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Semi-implicit iterative schemes with perturbed operators for infinite accretive mappings and infinite nonexpansive mappings and their applications to parabolic systems

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Abstract

In a real uniformly convex and uniformly smooth Banach space, we first prove a new path convergence theorem and then present some new semi-implicit iterative schemes with errors which are proved to be convergent strongly to the common element of the set of zero points of infinite m-accretive mappings and the set of fixed points of infinite nonexpansive mappings. The superposition of perturbed operators are considered in the construction of the iterative schemes and new proof techniques are employed compared to some of the recent work. Some examples are listed and computational experiments are conducted, which guarantee the effectiveness of the proposed iterative schemes. Moreover, a kind of parabolic systems is exemplified, which sets up the relationship among iterative schemes, nonlinear systems and variational inequalities.

Keywords: M-accretive mapping, τ_i-strongly accretive mapping, contractive mapping, λ_i-strictly pseudocontractive mapping, semi-implicit iterative scheme, parabolic systems.
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1. Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and C be a nonempty closed convex subset of E. A mapping $S : C \to C$ is said to be nonexpansive [1], if for all $x, y \in C$,

$$\|Sx - Sy\| \leqslant \|x - y\|.$$

We denote by $Fix(S)$ the set of fixed points of S, that is, $Fix(S) = \{x \in C : x = Sx\}$. A mapping $f : C \to C$ is said to be a contractive mapping with coefficient $k \in (0, 1)$, if

$$\|f(x) - f(y)\| \leqslant k\|x - y\|, \quad \forall x, y \in C.$$

Let Q be a mapping of E onto C. Then Q is said to be sunny [12], if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geqslant 0$. A mapping Q of E into E is said to be a retraction [12], if $Q^2 = Q$. If a mapping Q is a

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retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset C of E is said to be a sunny nonexpansive retract of E [12], if there exists a sunny nonexpansive retraction of E onto C and it is called a nonexpansive retract of E, if there exists a nonexpansive retraction of E onto C.

Let E^* be the dual space of E and let $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx := \{f \in E^* : \langle x, f\rangle = \|x\|^2,\ \|f\| = \|x\|\}, \quad x \in E.$$

If E is uniformly smooth, then J is norm-to-norm continuous from E to E^*, c.f. [1].

A mapping $A : D(A) \sqsubseteq E \to E$ is said to be accretive [1], if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, J(x_1 - x_2)\rangle \geqslant 0$. An accretive mapping A is said to be m-accretive, if $R(I + \lambda A) = E$, for all $\lambda > 0$.

If A is accretive, then for each $r > 0$, the nonexpansive single-valued mapping $J^A_r : R(I + rA) \to D(A)$ defined by $J^A_r := (I + rA)^{-1}$ is called the resolvent of A, c.f. [1]. We denote by $A^{-1}0$ the set of zero points of A, that is, $A^{-1}0 = \{x \in D(A) : Ax = 0\}$. Then $A^{-1}0 = \mathrm{Fix}(J^A_r)$.

A mapping $B : E \to E$ is said to be τ-strongly accretive [8], if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Bx - By, j(x - y)\rangle \geqslant \tau\|x - y\|^2,$$

for some $\tau \in (0, 1)$. A mapping $B : E \to E$ is said to be λ-strictly pseudocontractive [8], if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Bx - By, j(x - y)\rangle \leqslant \|x - y\|^2 - \lambda\|x - y - (Bx - By)\|^2,$$

for some $\lambda \in (0, 1)$.

Designing iterative schemes to approximate zero point of accretive mappings or fixed point of nonexpansive mappings is still a hot topic in applied mathematics due to the practical background. Some of the related works can be seen in [11, 13, 14, 17] and the references therein.

In 2016, using the idea of implicit midpoint rule [2], Wei, et al. [16], presented the following iterative scheme to approximate the common element of the set of zero points of an m-accretive mapping and the set of fixed points of infinite nonexpansive mappings:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)\displaystyle\sum_{i=1}^{\infty} b_{n,i}[(1 - \alpha_{n,i})J^A_{r_n} + \alpha_{n,i}S_i]Q_C x_n, \\ u_n = (1 - \delta_n)y_n + \delta_n J^A_{r_n}(\dfrac{u_n + y_n}{2}), \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)u_n, \\ z_{n+1} := \dfrac{1}{\sum_{k=1}^{n+1} a_k}\displaystyle\sum_{k=1}^{n+1} a_k x_k, \quad n \geqslant 0, \end{cases}$$

where $T : C \to C$ is a strongly positive linear bounded operator, $f : C \to C$ is a contractive mapping, $A : C \to E$ is m-accretive and $S_i : C \to C$ is nonexpansive, for $i \in N$. Under some assumptions, both $\{x_n\}$ and $\{z_n\}$ are proved to be convergent strongly to the unique element $q_0 \in (\bigcap_{i=1}^{\infty} F(S_i)) \bigcap A^{-1}0$, which is also the solution of the following variational inequality: for all y in $(\bigcap_{i=1}^{\infty} F(S_i)) \bigcap A^{-1}0$,

