambda_t L}{4}T_t = s_t I + (1-s_t)T_t,$$

where T_t is nonexpansive and $s_t := s_t(\lambda_t) = \frac{2-\lambda_t L}{4} \in (0, \frac{1}{2})$ for each $\lambda_t \in (0, \frac{2}{L})$. It is clear that $\lambda_t \to \frac{2}{L}$ iff $s_t \to 0$. Similarly, for each $n \geqslant 0$, $P_C(I-\lambda_n\nabla f)$ is $\frac{2+\lambda_n L}{4}$-averaged for each $\lambda_n \in (0, \frac{2}{L})$. Therefore, we can write

$$P_C(I-\lambda_n\nabla f) = \frac{2-\lambda_n L}{4}I + \frac{2+\lambda_n L}{4}T_n = s_n I + (1-s_n)T_n,$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$. It is clear that $\lambda_n \to \frac{2}{L}$ iff $s_n \to 0$. Note that $\mathrm{Fix}(T_t) = \mathrm{Fix}(T_n) = \Gamma$. By Proposition 2.4, we know that G is nonexpansive and $\Xi = \mathrm{Fix}(G)$. Since $\{\lambda_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, utilizing (2.1) and Lemma 2.18 we have that for all $x, y \in C$

$$\begin{aligned}
\|\Lambda_t^N x - \Lambda_t^N y\| &= \|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}x - J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}y\| \\
&\leqslant \|(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}x - (I-\lambda_{N,t}B_N)\Lambda_t^{N-1}y\| \\
&\leqslant \|\Lambda_t^{N-1}x - \Lambda_t^{N-1}y\| \\
&\ \ \vdots \\
&\leqslant \|\Lambda_t^i x - \Lambda_t^i y\| \\
&\ \ \vdots \\
&\leqslant \|\Lambda_t^0 x - \Lambda_t^0 y\| = \|x-y\|,
\end{aligned}$$

which implies that $\Lambda_t^i : C \to C$ is a nonexpansive mapping for all $t \in (0,1)$. Similarly, we have that for all $x, y \in C$,

$$\|\Lambda_n^N x - \Lambda_n^N y\| = \|J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}x - J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}y\|$$

$$\leqslant \|\Lambda_n^0 x - \Lambda_n^0 y\| \\ = \|x - y\|,$$

which implies that $\Lambda_n^i : C \to C$ is a nonexpansive mapping for all $n \geqslant 0$.

In this section, we introduce the first hybrid implicit steepest-descent scheme that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}$ in an implicit manner:

$$x_t = P_C[(I - s_t A)T_t\Lambda_t^N Gx_t + s_t(t\gamma Vx_t + (I - t\mu F)T_t\Lambda_t^N Gx_t)]. \tag{3.1}$$

We prove the strong convergence of $\{x_t\}$ as $t \to 0$ to a point $\tilde{x} \in \Omega$ which is a unique solution to the VIP

$$\langle (A - I)\tilde{x}, p - \tilde{x}\rangle \geqslant 0, \quad \forall p \in \Omega. \tag{3.2}$$

For arbitrarily given $x_0 \in C$, we also propose the second hybrid explicit steepest-descent scheme, which generates a sequence $\{x_n\}$ in an explicit way:

$$\begin{cases} y_n = \alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T_n\Lambda_n^N Gx_n, \\ x_{n+1} = P_C[(I - s_n A)T_n\Lambda_n^N Gx_n + s_n y_n], \quad \forall n \geqslant 0, \end{cases} \tag{3.3}$$

and establish the strong convergence of $\{x_n\}$ as $n \to \infty$ to the same point $\tilde{x} \in \Omega$, which is also the unique solution to VIP (3.2).

Now, for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$, and $s_t \in (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = P_C[(I - s_t A)T_t\Lambda_t^N Gx + s_t(t\gamma Vx + (I - t\mu F)T_t\Lambda_t^N Gx)], \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - s_t(\bar{\gamma} - 1 + t(\tau - \gamma l))$. Indeed, by Proposition 2.4 and Lemmas 2.11 and 2.14, we have

$$\begin{aligned} \|Q_t x - Q_t y\| &\leqslant \|(I - s_t A)T_t\Lambda_t^N Gx + s_t(t\gamma Vx + (I - t\mu F)T_t\Lambda_t^N Gx) \\ &\quad - (I - s_t A)T_t\Lambda_t^N Gy - s_t(t\gamma Vx + (I - t\mu F)T_t\Lambda_t^N Gy)\| \\ &\leqslant \|(I - s_t A)T_t\Lambda_t^N Gx - (I - s_t A)T_t\Lambda_t^N Gy\| \\ &\quad + s_t\|(t\gamma Vx + (I - t\mu F)T_t\Lambda_t^N Gx) - (t\gamma Vy + (I - t\mu F)T_t\Lambda_t^N Gy)\| \\ &\leqslant (1 - s_t\bar{\gamma})\|T_t\Lambda_t^N Gx - T_t\Lambda_t^N Gy\| + s_t[t\gamma\|Vx - Vy\| \\ &\quad + \|(I - t\mu F)T_t\Lambda_t^N Gx - (I - t\mu F)T_t\Lambda_t^N Gy\|] \\ &\leqslant (1 - s_t\bar{\gamma})\|x - y\| + s_t[t\gamma l\|x - y\| + (1 - t\tau)\|x - y\|] \\ &= [1 - s_t(\bar{\gamma} - 1 + t(\tau - \gamma l))]\|x - y\|. \end{aligned}$$

Since $\bar{\gamma} \in (1, 2)$, $\tau - \gamma l > 0$ and $0 < t < \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\} \leqslant \frac{2-\bar{\gamma}}{\tau-\gamma l}$, it follows that $0 < \bar{\gamma} - 1 + t(\tau - \gamma l) < 1$, which together with $0 < s_t < \min\{\frac{1}{2}, \|A\|^{-1}\} < 1$ yields $0 < 1 - s_t(\bar{\gamma} - 1 + t(\tau - \gamma l)) < 1$. Hence $Q_t : C \to C$ is a contractive mapping. By the Banach contraction principle, Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$.

Proposition 3.1. *Let $\{x_t\}$ be defined via (3.1). Then*

(i) *$\{x_t\}$ is bounded for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$;*

(ii) *$\lim_{t\to 0}\|x_t - T_t x_t\| = 0$, $\lim_{t\to 0}\|x_t - \Lambda_t^N x_t\| = 0$ and $\lim_{t\to 0}\|x_t - Gx_t\| = 0$ provided $\lim_{t\to 0} s_t = 0$;*

(iii) *$x_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to H$ is locally Lipschitzian provided $s_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$ is locally Lipschitzian and $\lambda_{i,t} : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [a_i, b_i]$ is locally Lipschitzian for each $i = 1, ..., N$;*

(iv) x_t *defines a continuous path from* $(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ *into* H *provided*

$$s_t : (0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0,\min\{\frac{1}{2},\|A\|^{-1}\})$$

is continuous and $\lambda_{i,t} : (0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [a_i,b_i]$ *is continuous for each* $i=1,...,N$.

Proof. (i) Let $p \in \Omega$. Noting that $\mathrm{Fix}(G)=\Xi$, $\mathrm{Fix}(T_t)=\Gamma$, and $\Lambda_t^i p = p$ for each $i=1,...,N$, by the nonexpansivity of G, T_t and Λ_t^i and Lemmas 2.11 and 2.14 we get

$$\begin{aligned}
\|x_t-p\| &\leqslant \|(I-s_tA)T_t\Lambda_t^NGx_t + s_t(t\gamma Vx_t+(I-t\mu F)T_t\Lambda_t^NGx_t)-p\| \\
&= \|(I-s_tA)T_t\Lambda_t^NGx_t-(I-s_tA)T_t\Lambda_t^NGp+s_t(t\gamma Vx_t+(I-t\mu F)T_t\Lambda_t^NGx_t-p)+s_t(I-A)p\| \\
&\leqslant \|(I-s_tA)T_t\Lambda_t^NGx_t-(I-s_tA)T_t\Lambda_t^NGp\|+s_t\|t\gamma Vx_t+(I-t\mu F)T_t\Lambda_t^NGx_t-p\|+s_t\|(I-A)p\| \\
&= \|(I-s_tA)T_t\Lambda_t^NGx_t-(I-s_tA)T_t\Lambda_t^NGp\| \\
&\quad + s_t\|(I-t\mu F)T_t\Lambda_t^NGx_t-(I-t\mu F)T_t\Lambda_t^NGp+t(\gamma Vx_t-\mu Fp)\|+s_t\|(I-A)p\| \\
&\leqslant (1-s_t\bar{\gamma})\|T_t\Lambda_t^NGx_t-T_t\Lambda_t^NGp\|+s_t[\|(I-t\mu F)T_t\Lambda_t^NGx_t-(I-t\mu F)T_t\Lambda_t^NGp\| \\
&\quad + t(\gamma\|Vx_t-Vp\|+\|\gamma Vp-\mu Fp\|)]+s_t\|(I-A)p\| \\
&\leqslant (1-s_t\bar{\gamma})\|x_t-p\|+s_t[(1-t\tau)\|x_t-p\|+t(\gamma l\|x_t-p\|+\|(\gamma V-\mu F)p\|)]+s_t\|I-A\|\|p\| \\
&= [1-s_t(\bar{\gamma}-1+t(\tau-\gamma l))]\|x_t-p\|+s_t[\|I-A\|\|p\|+t\|(\gamma V-\mu F)p\|].
\end{aligned}$$

So, it follows that

$$\|x_t-p\| \leqslant \frac{\|I-A\|\|p\|+t\|(\gamma V-\mu F)p\|}{\bar{\gamma}-1+t(\tau-\gamma l)} \leqslant \frac{\|I-A\|\|p\|+t\|(\gamma V-\mu F)p\|}{\bar{\gamma}-1} \leqslant \frac{\|I-A\|\|p\|+\|(\gamma V-\mu F)p\|}{\bar{\gamma}-1}.$$

Hence $\{x_t\}$ is bounded and so are $\{Vx_t\},\{\Lambda_t^NGx_t\},\{T_t\Lambda_t^NGx_t\}$, and $\{FT_t\Lambda_t^NGx_t\}$.

(ii) By the definition of $\{x_t\}$, we have

$$\begin{aligned}
\|x_t-T_t\Lambda_t^NGx_t\| &= \|P_C[(I-s_tA)T_t\Lambda_t^NGx_t+s_t(t\gamma Vx_t+(I-t\mu F)T_t\Lambda_t^NGx_t)]-P_CT_t\Lambda_t^NGx_t\| \\
&\leqslant \|(I-s_tA)T_t\Lambda_t^NGx_t+s_t(t\gamma Vx_t+(I-t\mu F)T_t\Lambda_t^NGx_t)-T_t\Lambda_t^NGx_t\| \\
&= \|s_t[(I-A)T_t\Lambda_t^NGx_t+t(\gamma Vx_t-\mu FT_t\Lambda_t^NGx_t)]\| \\
&= s_t\|(I-A)T_t\Lambda_t^NGx_t+t(\gamma Vx_t-\mu FT_t\Lambda_t^NGx_t)\| \\
&\leqslant s_t\|I-A\|\|T_t\Lambda_t^NGx_t\|+t\|\gamma Vx_t-\mu FT_t\Lambda_t^NGx_t\| \to 0 \quad \text{as } t\to 0,
\end{aligned}$$

by the boundedness of $\{Vx_t\},\{T_t\Lambda_t^NGx_t\}$ and $\{FT_t\Lambda_t^NGx_t\}$ in the assertion (i). That is,

$$\lim_{t\to 0}\|x_t-T_t\Lambda_t^NGx_t\|=0. \tag{3.4}$$

Since $p=Gp=P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)p$ and F_j is ζ_j-inverse-strongly monotone with $0<\nu_j<2\zeta_j$ for $j=1,2$, we deduce that

$$\begin{aligned}
\|Gx_t-p\|^2 &= \|P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)x_t-P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)p\|^2 \\
&\leqslant \|(I-\nu_1F_1)P_C(I-\nu_2F_2)x_t-(I-\nu_1F_1)P_C(I-\nu_2F_2)p\|^2 \\
&\leqslant \|P_C(I-\nu_2F_2)x_t-P_C(I-\nu_2F_2)p\|^2 \\
&\quad +\nu_1(\nu_1-2\zeta_1)\|F_1P_C(I-\nu_2F_2)x_t-F_1P_C(I-\nu_2F_2)p\|^2 \\
&\leqslant \|P_C(I-\nu_2F_2)x_t-P_C(I-\nu_2F_2)p\|^2 \\
&\leqslant \|(I-\nu_2F_2)x_t-(I-\nu_2F_2)p\|^2 \\
&= \|(x_t-p)-\nu_2(F_2x_t-F_2p)\|^2 \\
&\leqslant \|x_t-p\|^2+\nu_2(\nu_2-2\zeta_2)\|F_2x_t-F_2p\|^2 \\
&\leqslant \|x_t-p\|^2.
\end{aligned} \tag{3.5}$$

Also, since $B_i : C \to H$ is η_i-inverse-strongly monotone for each $i = 1, ..., N$, by Lemma 2.18 we have

$$\begin{aligned}\|\Lambda_t^i Gx_t - p\|^2 &= \|J_{R_i,\lambda_{i,t}}(I-\lambda_{i,t}B_i)\Lambda_t^{i-1}Gx_t - J_{R_i,\lambda_{i,t}}(I-\lambda_{i,t}B_i)p\|^2\\ &\leqslant \|(I-\lambda_{i,t}B_i)\Lambda_t^{i-1}Gx_t - (I-\lambda_{i,t}B_i)p\|^2\\ &\leqslant \|\Lambda_t^{i-1}Gx_t - p\|^2 + \lambda_{i,t}(\lambda_{i,t}-2\eta_i)\|B_i\Lambda_t^{i-1}Gx_t - B_ip\|^2\\ &\leqslant \|Gx_t - p\|^2 + \lambda_{i,t}(\lambda_{i,t}-2\eta_i)\|B_i\Lambda_t^{i-1}Gx_t - B_ip\|^2\\ &\leqslant \|Gx_t - p\|^2,\end{aligned} \tag{3.6}$$

for each $i \in \{1, 2, ..., N\}$. Simple calculations show that

$$\begin{aligned}x_t - p &= x_t - w_t + w_t - p\\ &= x_t - w_t + (I - s_tA)T_t\Lambda_t^N Gx_t + s_t(t\gamma Vx_t + (I-t\mu F)T_t\Lambda_t^N Gx_t) - p\\ &= x_t - w_t + (I - s_tA)(T_t\Lambda_t^N Gx_t - T_t\Lambda_t^N Gp) + s_t[t(\gamma Vx_t - \mu Fp)\\ &\quad + (I-t\mu F)T_t\Lambda_t^N Gx_t - (I-t\mu F)p] + s_t(I-A)p,\end{aligned} \tag{3.7}$$

where $w_t = (I - s_tA)T_t\Lambda_t^N Gx_t + s_t(t\gamma Vx_t + (I-t\mu F)T_t\Lambda_t^N Gx_t)$. For simplicity, we write $\tilde{x}_t = P_C(I - \nu_2F_2)x_t$, $y_t = P_C(I-\nu_1F_1)\tilde{x}_t$, and $\tilde{p} = P_C(I-\nu_2F_2)p$. Then $y_t = Gx_t$ and $p = P_C(I-\nu_1F_1)\tilde{p} = Gp$. Hence, by Lemmas 2.11 and 2.14 , from (3.5)-(3.7) we obtain that

$$\begin{aligned}\|x_t - p\|^2 &= \langle x_t - w_t, x_t - p\rangle + \langle (I-s_tA)(T_t\Lambda_t^N Gx_t - T_t\Lambda_t^N Gp), x_t - p\rangle + s_t[t\langle \gamma Vx_t - \mu Fp, x_t - p\rangle\\ &\quad + \langle (I-t\mu F)T_t\Lambda_t^N Gx_t - (I-t\mu F)p, x_t - p\rangle] + s_t\langle (I-A)p, x_t - p\rangle\\ &\leqslant \langle (I-s_tA)(T_t\Lambda_t^N Gx_t - T_t\Lambda_t^N Gp), x_t - p\rangle + s_t[t\langle \gamma Vx_t - \mu Fp, x_t - p\rangle\\ &\quad + \langle (I-t\mu F)T_t\Lambda_t^N Gx_t - (I-t\mu F)p, x_t - p\rangle] + s_t\langle (I-A)p, x_t - p\rangle\\ &= \langle (I-s_tA)(T_t\Lambda_t^N Gx_t - T_t\Lambda_t^N Gp), x_t - p\rangle + s_t[\langle (I-t\mu F)T_t\Lambda_t^N Gx_t - (I-t\mu F)p, x_t - p\rangle\\ &\quad + t(\gamma\langle Vx_t - Vp, x_t - p\rangle + \langle \gamma Vp - \mu Fp, x_t - p\rangle)] + s_t\langle (I-A)p, x_t - p\rangle\\ &\leqslant \|(I-s_tA)(T_t\Lambda_t^N Gx_t - T_t\Lambda_t^N Gp)\|\|x_t - p\| + s_t[\|(I-t\mu F)T_t\Lambda_t^N Gx_t - (I-t\mu F)p\|\|x_t - p\|\\ &\quad + t(\gamma\|Vx_t - Vp\|\|x_t - p\| + \|\gamma Vp - \mu Fp\|\|x_t - p\|)] + s_t\|(I-A)p\|\|x_t - p\|\\ &\leqslant (1 - s_t\bar{\gamma})\|T_t\Lambda_t^N Gx_t - T_t\Lambda_t^N Gp\|\|x_t - p\| + s_t[(1-t\tau)\|T_t\Lambda_t^N Gx_t - p\|\|x_t - p\|\\ &\quad + t(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|)] + s_t\|(I-A)p\|\|x_t - p\|\\ &\leqslant (1 - s_t\bar{\gamma})\|\Lambda_t^N Gx_t - p\|\|x_t - p\| + s_t[(1-t\tau)\|\Lambda_t^N Gx_t - p\|\|x_t - p\|\\ &\quad + t(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|)] + s_t\|(I-A)p\|\|x_t - p\|\\ &= (1 - s_t(\bar{\gamma} - 1 + t\tau))\|\Lambda_t^N Gx_t - p\|\|x_t - p\|\\ &\quad + s_tt(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|) + s_t\|(I-A)p\|\|x_t - p\|\\ &\leqslant (1 - s_t(\bar{\gamma} - 1 + t\tau))\frac{1}{2}(\|\Lambda_t^N Gx_t - p\|^2 + \|x_t - p\|^2)\\ &\quad + s_tt(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|) + s_t\|(I-A)p\|\|x_t - p\|\\ &\leqslant (1 - s_t(\bar{\gamma} - 1 + t\tau))\frac{1}{2}(\|\Lambda_t^i Gx_t - p\|^2 + \|x_t - p\|^2)\\ &\quad + s_tt(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|) + s_t\|(I-A)p\|\|x_t - p\|\\ &\leqslant (1 - s_t(\bar{\gamma} - 1 + t\tau))\frac{1}{2}[\|Gx_t - p\|^2 + \lambda_{i,t}(\lambda_{i,t}-2\eta_i)\|B_i\Lambda_t^{i-1}Gx_t - B_ip\|^2\\ &\quad + \|x_t - p\|^2] + s_tt(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|) + s_t\|(I-A)p\|\|x_t - p\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_t-p\|^2 &\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}[\|P_C(I-\nu_2F_2)x_t-P_C(I-\nu_2F_2)p\|^2\\&\quad+\nu_1(\nu_1-2\zeta_1)\|F_1\tilde{x}_t-F_1\tilde{p}\|^2+\lambda_{i,t}(\lambda_{i,t}-2\eta_i)\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|^2\\&\quad+\|x_t-p\|^2]+s_tt(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\&\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}[\|x_t-p\|^2+\nu_2(\nu_2-2\zeta_2)\|F_2x_t-F_2p\|^2\\&\quad+\nu_1(\nu_1-2\zeta_1)\|F_1\tilde{x}_t-F_1\tilde{p}\|^2+\lambda_{i,t}(\lambda_{i,t}-2\eta_i)\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|^2\\&\quad+\|x_t-p\|^2]+s_tt(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\&=[1-s_t(\bar{\gamma}-1+t(\tau-\gamma l)]\|x_t-p\|^2-\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}[\nu_2(2\zeta_2-\nu_2)\|F_2x_t-F_2p\|^2\\&\quad+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_t-F_1\tilde{p}\|^2+\lambda_{i,t}(2\eta_i-\lambda_{i,t})\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|^2]\\&\quad+s_t(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\|(I-A)p\|\|x_t-p\|)\\&\leqslant\|x_t-p\|^2-\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}[\nu_2(2\zeta_2-\nu_2)\|F_2x_t-F_2p\|^2\\&\quad+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_t-F_1\tilde{p}\|^2+\lambda_{i,t}(2\eta_i-\lambda_{i,t})\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|^2]\\&\quad+s_t(t\|(\gamma V-\mu F)p\|\|x_t-p\|+\|(I-A)p\|\|x_t-p\|),\end{aligned}\tag{3.8}$$

which together with $\nu_j\in(0,2\zeta_j), j=1,2$ and $\{\lambda_{i,t}\}\subset[a_i,b_i]\subset(0,2\eta_i), i=1,...,N$, implies that

$$\begin{aligned}&\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}[\nu_2(2\zeta_2-\nu_2)\|F_2x_t-F_2p\|^2\\&\quad+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_t-F_1\tilde{p}\|^2+a_i(2\eta_i-b_i)\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|^2]\\&\leqslant\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}[\nu_2(2\zeta_2-\nu_2)\|F_2x_t-F_2p\|^2\\&\quad+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_t-F_1\tilde{p}\|^2+\lambda_{i,t}(2\eta_i-\lambda_{i,t})\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|^2]\\&\leqslant s_t(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\|(I-A)p\|\|x_t-p\|).\end{aligned}$$

Since $\lim_{t\to0}s_t=0$ and $\{x_t\}$ is bounded, we have

$$\lim_{t\to0}\|F_2x_t-F_2p\|=\lim_{t\to0}\|F_1\tilde{x}_t-F_1\tilde{p}\|=\lim_{t\to0}\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|=0,\quad\forall i\in\{1,...,N\}.\tag{3.9}$$

Utilizing Lemmas 2.8 (a) and 2.18, we obtain that for each $i\in\{1,...,N\}$

$$\begin{aligned}\|\Lambda_t^iGx_t-p\|^2&=\|J_{R_i,\lambda_{i,t}}(I-\lambda_{i,t}B_i)\Lambda_t^{i-1}Gx_t-J_{R_i,\lambda_{i,t}}(I-\lambda_{i,t}B_i)p\|^2\\&\leqslant\langle(I-\lambda_{i,t}B_i)\Lambda_t^{i-1}Gx_t-(I-\lambda_{i,t}B_i)p,\Lambda_t^iGx_t-p\rangle\\&=\frac{1}{2}(\|(I-\lambda_{i,t}B_i)\Lambda_t^{i-1}Gx_t-(I-\lambda_{i,t}B_i)p\|^2+\|\Lambda_t^iGx_t-p\|^2\\&\quad-\|(I-\lambda_{i,t}B_i)\Lambda_t^{i-1}Gx_t-(I-\lambda_{i,t}B_i)p-(\Lambda_t^iGx_t-p)\|^2)\\&\leqslant\frac{1}{2}(\|\Lambda_t^{i-1}Gx_t-p\|^2+\|\Lambda_t^iGx_t-p\|^2-\|\Lambda_t^{i-1}Gx_t-\Lambda_t^iGx_t-\lambda_{i,t}(B_i\Lambda_t^{i-1}Gx_t-B_ip)\|^2)\\&\leqslant\frac{1}{2}(\|x_t-p\|^2+\|\Lambda_t^iGx_t-p\|^2-\|\Lambda_t^{i-1}Gx_t-\Lambda_t^iGx_t-\lambda_{i,t}(B_i\Lambda_t^{i-1}Gx_t-B_ip)\|^2),\end{aligned}$$

which immediately leads to

$$\begin{aligned}\|\Lambda_t^iGx_t-p\|^2&\leqslant\|x_t-p\|^2-\|\Lambda_t^{i-1}Gx_t-\Lambda_t^iGx_t-\lambda_{i,t}(B_i\Lambda_t^{i-1}Gx_t-B_ip)\|^2\\&=\|x_t-p\|^2-\|\Lambda_t^{i-1}Gx_t-\Lambda_t^iGx_t\|^2-\lambda_{i,t}^2\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|^2\\&\quad+2\lambda_{i,t}\langle\Lambda_t^{i-1}Gx_t-\Lambda_t^iGx_t,B_i\Lambda_t^{i-1}Gx_t-B_ip\rangle\\&\leqslant\|x_t-p\|^2-\|\Lambda_t^{i-1}Gx_t-\Lambda_t^iGx_t\|^2+2\lambda_{i,t}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^iGx_t\|\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|.\end{aligned}\tag{3.10}$$

Combining (3.8) and (3.10) we conclude that

$$\begin{aligned}\|x_t-p\|^2 &\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}(\|\Lambda_t^i Gx_t-p\|^2+\|x_t-p\|^2)\\&\quad+s_t t(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\&\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}\{\|x_t-p\|^2-\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|^2\\&\quad+2\lambda_{i,t}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|+\|x_t-p\|^2\}\\&\quad+s_t t(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\&=[1-s_t(\bar{\gamma}-1+t(\tau-\gamma l))]\|x_t-p\|^2-\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|^2\\&\quad+(1-s_t(\bar{\gamma}-1+t\tau))\lambda_{i,t}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|\\&\quad+s_t(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\|(I-A)p\|\|x_t-p\|)\\&\leqslant \|x_t-p\|^2-\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|^2\\&\quad+\lambda_{i,t}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|\\&\quad+s_t(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\|(I-A)p\|\|x_t-p\|),\end{aligned}$$

which hence yields

$$\begin{aligned}\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|^2 &\leqslant \lambda_{i,t}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|\\&\quad+s_t(t\|\gamma Vp-\mu Fp\|\|x_t-p\|+\|(I-A)p\|\|x_t-p\|).\end{aligned}$$

Since $\{\lambda_{i,t}\}\subset[a_i,b_i]\subset(0,2\eta_i)$, $\lim_{t\to0}s_t=0$ and $\lim_{t\to0}\|B_i\Lambda_t^{i-1}Gx_t-B_ip\|=0$ (due to (3.9)), we deduce from the boundedness of $\{x_t\}$ and $\{\Lambda_t^i Gx_t\}$ that

$$\lim_{t\to0}\|\Lambda_t^{i-1}Gx_t-\Lambda_t^i Gx_t\|=0,\quad \forall i\in\{1,\dots,N\}. \tag{3.11}$$

Furthermore, in terms of the firm nonexpansivity of P_C and the ζ_j-inverse strong monotonicity of F_j for $j=1,2$, we obtain from $\nu_j\in(0,2\zeta_j), j=1,2$ and (3.5) that

$$\begin{aligned}\|\tilde{x}_t-\tilde{p}\|^2&=\|P_C(I-\nu_2F_2)x_t-P_C(I-\nu_2F_2)p\|^2\\&\leqslant\langle(I-\nu_2F_2)x_t-(I-\nu_2F_2)p,\tilde{x}_t-\tilde{p}\rangle\\&=\frac{1}{2}[\|(I-\nu_2F_2)x_t-(I-\nu_2F_2)p\|^2+\|\tilde{x}_t-\tilde{p}\|^2-\|(I-\nu_2F_2)x_t-(I-\nu_2F_2)p-(\tilde{x}_t-\tilde{p})\|^2]\\&\leqslant\frac{1}{2}[\|x_t-p\|^2+\|\tilde{x}_t-\tilde{p}\|^2-\|(x_t-\tilde{x}_t)-\nu_2(F_2x_t-F_2p)-(p-\tilde{p})\|^2]\\&=\frac{1}{2}[\|x_t-p\|^2+\|\tilde{x}_t-\tilde{p}\|^2-\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|^2\\&\quad+2\nu_2\langle(x_t-\tilde{x}_t)-(p-\tilde{p}),F_2x_t-F_2p\rangle-\nu_2^2\|F_2x_t-F_2p\|^2],\end{aligned}$$

and

$$\begin{aligned}\|y_t-p\|^2&=\|P_C(I-\nu_1F_1)\tilde{x}_t-P_C(I-\nu_1F_1)\tilde{p}\|^2\\&\leqslant\langle(I-\nu_1F_1)\tilde{x}_t-(I-\nu_1F_1)\tilde{p},y_t-p\rangle\\&=\frac{1}{2}[\|(I-\nu_1F_1)\tilde{x}_t-(I-\nu_1F_1)\tilde{p}\|^2+\|y_t-p\|^2-\|(I-\nu_1F_1)\tilde{x}_t-(I-\nu_1F_1)\tilde{p}-(y_t-p)\|^2]\\&\leqslant\frac{1}{2}[\|\tilde{x}_t-\tilde{p}\|^2+\|y_t-p\|^2-\|(\tilde{x}_t-y_t)+(p-\tilde{p})\|^2\\&\quad+2\nu_1\langle F_1\tilde{x}_t-F_1\tilde{p},(\tilde{x}_t-y_t)+(p-\tilde{p})\rangle-\nu_1^2\|F_1\tilde{x}_t-F_1\tilde{p}\|^2]\\&\leqslant\frac{1}{2}[\|x_t-p\|^2+\|y_t-p\|^2-\|(\tilde{x}_t-y_t)+(p-\tilde{p})\|^2+2\nu_1\langle F_1\tilde{x}_t-F_1\tilde{p},(\tilde{x}_t-y_t)+(p-\tilde{p})\rangle].\end{aligned}$$

Thus, we have

$$\|\tilde{x}_t-\tilde{p}\|^2 \leqslant \|x_t-p\|^2-\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|^2+2\nu_2\langle(x_t-\tilde{x}_t)-(p-\tilde{p}),F_2x_t-F_2p\rangle \\ -\nu_2^2\|F_2x_t-F_2p\|^2, \tag{3.12}$$

and

$$\|y_t-p\|^2 \leqslant \|x_t-p\|^2-\|(\tilde{x}_t-y_t)+(p-\tilde{p})\|^2+2\nu_1\|F_1\tilde{x}_t-F_1\tilde{p}\|\|(\tilde{x}_t-y_t)+(p-\tilde{p})\|. \tag{3.13}$$

Consequently, from (3.5), (3.6), (3.5), and (3.12) it follows that

$$\begin{aligned}
\|x_t-p\|^2 &\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}(\|\Lambda_t^i Gx_t-p\|^2+\|x_t-p\|^2)\\
&\quad +s_tt(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\
&\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}(\|Gx_t-p\|^2+\|x_t-p\|^2)\\
&\quad +s_tt(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\
&\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}(\|\tilde{x}_t-\tilde{p}\|^2+\|x_t-p\|^2)\\
&\quad +s_tt(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\
&\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}[\|x_t-p\|^2-\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|^2\\
&\quad +2\nu_2\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|\|F_2x_t-F_2p\|+\|x_t-p\|^2]\\
&\quad +s_tt(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\
&= [1-s_t(\bar{\gamma}-1+t(\tau-\gamma l))]\|x_t-p\|^2-\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|^2\\
&\quad +(1-s_t(\bar{\gamma}-1+t\tau))\nu_2\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|\|F_2x_t-F_2p\|\\
&\quad +s_t(t\|\gamma Vp-\mu Fp\|\|x_t-p\|)+\|(I-A)p\|\|x_t-p\|)\\
&\leqslant \|x_t-p\|^2-\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|^2+\nu_2\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|\|F_2x_t-F_2p\|\\
&\quad +s_t(t\|\gamma Vp-\mu Fp\|\|x_t-p\|)+\|(I-A)p\|\|x_t-p\|),
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{1-s_t(\bar{\gamma}-1+t\tau)}{2}\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|^2\\
&\quad\leqslant \nu_2\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|\|F_2x_t-F_2p\|+s_t(t\|\gamma Vp-\mu Fp\|\|x_t-p\|)+\|(I-A)p\|\|x_t-p\|).
\end{aligned}$$

Since $\lim_{t\to 0} s_t=0$ and $\lim_{t\to 0}\|F_2x_t-F_2p\|=0$ (due to (3.9)), we deduce from the boundedness of $\{x_t\}$ and $\{\tilde{x}_t\}$ that

$$\lim_{t\to 0}\|(x_t-\tilde{x}_t)-(p-\tilde{p})\|=0. \tag{3.14}$$

In the meantime, from (3.5), (3.6), (3.8), and (3.13) it follows that

$$\begin{aligned}
\|x_t-p\|^2 &\leqslant (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}(\|Gx_t-p\|^2+\|x_t-p\|^2)\\
&\quad +s_tt(\gamma l\|x_t-p\|^2+\|\gamma Vp-\mu Fp\|\|x_t-p\|)+s_t\|(I-A)p\|\|x_t-p\|\\
&= (1-s_t(\bar{\gamma}-1+t\tau))\frac{1}{2}(\|y_t-p\|^2+\|x_t-p\|^2)
\end{aligned}$$

$$
\begin{aligned}
&\quad + s_t t(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|) + s_t\|(I-A)p\|\|x_t - p\| \\
&\leqslant (1 - s_t(\bar{\gamma} - 1 + t\tau))\frac{1}{2}[\|x_t - p\|^2 - \|(\tilde{x}_t - y_t) + (p - \tilde{p})\|^2 \\
&\quad + 2\nu_1\|F_1\tilde{x}_t - F_1\tilde{p}\|\|(\tilde{x}_t - y_t) + (p - \tilde{p})\| + \|x_t - p\|^2] \\
&\quad + s_t t(\gamma l\|x_t - p\|^2 + \|\gamma Vp - \mu Fp\|\|x_t - p\|) + s_t\|(I-A)p\|\|x_t - p\| \\
&= [1 - s_t(\bar{\gamma} - 1 + t(\tau - \gamma l))]\|x_t - p\|^2 - \frac{1 - s_t(\bar{\gamma} - 1 + t\tau)}{2}\|(\tilde{x}_t - y_t) + (p - \tilde{p})\|^2 \\
&\quad + (1 - s_t(\bar{\gamma} - 1 + t\tau))\nu_1\|F_1\tilde{x}_t - F_1\tilde{p}\|\|(\tilde{x}_t - y_t) + (p - \tilde{p})\| \\
&\quad + s_t(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \|(I-A)p\|\|x_t - p\|) \\
&\leqslant \|x_t - p\|^2 - \frac{1 - s_t(\bar{\gamma} - 1 + t\tau)}{2}\|(\tilde{x}_t - y_t) + (p - \tilde{p})\|^2 + \nu_1\|F_1\tilde{x}_t - F_1\tilde{p}\|\|(\tilde{x}_t - y_t) + (p - \tilde{p})\| \\
&\quad + s_t(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \|(I-A)p\|\|x_t - p\|),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\frac{1 - s_t(\bar{\gamma} - 1 + t\tau)}{2}\|(\tilde{x}_t - y_t) + (p - \tilde{p})\|^2 \\
&\quad \leqslant \nu_1\|F_1\tilde{x}_t - F_1\tilde{p}\|\|(\tilde{x}_t - y_t) + (p - \tilde{p})\| + s_t(t\|\gamma Vp - \mu Fp\|\|x_t - p\| + \|(I-A)p\|\|x_t - p\|).
\end{aligned}
$$

Since $\lim_{t\to 0} s_t = 0$ and $\lim_{t\to 0}\|F_1\tilde{x}_t - F_1\tilde{p}\| = 0$ (due to (3.9)), we deduce from the boundedness of $\{x_t\}, \{\tilde{x}_t\}$, and $\{y_t\}$ that

$$
\lim_{t\to 0}\|(\tilde{x}_t - y_t) + (p - \tilde{p})\| = 0. \tag{3.15}
$$

Note that

$$
\|x_t - y_t\| \leqslant \|(x_t - \tilde{x}_t) - (p - \tilde{p})\| + \|(\tilde{x}_t - y_t) + (p - \tilde{p})\|.
$$

Hence from (3.14) and (3.15) we get

$$
\lim_{n\to\infty}\|x_t - y_t\| = \lim_{n\to\infty}\|x_t - Gx_t\| = 0. \tag{3.16}
$$

Also, observe that

$$
\begin{aligned}
\|Gx_t - \Lambda_t^N Gx_t\| &= \|\Lambda_t^0 Gx_t - \Lambda_t^N Gx_t\| \\
&\leqslant \|\Lambda_t^0 Gx_t - \Lambda_t^1 Gx_t\| + \|\Lambda_t^1 Gx_t - \Lambda_t^2 Gx_t\| + \cdots + \|\Lambda_t^{N-1} Gx_t - \Lambda_t^N Gx_t\|.
\end{aligned}
$$

Thus, from (3.11) we get

$$
\lim_{t\to 0}\|Gx_t - \Lambda_t^N Gx_t\| = 0. \tag{3.17}
$$

In addition, it is easy to see that

$$
\|x_t - \Lambda_t^N x_t\| \leqslant \|x_t - Gx_t\| + \|Gx_t - \Lambda_t^N Gx_t\| + \|\Lambda_t^N Gx_t - \Lambda_t^N x_t\| \leqslant 2\|x_t - Gx_t\| + \|Gx_t - \Lambda_t^N Gx_t\|.
$$

So, from (3.16) and (3.17) it follows that

$$
\lim_{t\to 0}\|x_t - \Lambda_t^N x_t\| = 0.
$$

Further, it is not hard to find that

$$
\begin{aligned}
\|x_t - T_t x_t\| &\leqslant \|x_t - T_t\Lambda_t^N Gx_t\| + \|T_t\Lambda_t^N Gx_t - T_t Gx_t\| + \|T_t Gx_t - T_t x_t\| \\
&\leqslant \|x_t - T_t\Lambda_t^N Gx_t\| + \|\Lambda_t^N Gx_t - Gx_t\| + \|Gx_t - x_t\|.
\end{aligned}
$$

Consequently, from (3.4), (3.16), and (3.17) it follows that

$$
\lim_{t\to 0}\|x_t - T_t x_t\| = 0.
$$

(iii) Let $t, t_0 \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$. For simplicity, we write $v_t = \Lambda_t^N Gx_t$. Then we know that $x_t = P_C[(I - s_t A)T_t v_t + s_t(t\gamma V x_t + (I - t\mu F)T_t v_t)]$. Since ∇f is $\frac{1}{L}$-ism, $P_C(I - \lambda_t \nabla f)$ is nonexpansive for $\lambda_t \in (0, \frac{2}{L})$. So, it follows that for any given $p \in \Omega$,

$$\begin{aligned}\|P_C(I-\lambda_t\nabla f)v_{t_0}\| &\leqslant \|P_C(I-\lambda_t\nabla f)v_{t_0} - p\| + \|p\| \\ &= \|P_C(I-\lambda_t\nabla f)v_{t_0} - P_C(I-\lambda_t\nabla f)p\| + \|p\| \leqslant \|v_{t_0} - p\| + \|p\| \leqslant \|v_{t_0}\| + 2\|p\|.\end{aligned}$$

This implies that $\{P_C(I-\lambda_t\nabla f)v_{t_0}\}$ is bounded. Also, observe that

$$\begin{aligned}&\|T_t v_{t_0} - T_{t_0} v_{t_0}\| \\ &= \|\frac{4P_C(I-\lambda_t\nabla f)-(2-\lambda_t L)I}{2+\lambda_t L}v_{t_0} - \frac{4P_C(I-\lambda_{t_0}\nabla f)-(2-\lambda_{t_0} L)I}{2+\lambda_{t_0} L}v_{t_0}\| \\ &\leqslant \|\frac{4P_C(I-\lambda_t\nabla f)}{2+\lambda_t L}v_{t_0} - \frac{4P_C(I-\lambda_{t_0}\nabla f)}{2+\lambda_{t_0} L}v_{t_0}\| + \|\frac{2-\lambda_{t_0}L}{2+\lambda_{t_0}L}v_{t_0} - \frac{2-\lambda_t L}{2+\lambda_t L}v_{t_0}\| \\ &= \|\frac{4(2+\lambda_{t_0}L)P_C(I-\lambda_t\nabla f)v_{t_0} - 4(2+\lambda_t L)P_C(I-\lambda_{t_0}\nabla f)v_{t_0}}{(2+\lambda_t L)(2+\lambda_{t_0}L)}\| + \frac{4L|\lambda_t-\lambda_{t_0}|}{(2+\lambda_t L)(2+\lambda_{t_0}L)}\|v_{t_0}\| \\ &= \|\frac{4L(\lambda_{t_0}-\lambda_t)P_C(I-\lambda_t\nabla f)v_{t_0} + 4(2+\lambda_t L)(P_C(I-\lambda_t\nabla f)v_{t_0} - P_C(I-\lambda_{t_0}\nabla f)v_{t_0})}{(2+\lambda_t L)(2+\lambda_{t_0}L)}\| \\ &\quad + \frac{4L|\lambda_t-\lambda_{t_0}|}{(2+\lambda_t L)(2+\lambda_{t_0}L)}\|v_{t_0}\| \\ &\leqslant \frac{4L|\lambda_{t_0}-\lambda_t|\|P_C(I-\lambda_t\nabla f)v_{t_0}\|}{(2+\lambda_t L)(2+\lambda_{t_0}L)} + \frac{4(2+\lambda_t L)\|P_C(I-\lambda_t\nabla f)v_{t_0} - P_C(I-\lambda_{t_0}\nabla f)v_{t_0}\|}{(2+\lambda_t L)(2+\lambda_{t_0}L)} \\ &\quad + \frac{4L|\lambda_t-\lambda_{t_0}|}{(2+\lambda_t L)(2+\lambda_{t_0}L)}\|v_{t_0}\| \\ &\leqslant |\lambda_t-\lambda_{t_0}|[L\|P_C(I-\lambda_t\nabla f)v_{t_0}\| + 4\|\nabla f(v_{t_0})\| + L\|v_{t_0}\|] \\ &\leqslant \widetilde{M}|\lambda_t-\lambda_{t_0}|,\end{aligned} \tag{3.18}$$

where $\sup_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}\{L\|P_C(I-\lambda_t\nabla f)v_{t_0}\| + 4\|\nabla f(v_{t_0})\| + L\|v_{t_0}\|\} \leqslant \widetilde{M}$ for some $\widetilde{M} > 0$. So, by (3.18), we have that

$$\begin{aligned}\|T_t v_t - T_{t_0}v_{t_0}\| &\leqslant \|T_t v_t - T_t v_{t_0}\| + \|T_t v_{t_0} - T_{t_0}v_{t_0}\| \\ &\leqslant \|v_t - v_{t_0}\| + \widetilde{M}|\lambda_t - \lambda_{t_0}| \leqslant \|v_t - v_{t_0}\| + \frac{4\widetilde{M}}{L}|s_t - s_{t_0}|.\end{aligned} \tag{3.19}$$

Utilizing (2.1) and (2.2), we obtain that

$$\begin{aligned}\|v_t - v_{t_0}\| &= \|\Lambda_t^N Gx_t - \Lambda_{t_0}^N Gx_{t_0}\| \\ &= \|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}Gx_t - J_{R_N,\lambda_{N,t_0}}(I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}Gx_{t_0}\| \\ &\leqslant \|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}Gx_t - J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}Gx_t\| \\ &\quad + \|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}Gx_t - J_{R_N,\lambda_{N,t_0}}(I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}Gx_{t_0}\| \\ &\leqslant \|(I-\lambda_{N,t}B_N)\Lambda_t^{N-1}Gx_t - (I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}Gx_t\| \\ &\quad + \|(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}Gx_t - (I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}Gx_{t_0}\| + |\lambda_{N,t}-\lambda_{N,t_0}|\times \\ &\quad \times (\frac{1}{\lambda_{N,t}}\|J_{R_N,\lambda_{N,t}}(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}Gx_t - (I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}Gx_{t_0}\| \\ &\quad + \frac{1}{\lambda_{N,t_0}}\|(I-\lambda_{N,t_0}B_N)\Lambda_t^{N-1}Gx_t - J_{R_N,\lambda_{N,t_0}}(I-\lambda_{N,t_0}B_N)\Lambda_{t_0}^{N-1}Gx_{t_0}\|) \\ &\leqslant |\lambda_{N,t}-\lambda_{N,t_0}|(\|B_N\Lambda_t^{N-1}Gx_t\| + \widehat{M}) + \|\Lambda_t^{N-1}Gx_t - \Lambda_{t_0}^{N-1}Gx_{t_0}\|\end{aligned}$$

$$
\begin{aligned}
&\leqslant |\lambda_{N,t}-\lambda_{N,t_0}|(\|B_N\Lambda_t^{N-1}Gx_t\|+\widehat{M})\\
&\quad+|\lambda_{N-1,t}-\lambda_{N-1,t_0}|(\|B_{N-1}\Lambda_t^{N-2}Gx_t\|+\widehat{M})+\|\Lambda_t^{N-2}Gx_t-\Lambda_{t_0}^{N-2}Gx_{t_0}\|\\
&\leqslant \\
&\ \vdots\\
&\leqslant |\lambda_{N,t}-\lambda_{N,t_0}|(\|B_N\Lambda_t^{N-1}Gx_t\|+\widehat{M})+|\lambda_{N-1,t}-\lambda_{N-1,t_0}|(\|B_{N-1}\Lambda_t^{N-2}Gx_t\|+\widehat{M})\\
&\quad+\cdots+|\lambda_{1,t}-\lambda_{1,t_0}|(\|B_1\Lambda_t^{0}Gx_t\|+\widehat{M})+\|\Lambda_t^{0}Gx_t-\Lambda_{t_0}^{0}Gx_{t_0}\|\\
&\leqslant \widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}|+\|x_t-x_{t_0}\|,
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{aligned}
\sup_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\}),1\leqslant i\leqslant N}&\{\frac{1}{\lambda_{i,t}}\|J_{R_i,\lambda_{i,t}}(I-\lambda_{i,t_0}B_i)\Lambda_t^{i-1}Gx_t-(I-\lambda_{i,t_0}B_i)\Lambda_{t_0}^{i-1}Gx_{t_0}\|\\
&\quad+\frac{1}{\lambda_{i,t_0}}\|(I-\lambda_{i,t_0}B_i)\Lambda_t^{i-1}Gx_t-J_{R_i,\lambda_{i,t_0}}(I-\lambda_{i,t_0}B_i)\Lambda_{t_0}^{i-1}Gx_{t_0}\|\}\leqslant\widehat{M},
\end{aligned}
$$

for some $\widehat{M}>0$ and $\sup_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}\{\sum_{i=1}^{N}\|B_i\Lambda_t^{i-1}Gx_t\|+\widehat{M}\}\leqslant\widetilde{M}_0$ for some $\widetilde{M}_0>0$. By combining (3.19) and (3.20) we get

$$
\|T_tv_t-T_{t_0}v_{t_0}\|\leqslant\|v_t-v_{t_0}\|+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|\leqslant\|x_t-x_{t_0}\|+\widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}|+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|.
\tag{3.21}
$$

In terms of (3.21), we calculate

$$
\begin{aligned}
\|x_t-x_{t_0}\|&\leqslant\|(I-s_tA)T_tv_t+s_t(t\gamma Vx_t+(I-t\mu F)T_tv_t)\\
&\quad-(I-s_{t_0}A)T_{t_0}v_{t_0}-s_{t_0}(t_0\gamma Vx_{t_0}+(I-t_0\mu F)T_{t_0}v_{t_0})\|\\
&\leqslant\|(I-s_tA)T_tv_t-(I-s_{t_0}A)T_tv_t\|+\|(I-s_{t_0}A)T_tv_t-(I-s_{t_0}A)T_{t_0}v_{t_0}\|\\
&\quad+|s_t-s_{t_0}|\|t\gamma Vx_t+(I-t\mu F)T_tv_t\|+s_{t_0}\|[t\gamma Vx_t+(I-t\mu F)T_tv_t]\\
&\quad-[t_0\gamma Vx_{t_0}+(I-t_0\mu F)T_{t_0}v_{t_0}]\|\\
&\leqslant|s_t-s_{t_0}|\|A\|\|T_tv_t\|+(1-s_{t_0}\bar{\gamma})\|T_tv_t-T_{t_0}v_{t_0}\|\\
&\quad+|s_t-s_{t_0}|\|t\gamma Vx_t+(I-t\mu F)T_tv_t\|+s_{t_0}\|(t-t_0)\gamma Vx_t+t_0\gamma(Vx_t-Vx_{t_0})\\
&\quad-(t-t_0)\mu FT_tv_t+(I-t_0\mu F)T_tv_t-(I-t_0\mu F)T_{t_0}v_{t_0}\|\\
&\leqslant|s_t-s_{t_0}|\|A\|\|T_tv_t\|+(1-s_{t_0}\bar{\gamma})[\|x_t-x_{t_0}\|+\widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}|\\
&\quad+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|]+|s_t-s_{t_0}|[\|T_tv_t\|+t(\gamma\|Vx_t\|+\mu\|FT_tv_t\|)]\\
&\quad+s_{t_0}[(\gamma\|Vx_t\|+\mu\|FT_tv_t\|)|t-t_0|+t_0\gamma l\|x_t-x_{t_0}\|+(1-t_0\tau)\|T_tv_t-T_{t_0}v_{t_0}\|]\\
&\leqslant|s_t-s_{t_0}|\|A\|\|T_tv_t\|+(1-s_{t_0}\bar{\gamma})[\|x_t-x_{t_0}\|+\widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}|\\
&\quad+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|]+|s_t-s_{t_0}|(\|T_tv_t\|+\gamma\|Vx_t\|+\mu\|FT_tv_t\|)\\
&\quad+s_{t_0}(\gamma\|Vx_t\|+\mu\|FT_tv_t\|)|t-t_0|+s_{t_0}t_0\gamma l\|x_t-x_{t_0}\|+s_{t_0}(1-t_0\tau)[\|x_t-x_{t_0}\|
\end{aligned}
$$

$$+\widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}|+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|].$$

This immediately implies that

$$\begin{aligned}\|x_t-x_{t_0}\| &\leqslant \frac{\|A\|\|T_t v_t\|+\|T_t v_t\|+\gamma\|Vx_t\|+\mu\|FT_t v_t\|}{s_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma l))}|s_t-s_{t_0}|+\frac{\gamma\|Vx_t\|+\mu\|FT_t v_t\|}{\bar{\gamma}-1+t_0(\tau-\gamma l)}|t-t_0| \\ &\quad+\frac{1-s_{t_0}(\bar{\gamma}-1+t_0\tau)}{s_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma l))}[\widetilde{M}_0\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}|+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|] \\ &\leqslant \frac{\|A\|\|T_t v_t\|+\|T_t v_t\|+\gamma\|Vx_t\|+\mu\|FT_t v_t\|+\frac{4\widetilde{M}}{L}}{s_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma l))}|s_t-s_{t_0}|+\frac{\gamma\|Vx_t\|+\mu\|FT_t v_t\|}{\bar{\gamma}-1+t_0(\tau-\gamma l)}|t-t_0| \\ &\quad+\frac{\widetilde{M}_0}{s_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma l))}\sum_{i=1}^{N}|\lambda_{i,t}-\lambda_{i,t_0}|.\end{aligned}$$

Since $s_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})\to(0,\min\{\frac{1}{2},\|A\|^{-1}\})$ is locally Lipschitzian and $\lambda_{i,t}:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})\to[a_i,b_i]$ is locally Lipschitzian for each $i=1,...,N$, we conclude that $x_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})\to C$ is locally Lipschitzian.

(iv) From the last inequality in (iii), the result follows immediately. □

We prove the following theorem for strong convergence of the net $\{x_t\}$ as $t\to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.2. *Let the net $\{x_t\}$ be defined via (3.1). If $\lim_{t\to 0}s_t=0$, then x_t converges strongly to a point $\tilde{x}\in\Omega$ as $t\to 0$, which solves the VIP (3.2). Equivalently, we have $P_\Omega(2I-A)\tilde{x}=\tilde{x}$.*

Proof. We first show the uniqueness of solutions of the VIP (3.2), which is indeed a consequence of the strong monotonicity of $A-I$. In fact, since A is a $\bar{\gamma}$-strongly positive bounded linear operator with $\bar{\gamma}\in(1,2)$, we know that $A-I$ is $(\bar{\gamma}-1)$-strongly monotone with constant $\bar{\gamma}-1\in(0,1)$. Suppose that $\tilde{x}\in\Omega$ and $\hat{x}\in\Omega$ both are solutions to the VIP (3.2). Then we have

$$\langle (A-I)\tilde{x},\tilde{x}-\hat{x}\rangle\leqslant 0, \tag{3.22}$$

and

$$\langle (A-I)\hat{x},\hat{x}-\tilde{x}\rangle\leqslant 0. \tag{3.23}$$

Adding up (3.22) and (3.23) yields

$$\langle (A-I)\tilde{x}-(A-I)\hat{x},\tilde{x}-\hat{x}\rangle\leqslant 0.$$

The strong monotonicity of $A-I$ implies that $\tilde{x}=\hat{x}$ and the uniqueness is proved.

Next, we prove that $x_t\to\tilde{x}$ as $t\to 0$. Observing $\mathrm{Fix}(T_t)=\Gamma$, $\mathrm{Fix}(G)=\Xi$, and $\Lambda_t^N p=p$, from (3.1), we write, for given $p\in\Omega$,

$$\begin{aligned}x_t-p&=x_t-w_t+w_t-p\\ &=x_t-w_t+(I-s_tA)T_t\Lambda_t^N Gx_t+s_t(t\gamma Vx_t+(I-t\mu F)T_t\Lambda_t^N Gx_t)-p\\ &=x_t-w_t+(I-s_tA)(T_t\Lambda_t^N Gx_t-T_t\Lambda_t^N Gp)+s_t[t\gamma Vx_t+(I-t\mu F)T_t\Lambda_t^N Gx_t-p]+s_t(I-A)p\\ &=x_t-w_t+(I-s_tA)(T_t\Lambda_t^N Gx_t-T_t\Lambda_t^N Gp)\\ &\quad+s_t[t(\gamma Vx_t-\mu Fp)+(I-t\mu F)T_t\Lambda_t^N Gx_t-(I-t\mu F)p]+s_t(I-A)p,\end{aligned}$$

where $w_t = (I - s_tA)T_t\Lambda_t^N Gx_t + s_t(t\gamma Vx_t + (I - t\mu F)T_t\Lambda_t^N Gx_t)$. Then, by Proposition 2.1 (i), we have

$$\begin{aligned}\|x_t - p\|^2 &= \langle x_t - w_t, x_t - p\rangle + \langle (I - s_tA)(T_t\Lambda_t^N Gx_t - T_t\Lambda_t^N Gp, x_t - p\rangle + s_t[t\langle \gamma Vx_t - \mu Fp, x_t - p\rangle \\ &\quad + \langle (I - t\mu F)T_t\Lambda_t^N Gx_t - (I - t\mu F)p, x_t - p\rangle] + s_t\langle (I - A)p, x_t - p\rangle \\ &\leqslant (1 - s_t\bar{\gamma})\|x_t - p\|^2 + s_t[(1 - t\tau)\|x_t - p\|^2 + t\gamma l\|x_t - p\|^2 \\ &\quad + t\langle (\gamma V - \mu F)p, x_t - p\rangle] + s_t\langle (I - A)p, x_t - p\rangle \\ &= [1 - s_t(\bar{\gamma} - 1 + t(\tau - \gamma l))]\|x_t - p\|^2 + s_t(t\langle (\gamma V - \mu F)p, x_t - p\rangle + \langle (I - A)p, x_t - p\rangle).\end{aligned}$$

Therefore,

$$\|x_t - p\|^2 \leqslant \frac{1}{\bar{\gamma} - 1 + t(\tau - \gamma l)}(t\langle (\gamma V - \mu F)p, x_t - p\rangle + \langle (I - A)p, x_t - p\rangle). \tag{3.24}$$

Since the net $\{x_t\}_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}$ is bounded (due to Proposition 3.1 (i)), we know that if $\{t_n\}$ is a subsequence in $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup x^*$, then from (3.24), we obtain $x_{t_n} \to x^*$. Let us show that $x^* \in \Omega$. Indeed, by Proposition 3.1 (ii), we know that $\lim_{n\to\infty} \|x_{t_n} - Gx_{t_n}\| = 0$. Hence, according to Lemma 2.9 we get $x^* \in \mathrm{Fix}(G) = \Xi$. In the meantime, by Proposition 3.1 (ii), we know that $\lim_{n\to\infty} \|x_{t_n} - T_{t_n}x_{t_n}\| = 0$. Observe that

$$\|P_C(I - \lambda_{t_n}\nabla f)x_{t_n} - x_{t_n}\| = \|s_{t_n}x_{t_n} + (1 - s_{t_n})T_{t_n}x_{t_n} - x_{t_n}\| = (1 - s_{t_n})\|T_{t_n}x_{t_n} - x_{t_n}\| \leqslant \|T_{t_n}x_{t_n} - x_{t_n}\|,$$

where $s_{t_n} = \frac{2-\lambda_{t_n}L}{4} \in (0, \frac{1}{2})$ for $\lambda_{t_n} \in (0, \frac{2}{L})$. Hence we have

$$\begin{aligned}\|P_C(I - \frac{2}{L}\nabla f)x_{t_n} - x_{t_n}\| &\leqslant \|P_C(I - \frac{2}{L}\nabla f)x_{t_n} - P_C(I - \lambda_{t_n}\nabla f)x_{t_n}\| + \|P_C(I - \lambda_{t_n}\nabla f)x_{t_n} - x_{t_n}\| \\ &\leqslant \|(I - \frac{2}{L}\nabla f)x_{t_n} - (I - \lambda_{t_n}\nabla f)x_{t_n}\| + \|P_C(I - \lambda_{t_n}\nabla f)x_{t_n} - x_{t_n}\| \\ &\leqslant (\frac{2}{L} - \lambda_{t_n})\|\nabla f(x_{t_n})\| + \|T_{t_n}x_{t_n} - x_{t_n}\|.\end{aligned}$$

From the boundedness of $\{x_{t_n}\}$, $s_{t_n} \to 0$ ($\Leftrightarrow$ $\lambda_{t_n} \to \frac{2}{L}$) and $\|T_{t_n}x_{t_n} - x_{t_n}\| \to 0$, it follows that

$$\|x^* - P_C(I - \frac{2}{L}\nabla f)x^*\| = \lim_{n\to\infty} \|x_{t_n} - P_C(I - \frac{2}{L}\nabla f)x_{t_n}\| = 0.$$

So, $x^* \in \mathrm{VI}(C, \nabla f) = \Gamma$. Next we prove that $x^* \in \bigcap_{m=1}^N I(B_m, R_m)$. As a matter of fact, it is easy to see from (3.16) and (3.17) that $\Lambda_{t_n}^m Gx_{t_n} \to x^*$ for each $m = 1, ..., N$. Since B_m is η_m-inverse strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 2.21 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$, i.e., $g - B_m v \in R_m v$. Again, since $\Lambda_{t_n}^m Gx_{t_n} = J_{R_m,\lambda_{m,t_n}}(I - \lambda_{m,t_n}B_m)\Lambda_{t_n}^{m-1}Gx_{t_n}, m \in \{1, 2, ..., N\}$, we have

$$\Lambda_{t_n}^{m-1}Gx_{t_n} - \lambda_{m,t_n}B_m\Lambda_{t_n}^{m-1}Gx_{t_n} \in (I + \lambda_{m,t_n}R_m)\Lambda_{t_n}^m Gx_{t_n},$$

that is,

$$\frac{1}{\lambda_{m,t_n}}(\Lambda_{t_n}^{m-1}Gx_{t_n} - \Lambda_{t_n}^m Gx_{t_n} - \lambda_{m,t_n}B_m\Lambda_{t_n}^{m-1}Gx_{t_n}) \in R_m\Lambda_{t_n}^m Gx_{t_n}.$$

In terms of the monotonicity of R_m, we get

$$\langle v - \Lambda_{t_n}^m Gx_{t_n}, g - B_m v - \frac{1}{\lambda_{m,t_n}}(\Lambda_{t_n}^{m-1}Gx_{t_n} - \Lambda_{t_n}^m Gx_{t_n} - \lambda_{m,t_n}B_m\Lambda_{t_n}^{m-1}Gx_{t_n})\rangle \geqslant 0,$$

and hence

$$
\begin{aligned}
&\langle v-\Lambda_{t_n}^m Gx_{t_n}, g\rangle \\
&\quad \geqslant \langle v-\Lambda_{t_n}^m Gx_{t_n}, B_m v+\frac{1}{\lambda_{m,t_n}}(\Lambda_{t_n}^{m-1}Gx_{t_n}-\Lambda_{t_n}^m Gx_{t_n}-\lambda_{m,t_n}B_m\Lambda_{t_n}^{m-1}Gx_{t_n})\rangle \\
&\quad = \langle v-\Lambda_{t_n}^m Gx_{t_n}, B_m v-B_m\Lambda_{t_n}^m Gx_{t_n}+B_m\Lambda_{t_n}^m Gx_{t_n}-B_m\Lambda_{t_n}^{m-1}Gx_{t_n}+\frac{\Lambda_{t_n}^{m-1}Gx_{t_n}-\Lambda_{t_n}^m Gx_{t_n}}{\lambda_{m,t_n}}\rangle \\
&\quad \geqslant \langle v-\Lambda_{t_n}^m Gx_{t_n}, B_m\Lambda_{t_n}^m Gx_{t_n}-B_m\Lambda_{t_n}^{m-1}Gx_{t_n}\rangle+\langle v-\Lambda_{t_n}^m Gx_{t_n}, \frac{\Lambda_{t_n}^{m-1}Gx_{t_n}-\Lambda_{t_n}^m Gx_{t_n}}{\lambda_{m,t_n}}\rangle.
\end{aligned}
$$

Since $\|\Lambda_{t_n}^m Gx_{t_n}-\Lambda_{t_n}^{m-1}Gx_{t_n}\|\to 0$ (due to (3.11)) and $\|B_m\Lambda_{t_n}^m Gx_{t_n}-B_m\Lambda_{t_n}^{m-1}Gx_{t_n}\|\to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Lambda_{t_n}^m Gx_{t_n}\to x^*$ and $\{\lambda_{m,t_n}\}\subset[a_m,b_m]\subset(0,2\eta_m)$ that

$$\lim_{n\to\infty}\langle v-\Lambda_{t_n}^m Gx_{t_n}, g\rangle=\langle v-x^*, g\rangle\geqslant 0.$$

It follows from the maximal monotonicity of B_m+R_m that $0\in(R_m+B_m)x^*$, i.e., $x^*\in I(B_m,R_m)$. Thus, $x^*\in\bigcap_{m=1}^N I(B_m,R_m)$. Consequently, it is known that

$$x^*\in\bigcap_{m=1}^N I(B_m,R_m)\cap\Xi\cap\Gamma=:\Omega.$$

Finally, let us show that x^* is a solution of the VIP (3.2). As a matter of fact, since

$$x_t=x_t-w_t+(I-s_tA)T_t\Lambda_t^N Gx_t+s_t(t\gamma Vx_t+(I-t\mu F)T_t\Lambda_t^N Gx_t),$$

we have

$$x_t-T_t\Lambda_t^N Gx_t=x_t-w_t+s_t(I-A)T_t\Lambda_t^N Gx_t+s_t t(\gamma Vx_t-\mu FT_t\Lambda_t^N Gx_t).$$

Since G, T_t, and Λ_t^N are nonexpansive, $I-T_t\Lambda_t^N G$ is monotone. So, from the monotonicity of $I-T_t\Lambda_t^N G$, it follows that, for $p\in\Omega$,

$$
\begin{aligned}
0&\leqslant\langle(I-T_t\Lambda_t^N G)x_t-(I-T_t\Lambda_t^N G)p, x_t-p\rangle\\
&=\langle(I-T_t\Lambda_t^N G)x_t, x_t-p\rangle\\
&=\langle x_t-w_t, x_t-p\rangle+s_t\langle(I-A)T_t\Lambda_t^N Gx_t, x_t-p\rangle+s_t t\langle\gamma Vx_t-\mu FT_t\Lambda_t^N Gx_t, x_t-p\rangle\\
&\leqslant s_t\langle(I-A)T_t\Lambda_t^N Gx_t, x_t-p\rangle+s_t t\langle\gamma Vx_t-\mu FT_t\Lambda_t^N Gx_t, x_t-p\rangle\\
&=s_t\langle(I-A)x_t, x_t-p\rangle+s_t\langle(I-A)(T_t\Lambda_t^N G-I)x_t, x_t-p\rangle+s_t t\langle\gamma Vx_t-\mu FT_t\Lambda_t^N Gx_t, x_t-p\rangle.
\end{aligned}
$$

This implies that

$$\langle(A-I)x_t, x_t-p\rangle\leqslant\langle(I-A)(T_t\Lambda_t^N G-I)x_t, x_t-p\rangle+t\langle\gamma Vx_t-\mu FT_t\Lambda_t^N Gx_t, x_t-p\rangle. \tag{3.25}$$

Now, replacing t in (3.25) with t_n and letting $n\to\infty$, noticing the boundedness of $\{\gamma Vx_{t_n}-\mu FT_{t_n}\Lambda_{t_n}^N Gx_{t_n}\}$ and the fact that $(I-A)(T_{t_n}\Lambda_{t_n}^N G-I)x_{t_n}\to 0$ as $n\to\infty$ (due to (3.4)), we obtain

$$\langle(A-I)x^*, x^*-p\rangle\leqslant 0.$$

That is, $x^*\in\Omega$ is a solution of the VIP (3.2); hence $x^*=\tilde{x}$ by uniqueness. In summary, we have proven that each cluster point of $\{x_t\}$ (as $t\to 0$) equals to $\tilde{x}$. Consequently, $x_t\to\tilde{x}$ as $t\to 0$.

The VIP (3.2) can be rewritten as

$$\langle(2I-A)\tilde{x}-\tilde{x}, \tilde{x}-p\rangle\geqslant 0, \quad \forall p\in\Omega.$$

Recalling Proposition 2.1 (i), the last inequality is equivalent to the fixed point equation

$$P_\Omega(2I-A)\tilde{x}=\tilde{x}.$$

□

Taking $F = \frac{1}{2}I$, $\mu = 2$ and $\gamma = 1$ in Theorem 3.2, we get

Corollary 3.3. *Let $\{x_t\}$ be defined by*

$$x_t = P_C[(I - s_t A)T_t \Lambda_t^N Gx_t + s_t(tVx_t + (1-t)T_t\Lambda_t^N Gx_t)].$$

If $\lim_{t\to 0} s_t = 0$, then $\{x_t\}$ converges strongly as $t \to 0$ to a point $\tilde{x} \in \Omega$, which is the unique solution of the VIP (3.2).

First, we prove the following result in order to establish the strong convergence of the sequence $\{x_n\}$ generated by the hybrid explicit steepest-descent scheme (3.3).

Theorem 3.4. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3), where $\{\alpha_n\}$ and $\{s_n\}$ satisfy the following condition:*

(C1) $\{\alpha_n\} \subset [0,1]$, $\{s_n\} \subset (0, \frac{1}{2})$, and $\alpha_n \to 0$, $s_n \to 0$ as $n \to \infty$.

Let LIM be a Banach limit. Then

$$\text{LIM}_n \langle (A-I)\tilde{x}, \tilde{x} - x_n\rangle \leqslant 0,$$

where $\tilde{x} = \lim_{t\to 0^+} x_t$ with x_t being defined by

$$x_t = P_C[(I - s_t A)T\Lambda^N Gx_t + s_t(t\gamma Vx_t + (I - t\mu F)T\Lambda^N Gx_t)], \tag{3.26}$$

where $T, G, \Lambda^N : C \to C$ are defined by $Tx = P_C(I - \frac{2}{L}\nabla f)x$, $Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x$ and $\Lambda^N x = J_{R_N,\lambda_N}(I - \lambda_N B_N)\cdots J_{R_1,\lambda_1}(I - \lambda_1 B_1)x$ with $\nu_j \in (0, 2\zeta_j), j = 1,2$ and $\lambda_i \in [a_i, b_i] \subset (0, 2\eta_i)$ for each $i = 1, ..., N$.

Proof. First, note that from the condition (C1), without loss of generality, we may assume that $0 < s_n \leqslant \|A\|^{-1}$ for all $n \geqslant 0$.

Let $\{x_t\}$ be the net generated by (3.26). Since T, G, and Λ^N are nonexpansive self-mappings on C, by Theorem 3.2 with $T_t = T$ and $\Lambda_t^N = \Lambda^N$, there exists $\lim_{t\to 0} x_t \in \Omega$. Denote it by $\tilde{x}$. Moreover, $\tilde{x}$ is the unique solution of the VIP (3.2). From Proposition 3.1 (i) with $T_t = T$ and $\Lambda_t^N = \Lambda^N$, we know that $\{x_t\}$ is bounded and so are the nets $\{Vx_t\}, \{\Lambda^N Gx_t\}, \{T\Lambda^N Gx_t\}$, and $\{FT\Lambda^N Gx_t\}$.

First of all, let us show that $\{x_n\}$ is bounded. To this end, take $p \in \Omega$. Then we get

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T_n\Lambda_n^N Gx_n - p\| \\
&= \|\alpha_n(\gamma Vx_n - \mu Fp) + (I - \alpha_n\mu F)T_n\Lambda_n^N Gx_n - (I - \alpha_n\mu F)T_n\Lambda_n^N Gp\| \\
&\leqslant \alpha_n\gamma l\|x_n - p\| + \alpha_n\|(\gamma V - \mu F)p\| + (1 - \alpha_n\tau)\|x_n - p\| \\
&= (1 - \alpha_n(\tau - \gamma l))\|x_n - p\| + \alpha_n\|(\gamma V - \mu F)p\|,
\end{aligned}$$

which together with Lemma 2.14, implies that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|P_C[(I - s_n A)T_n\Lambda_n^N Gx_n + s_n y_n] - p\| \\
&\leqslant \|(I - s_n A)T_n\Lambda_n^N Gx_n + s_n y_n - p\| \\
&= \|(I - s_n A)T_n\Lambda_n^N Gx_n - (I - s_n A)T_n\Lambda_n^N Gp + s_n(y_n - p) + s_n(I - A)p\| \\
&\leqslant \|(I - s_n A)T_n\Lambda_n^N Gx_n - (I - s_n A)T_n\Lambda_n^N Gp\| + s_n\|y_n - p\| + s_n\|I - A\|\|p\| \\
&\leqslant (1 - s_n\bar{\gamma})\|x_n - p\| + s_n[(1 - \alpha_n(\tau - \gamma l))\|x_n - p\| \\
&\quad + \alpha_n\|(\gamma V - \mu F)p\|] + s_n\|I - A\|\|p\| \\
&\leqslant (1 - s_n(\bar{\gamma} - 1))\|x_n - p\| + s_n(\|(\gamma V - \mu F)p\| + \|I - A\|\|p\|) \\
&= (1 - s_n(\bar{\gamma} - 1))\|x_n - p\| + s_n(\bar{\gamma} - 1)\frac{\|(\gamma V - \mu F)p\| + \|I - A\|\|p\|}{\bar{\gamma} - 1}
\end{aligned}$$

$$\leqslant \max\{\|x_n - p\|, \frac{\|(\gamma V - \mu F)p\| + \|I - A\|\|p\|}{\bar{\gamma} - 1}\}.$$

By induction

$$\|x_n - p\| \leqslant \max\{\|x_0 - p\|, \frac{\|(\gamma V - \mu F)p\| + \|I - A\|\|p\|}{\bar{\gamma} - 1}\}, \quad \forall n \geqslant 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{Vx_n\}, \{T_n \Lambda_n^N Gx_n\}, \{FT_n \Lambda_n^N Gx_n\}$, and $\{y_n\}$. Thus, utilizing the control condition (C1), we get

$$\begin{aligned} \|x_{n+1} - T_n \Lambda_n^N Gx_n\| &= \|P_C[(I - s_n A)T_n \Lambda_n^N Gx_n + s_n y_n] - T_n \Lambda_n^N Gx_n\| \\ &\leqslant \|(I - s_n A)T_n \Lambda_n^N Gx_n + s_n y_n - T_n \Lambda_n^N Gx_n\| \\ &= s_n \|y_n - AT_n \Lambda_n^N Gx_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

For simplicity, we write $v_n = \Lambda_n^N Gx_n$ for all $n \geqslant 0$. Then, utilizing the similar arguments to those of (3.19), we have that

$$\|T\Lambda^N Gx_n - T_n \Lambda_n^N Gx_n\| \leqslant \|\Lambda^N Gx_n - \Lambda_n^N Gx_n\| + \widehat{M}|\frac{2}{L} - \lambda_n|, \tag{3.27}$$

where $\sup_{n \geqslant 0}\{L\|P_C(I - \frac{2}{L}\nabla f)v_n\| + 4\|\nabla f(v_n)\| + L\|v_n\|\} \leqslant \widehat{M}$ for some $\widehat{M} > 0$. Utilizing the similar arguments to those of (3.20), we have that

$$\|\Lambda_n^N Gx_n - \Lambda^N Gx_n\| \leqslant \widehat{M}_0 \sum_{i=1}^{N} |\lambda_{i,n} - \lambda_i|, \tag{3.28}$$

where

$$\begin{aligned} \sup_{n \geqslant 0, 1 \leqslant i \leqslant N} \{ & \frac{1}{\lambda_{i,n}} \|J_{R_i, \lambda_{i,n}}(I - \lambda_i B_i)\Lambda_n^{i-1} Gx_n - (I - \lambda_i B_i)\Lambda^{i-1} Gx_n\| \\ & + \frac{1}{\lambda_i} \|(I - \lambda_i B_i)\Lambda_n^{i-1} Gx_n - J_{R_i, \lambda_i}(I - \lambda_i B_i)\Lambda^{i-1} Gx_n\|\} \leqslant \widehat{N} \end{aligned}$$

for some $\widehat{N} > 0$ and $\sup_{n \geqslant 0}\{\sum_{i=1}^{N} \|B_i \Lambda_n^{i-1} x_n\| + \widehat{N}\} \leqslant \widehat{M}_0$ for some $\widehat{M}_0 > 0$. In terms of (3.27)-(3.28) we calculate

$$\|T\Lambda^N Gx_n - T_n \Lambda_n^N Gx_n\| \leqslant \|\Lambda^N Gx_n - \Lambda_n^N Gx_n\| + \widehat{M}|\frac{2}{L} - \lambda_n| \leqslant \widehat{M}_0 \sum_{i=1}^{N} |\lambda_i - \lambda_{i,n}| + \widehat{M}|\frac{2}{L} - \lambda_n|.$$

Consequently, it is not hard to find that

$$\begin{aligned} \|T\Lambda^N Gx_t - x_{n+1}\| &\leqslant \|T\Lambda^N Gx_t - T\Lambda^N Gx_n\| + \|T\Lambda^N Gx_n - T_n \Lambda_n^N Gx_n\| + \|T_n \Lambda_n^N Gx_n - x_{n+1}\| \\ &\leqslant \|x_t - x_n\| + \widehat{M}_0 \sum_{i=1}^{N} |\lambda_i - \lambda_{i,n}| + \widehat{M}|\frac{2}{L} - \lambda_n| + \|T_n \Lambda_n^N Gx_n - x_{n+1}\| \\ &= \|x_t - x_n\| + \epsilon_n, \end{aligned} \tag{3.29}$$

where $\epsilon_n = \widehat{M}_0 \sum_{i=1}^{N} |\lambda_i - \lambda_{i,n}| + \widehat{M}|\frac{2}{L} - \lambda_n| + \|T_n \Lambda_n^N Gx_n - x_{n+1}\| \to 0$ as $n \to \infty$. Also observing that A is strongly positive, we have

$$\langle Ax_t - Ax_n, x_t - x_n \rangle = \langle A(x_t - x_n), x_t - x_n \rangle \geqslant \bar{\gamma}\|x_t - x_n\|^2. \tag{3.30}$$

For simplicity, we write $w_t = (I - s_t A)T\Lambda^N Gx_t + s_t(t\gamma Vx_t + (I - t\mu F)T\Lambda^N Gx_t)$. Then we obtain that $x_t = P_C w_t$ and

$$\begin{aligned} x_t - x_{n+1} &= x_t - w_t + (I - s_t A)T\Lambda^N Gx_t + s_t(t\gamma Vx_t + (I - t\mu F)T\Lambda^N Gx_t) - x_{n+1} \\ &= (I - s_t A)T\Lambda^N Gx_t - (I - s_t A)x_{n+1} + s_t(t\gamma Vx_t + (I - t\mu F)T\Lambda^N Gx_t - Ax_{n+1}) + x_t - w_t. \end{aligned}$$

Observe that

$$\begin{aligned} \|x_t - x_{n+1}\|^2 &\leqslant \|(I - s_t A)T\Lambda^N Gx_t - (I - s_t A)x_{n+1}\|^2 + 2\langle x_t - w_t, x_t - x_{n+1}\rangle \\ &\quad + 2s_t\langle T\Lambda^N Gx_t - t(\mu FT\Lambda^N Gx_t - \gamma Vx_t) - Ax_{n+1}, x_t - x_{n+1}\rangle \\ &\leqslant \|(I - s_t A)T\Lambda^N Gx_t - (I - s_t A)x_{n+1}\|^2 \\ &\quad + 2s_t\langle T\Lambda^N Gx_t - t(\mu FT\Lambda^N Gx_t - \gamma Vx_t) - Ax_{n+1}, x_t - x_{n+1}\rangle \\ &\leqslant (1 - s_t\bar{\gamma})^2\|T\Lambda^N Gx_t - x_{n+1}\|^2 + 2s_t\langle T\Lambda^N Gx_t - x_t, x_t - x_{n+1}\rangle \\ &\quad - 2s_t t\langle \mu FT\Lambda^N Gx_t - \gamma Vx_t, x_t - x_{n+1}\rangle + 2s_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle. \end{aligned} \tag{3.31}$$

Using (3.29) and (3.30) in (3.31), we obtain

$$\begin{aligned} \|x_t - x_{n+1}\|^2 &\leqslant (1 - s_t\bar{\gamma})^2\|T\Lambda^N Gx_t - x_{n+1}\|^2 + 2s_t\langle T\Lambda^N Gx_t - x_t, x_t - x_{n+1}\rangle \\ &\quad + 2s_t t\langle \gamma Vx_t - \mu FT\Lambda^N Gx_t, x_t - x_{n+1}\rangle + 2s_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\ &\leqslant (1 - s_t\bar{\gamma})^2(\|x_t - x_n\| + \epsilon_n)^2 + 2s_t\|T\Lambda^N Gx_t - x_t\|\|x_t - x_{n+1}\| \\ &\quad + 2s_t t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\|x_t - x_{n+1}\| + 2s_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\ &= (s_t^2\bar{\gamma} - 2s_t)\bar{\gamma}\|x_t - x_n\|^2 + \|x_t - x_n\|^2 \\ &\quad + (1 - s_t\bar{\gamma})^2[2\|x_t - x_n\|\epsilon_n + \epsilon_n^2] + 2s_t\|T\Lambda^N Gx_t - x_t\|\|x_t - x_{n+1}\| \\ &\quad + 2s_t t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\|x_t - x_{n+1}\| + 2s_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\ &= (s_t^2\bar{\gamma} - 2s_t)\bar{\gamma}\|x_t - x_n\|^2 + \|x_t - x_n\|^2 + (1 - s_t\bar{\gamma})^2[2\|x_t - x_n\|\epsilon_n + \epsilon_n^2] \\ &\quad + 2s_t\|T\Lambda^N Gx_t - x_t\|\|x_t - x_{n+1}\| + 2s_t t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\|x_t - x_{n+1}\| \\ &\quad + 2s_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\ &\leqslant (s_t^2\bar{\gamma} - 2s_t)\langle Ax_t - Ax_n, x_t - x_n\rangle + \|x_t - x_n\|^2 + (1 - s_t\bar{\gamma})^2[2\|x_t - x_n\|\epsilon_n + \epsilon_n^2] \\ &\quad + 2s_t\|T\Lambda^N Gx_t - x_t\|\|x_t - x_{n+1}\| + 2s_t t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\|x_t - x_{n+1}\| \\ &\quad + 2s_t\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle \\ &= s_t^2\bar{\gamma}\langle Ax_t - Ax_n, x_t - x_n\rangle + \|x_t - x_n\|^2 + (1 - s_t\bar{\gamma})^2[2\|x_t - x_n\|\epsilon_n + \epsilon_n^2] \\ &\quad + 2s_t\|T\Lambda^N Gx_t - x_t\|\|x_t - x_{n+1}\| + 2s_t t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\|x_t - x_{n+1}\| \\ &\quad + 2s_t[\langle x_t - Ax_{n+1}, x_t - x_{n+1}\rangle - \langle Ax_t - Ax_n, x_t - x_n\rangle] \\ &= s_t^2\bar{\gamma}\langle A(x_t - x_n), x_t - x_n\rangle + \|x_t - x_n\|^2 + (1 - s_t\bar{\gamma})^2[2\|x_t - x_n\|\epsilon_n + \epsilon_n^2] \\ &\quad + 2s_t\|T\Lambda^N Gx_t - x_t\|\|x_t - x_{n+1}\| + 2s_t t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\|x_t - x_{n+1}\| \\ &\quad + 2s_t[\langle (I - A)x_t, x_t - x_{n+1}\rangle + \langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle - \langle A(x_t - x_n), x_t - x_n\rangle]. \end{aligned} \tag{3.32}$$

Applying the Banach limit LIM to (3.32), from $\epsilon_n \to 0$ we have

$$\begin{aligned} &\mathrm{LIM}_n\|x_t - x_{n+1}\|^2 \\ &\leqslant s_t^2\bar{\gamma}\mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle + \mathrm{LIM}_n\|x_t - x_n\|^2 \\ &\quad + 2s_t\|T\Lambda^N Gx_t - x_t\|\mathrm{LIM}_n\|x_t - x_{n+1}\| + 2s_t t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\mathrm{LIM}_n\|x_t - x_{n+1}\| \\ &\quad + 2s_t[\mathrm{LIM}_n\langle (I - A)x_t, x_t - x_{n+1}\rangle + \mathrm{LIM}_n\langle A(x_t - x_{n+1}), x_t - x_{n+1}\rangle \\ &\quad - \mathrm{LIM}_n\langle A(x_t - x_n), x_t - x_n\rangle]. \end{aligned} \tag{3.33}$$

Utilizing the property $\mathrm{LIM}_n a_n = \mathrm{LIM}_n a_{n+1}$ of the Banach limit in (3.33), we obtain

$$\begin{aligned}
&\mathrm{LIM}_n\langle (A-I)x_t, x_t - x_n\rangle \\
&\quad = \mathrm{LIM}_n\langle (A-I)x_t, x_t - x_{n+1}\rangle \\
&\quad \leqslant \frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n), x_t-x_n\rangle + \frac{1}{2s_t}[\mathrm{LIM}_n\|x_t-x_n\|^2 - \mathrm{LIM}_n\|x_t-x_{n+1}\|^2] \\
&\qquad + \|T\Lambda^N Gx_t - x_t\|\mathrm{LIM}_n\|x_t-x_n\| + t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\mathrm{LIM}_n\|x_t-x_n\| \\
&\qquad + \mathrm{LIM}_n\langle A(x_t-x_{n+1}), x_t-x_{n+1}\rangle - \mathrm{LIM}_n\langle A(x_t-x_n), x_t-x_n\rangle \\
&\quad \leqslant \frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n), x_t-x_n\rangle + \|T\Lambda^N Gx_t - x_t\|\mathrm{LIM}_n\|x_t-x_n\| \\
&\qquad + t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\mathrm{LIM}_n\|x_t-x_n\|.
\end{aligned} \tag{3.34}$$

Since as $t \to 0$,

$$s_t\langle A(x_t-x_n), x_t-x_n\rangle \leqslant s_t\|A\|\|x_t-x_n\|^2 \leqslant s_t K \to 0, \tag{3.35}$$

where $\|A\|\|x_t-x_n\|^2 \leqslant K$,

$$\|T\Lambda^N Gx_t - x_t\| \to 0 \text{ (see (3.4))} \quad \text{and} \quad t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\| \to 0 \quad \text{as } t \to 0, \tag{3.36}$$

we conclude from (3.34)-(3.36) that

$$\begin{aligned}
\mathrm{LIM}_n\langle (A-I)\tilde{x}, \tilde{x}-x_n\rangle &\leqslant \limsup_{t\to 0}\mathrm{LIM}_n\langle (A-I)x_t, x_t-x_n\rangle \\
&\leqslant \limsup_{t\to 0}\frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n), x_t-x_n\rangle + \limsup_{t\to 0}\|T\Lambda^N Gx_t - x_t\|\mathrm{LIM}_n\|x_t-x_n\| \\
&\quad + \limsup_{t\to 0} t\|\gamma Vx_t - \mu FT\Lambda^N Gx_t\|\mathrm{LIM}_n\|x_t-x_n\| \\
&= 0.
\end{aligned}$$

This completes the proof. □

Now, using Theorem 3.4, we establish the strong convergence of the sequence $\{x_n\}$ generated by the hybrid explicit steepest-descent scheme (3.3) to a point $\tilde{x} \in \Omega$, which is also the unique solution of the VIP (3.2).

Theorem 3.5. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3), where $\{\alpha_n\}$ and $\{s_n\}$ satisfy the following conditions:*

(C1) $\{\alpha_n\} \subset [0,1]$, $\{s_n\} \subset (0, \frac{1}{2})$, *and* $\alpha_n \to 0$, $s_n \to 0$ *as* $n \to \infty$;

(C2) $\sum_{n=0}^{\infty} s_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1} - x_n \rightharpoonup 0$), then x_n converges strongly to a point $\tilde{x} \in \Omega$, which is the unique solution of the VIP (3.2).

Proof. First, note that from the condition (C1), without loss of generality, we may assume that $\alpha_n\tau < 1$ and $\frac{2s_n(\bar{\gamma}-1)}{1-s_n} < 1$ for all $n \geqslant 0$.

Let x_t be defined by (3.26), that is,

$$x_t = P_C[(I - s_t A)T\Lambda^N Gx_t + s_t(T\Lambda^N Gx_t - t(\mu FT\Lambda^N Gx_t - \gamma Vx_t)],$$

for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$, where $Tx = P_C(I - \frac{2}{L}\nabla f)x$, $Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x$, $\Lambda^N x = J_{R_N,\lambda_N}(I - \lambda_N B_N)\cdots J_{R_1,\lambda_1}(I - \lambda_1 B_1)x$ with $\nu_j \in (0, 2\zeta_j), j = 1,2$ and $\lambda_i \in [a_i, b_i] \subset (0, 2\eta_i), i = 1, ..., N$, and $\lim_{t\to 0} x_t := \tilde{x} \in \Omega$ (due to Theorem 3.2). Then $\tilde{x}$ is the unique solution of the VIP (3.2).

We divide the rest of the proof into several steps.

Step 1. We see that

$$\|x_n - p\| \leqslant \max\{\|x_0 - p\|, \frac{\|(\gamma V - \mu F)p\| + \|I - A\|\|p\|}{\bar{\gamma} - 1}\}, \quad \forall n \geqslant 0,$$

for all $p \in \Omega$ as in the proof of Theorem 3.4. Hence $\{x_n\}$ is bounded and so are $\{Vx_n\}$,$\{\Lambda_n^N Gx_n\}$, $\{FT_n\Lambda_n^N Gx_n\}$, and $\{y_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle (I-A)\tilde{x}, x_n - \tilde{x}\rangle \leqslant 0$. To this end, put

$$a_n := \langle (A-I)\tilde{x}, \tilde{x} - x_n\rangle, \quad \forall n \geqslant 0.$$

Then, by Theorem 3.4 we get $\mathrm{LIM}_n a_n \leqslant 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \limsup_{j\to\infty}(a_{n_j+1} - a_{n_j}),$$

and $x_{n_j} \rightharpoonup v \in H$. This implies that $x_{n_j+1} \rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w - \lim_{j\to\infty}(\tilde{x} - x_{n_j+1}) = w - \lim_{j\to\infty}(\tilde{x} - x_{n_j}) = (\tilde{x} - v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\langle (A-I)\tilde{x}, (\tilde{x} - x_{n_j+1}) - (\tilde{x} - x_{n_j})\rangle = 0.$$

Then, by Lemma 2.15, we obtain $\limsup_{n\to\infty} a_n \leqslant 0$, that is,

$$\limsup_{n\to\infty}\langle (I-A)\tilde{x}, x_n - \tilde{x}\rangle = \limsup_{n\to\infty}\langle (A-I)\tilde{x}, \tilde{x} - x_n\rangle \leqslant 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. Indeed, for simplicity, we write $w_n = (I - s_n A)T_n\Lambda_n^N Gx_n + s_n y_n$ for all $n \geqslant 0$. Then $x_{n+1} = P_C w_n$. Utilizing (3.3) and $T_n\Lambda_n^N G\tilde{x} = \tilde{x}$, we have

$$y_n - \tilde{x} = (I - \alpha_n\mu F)T_n\Lambda_n^N Gx_n - (I - \alpha_n\mu F)T_n\Lambda_n^N G\tilde{x} + \alpha_n(\gamma Vx_n - \mu F\tilde{x}),$$

and

$$x_{n+1} - \tilde{x} = x_{n+1} - w_n + (I - s_n A)(T_n\Lambda_n^N Gx_n - T_n\Lambda_n^N G\tilde{x}) + s_n(y_n - \tilde{x}) + s_n(I-A)\tilde{x}.$$

Applying Lemmas 2.14 and 2.18, we obtain

$$\begin{aligned}
\|y_n - \tilde{x}\|^2 &= \|(I - \alpha_n\mu F)T_n\Lambda_n^N Gx_n - (I - \alpha_n\mu F)T_n\Lambda_n^N G\tilde{x} + \alpha_n(\gamma Vx_n - \mu F\tilde{x})\|^2 \\
&\leqslant \|(I - \alpha_n\mu F)T_n\Lambda_n^N Gx_n - (I - \alpha_n\mu F)T_n\Lambda_n^N G\tilde{x}\|^2 + 2\alpha_n\langle \gamma Vx_n - \mu F\tilde{x}, y_n - \tilde{x}\rangle \\
&\leqslant (1 - \alpha_n\tau)^2\|x_n - \tilde{x}\|^2 + 2\alpha_n\|\gamma Vx_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\| \\
&\leqslant \|x_n - \tilde{x}\|^2 + 2\alpha_n\|\gamma Vx_n - \mu F\tilde{x}\|\|y_n - \tilde{x}\|,
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|(I - s_n A)(T_n\Lambda_n^N Gx_n - T_n\Lambda_n^N G\tilde{x}) + s_n(y_n - \tilde{x}) + s_n(I-A)\tilde{x} + x_{n+1} - w_n\|^2 \\
&\leqslant \|(I - s_n A)(T_n\Lambda_n^N Gx_n - T_n\Lambda_n^N G\tilde{x})\|^2 + 2s_n\langle y_n - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\quad + 2s_n\langle (I-A)\tilde{x}, x_{n+1} - \tilde{x}\rangle + 2\langle x_{n+1} - w_n, x_{n+1} - \tilde{x}\rangle \\
&\leqslant \|(I - s_n A)(T_n\Lambda_n^N Gx_n - T_n\Lambda_n^N G\tilde{x})\|^2 + 2s_n\langle y_n - \tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\quad + 2s_n\langle (I-A)\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\leqslant (1 - s_n\bar{\gamma})^2\|x_n - \tilde{x}\|^2 + 2s_n\|y_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| + 2s_n\langle (I-A)\tilde{x}, x_{n+1} - \tilde{x}\rangle
\end{aligned}$$

$$\begin{aligned}
&\leqslant (1-s_n\bar{\gamma})^2\|x_n-\tilde{x}\|^2+s_n(\|y_n-\tilde{x}\|^2+\|x_{n+1}-\tilde{x}\|^2)+2s_n\langle (I-A)\tilde{x},x_{n+1}-\tilde{x}\rangle\\
&\leqslant (1-s_n\bar{\gamma})^2\|x_n-\tilde{x}\|^2+s_n[\|x_n-\tilde{x}\|^2+2\alpha_n\|\gamma Vx_n-\mu F\tilde{x}\|\|y_n-\tilde{x}\|]\\
&\quad +s_n\|x_{n+1}-\tilde{x}\|^2+2s_n\langle (I-A)\tilde{x},x_{n+1}-\tilde{x}\rangle \qquad (3.37)\\
&=[(1-s_n\bar{\gamma})^2+s_n]\|x_n-\tilde{x}\|^2+2\alpha_n s_n\|\gamma Vx_n-\mu F\tilde{x}\|\|y_n-\tilde{x}\|\\
&\quad +s_n\|x_{n+1}-\tilde{x}\|^2+2s_n\langle (I-A)\tilde{x},x_{n+1}-\tilde{x}\rangle.
\end{aligned}$$

It then follows from (3.37) that

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\leqslant \frac{(1-s_n\bar{\gamma})^2+s_n}{1-s_n}\|x_n-\tilde{x}\|^2+\frac{s_n}{1-s_n}[2\alpha_n\|\gamma Vx_n-\mu F\tilde{x}\|\|y_n-\tilde{x}\|+2\langle (I-A)\tilde{x},x_{n+1}-\tilde{x}\rangle]\\
&=(1-\frac{2s_n(\bar{\gamma}-1)}{1-s_n})\|x_n-\tilde{x}\|^2+\frac{2s_n(\bar{\gamma}-1)}{1-s_n}\cdot\frac{1}{2(\bar{\gamma}-1)}[2\alpha_n\|\gamma Vx_n-\mu F\tilde{x}\|\|y_n-\tilde{x}\|\\
&\quad +s_n\bar{\gamma}^2\|x_n-\tilde{x}\|^2+2\langle (I-A)\tilde{x},x_{n+1}-\tilde{x}\rangle]\\
&=(1-\omega_n)\|x_n-\tilde{x}\|^2+\omega_n\delta_n,
\end{aligned}$$

where $\omega_n=\frac{2s_n(\bar{\gamma}-1)}{1-s_n}$ and

$$\delta_n=\frac{1}{2(\bar{\gamma}-1)}[2\alpha_n\|\gamma Vx_n-\mu F\tilde{x}\|\|y_n-\tilde{x}\|+s_n\bar{\gamma}^2\|x_n-\tilde{x}\|^2+2\langle (I-A)\tilde{x},x_{n+1}-\tilde{x}\rangle].$$

It can be readily seen from Step 2 and conditions (C1) and (C2) that $\omega_n\to 0$, $\sum_{n=0}^{\infty}\omega_n=\infty$, and $\limsup_{n\to\infty}\delta_n\leqslant 0$. By Lemma 2.13 with $r_n=0$, we conclude that $\lim_{n\to\infty}\|x_n-\tilde{x}\|=0$. This completes the proof. □

Corollary 3.6. *Let $\{x_n\}$ be the sequence generated by the explicit scheme* (3.3)*. Assume that the sequences $\{\alpha_n\}$ and $\{s_n\}$ satisfy the conditions* (C1) *and* (C2) *in Theorem* 3.5*. If $\{x_n\}$ is asymptotically regular (i.e., $x_{n+1}-x_n\to 0$), then $\{x_n\}$ converges strongly to a point $\tilde{x}\in\Omega$, which is the unique solution of the VIP* (3.2)*.*

Putting $\mu=2$, $F=\frac{1}{2}I$, and $\gamma=1$ in Theorem 3.5, we obtain the following.

Corollary 3.7. *Let $\{x_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} y_n=\alpha_n Vx_n+(1-\alpha_n)T_n\Lambda_n^N Gx_n,\\ x_{n+1}=P_C[(I-s_nA)T_n\Lambda_n^N Gx_n+s_ny_n], \quad \forall n\geqslant 0.\end{cases}$$

Assume that the sequences $\{\alpha_n\}$ and $\{s_n\}$ satisfy the conditions (C1) *and* (C2) *in Theorem* 3.5*. If $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1}-x_n\rightharpoonup 0$), then $\{x_n\}$ converges strongly to a point $\tilde{x}\in\Omega$, which is the unique solution of the VIP* (3.2)*.*

Putting $\alpha_n=0$, for all $n\geqslant 0$ in Corollary 3.7, we get the following.

Corollary 3.8. *Let $\{x_n\}$ be generated by the following iterative scheme:*

$$x_{n+1}=P_C[(I-s_n(A-I))T_n\Lambda_n^N Gx_n], \quad \forall n\geqslant 0.$$

Assume that the sequence $\{s_n\}$ satisfies the conditions (C1) *and* (C2) *in Theorem* 3.5 *with $\alpha_n=0$, for all $n\geqslant 0$. If $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1}-x_n\rightharpoonup 0$), then $\{x_n\}$ converges strongly to a point $\tilde{x}\in\Omega$, which is the unique solution of the VIP* (3.2)*.*

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