$$\langle (T - \eta f)q_0, J(q_0 - y)\rangle \leqslant 0.$$

In 2012, Ceng, et al. [5], presented the following iterative scheme to approximate zero point of an m-accretive mapping:

$$\begin{cases} x_0 \in E, \\ y_n = \alpha_n x_n + (1 - \alpha_n)J^A_{r_n} x_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)[J^A_{r_n} y_n - \lambda_n \mu_n F(J^A_{r_n} y_n)], \quad n \geqslant 0, \end{cases} \tag{1.1}$$

where $T : E \to E$ is a δ-strongly accretive and λ-strictly pseudocontractive mapping, with $\delta + \lambda > 1$, $f : E \to E$ is a contractive mapping and $A : E \to E$ is m-accretive. Under some assumptions, $\{x_n\}$ is proved to be convergent strongly to the unique element $p_0 \in A^{-1}0$, which solves the following variational inequality:

$$\langle p_0 - f(p_0), J(p_0 - u)\rangle \leqslant 0, \quad \forall u \in A^{-1}0. \tag{1.2}$$

A very interesting thing is considered in Ceng's work. The mapping F considered in (1.1) is called a perturbed operator which only plays a role in the construction of the iterative scheme for selecting a particular zero of A and it is not involved in the variational inequality (1.2).

Motivated by the work in [16] and [5], in Section 2, we shall construct a new semi-implicit iterative scheme for approximating the common element of the set of zero points of infinite m-accretive mappings and the set of fixed points of infinite nonexpansive mappings. New proof techniques can be found, the superposition of perturbed operators are considered instead of one perturbed operator, infinite families of m-accretive mappings and nonexpansive mappings are discussed instead of finite families of m-accretive mappings and nonexpansive mappings, and some restrictions on the parameters are weakened compared to the existing similar works. Moreover, the computational experiments are conducted to clarify the effectiveness of our new iterative schemes. In Section 3, we shall discuss one kind parabolic systems as an example to strengthen the validity of the iterative scheme presented in Section 2.

We need the following preliminaries in our paper:

Lemma 1.1 ([6]). *Let E be a real smooth Banach space and $B : E \to E$ be a λ-strictly pseudocontractive mapping and also be a τ-strongly accretive mapping with $\lambda + \tau > 1$. Then for any fixed number $\delta \in (0, 1)$, $I - \delta B$ is contractive with coefficient $1 - \delta(1 - \sqrt{\frac{1-\tau}{\lambda}})$.*

Lemma 1.2 ([1]). *Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $f : C \to C$ be a contractive mapping. Then f has a unique fixed point $u \in C$.*

Lemma 1.3 ([4]). *Let E be a real strictly convex Banach space and let C be a nonempty closed and convex subset of E. Let $T_m : C \to C$ be a nonexpansive mapping for each $m \geqslant 1$. Let $\{a_m\}$ be a real number sequence in $(0, 1)$ such that $\sum_{m=1}^{\infty} a_m = 1$. Suppose that $\bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m) \neq \emptyset$. Then the mapping $\sum_{m=1}^{\infty} a_m T_m$ is nonexpansive with $\mathrm{Fix}(\sum_{m=1}^{\infty} a_m T_m) = \bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m)$.*

Lemma 1.4 ([7]). *In a real Banach space E, the following inequality holds:*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 1.5 ([9]). *Let $r, t > 0$. If E is uniformly convex, then there exists a continuous strictly increasing and convex function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ so that*

$$\|J_r^A x - J_r^A y\|^2 \leqslant \|x - y\|^2 - \varphi(\|(I - J_r^A)x - (I - J_r^A)y\|),$$

for all $x, y \in R(I + rA)$ with $\max\{\|x\|, \|y\|\} \leqslant t$, where $A : E \to E$ is m-accretive.

Lemma 1.6 ([10]). *Let $\{a_n\}$ be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_k}\}$ so that $a_{n_k} \leqslant a_{n_k+1}$, for all $k \geqslant 0$. For every $n > n_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{n_0 \leqslant k \leqslant n : a_k < a_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$, $\max\{a_{\tau(n)}, a_n\} \leqslant a_{\tau(n)+1}$.

2. Path convergence theorem and iterative convergence theorem

Theorem 2.1. *Let E be a real uniformly convex and uniformly smooth Banach space and C be a nonempty closed and convex sunny nonexpansive retract of E. Let Q_C be the sunny nonexpansive retraction of E onto C. Let $f_i : E \to E$ be contractive mappings with coefficient $k_i \in (0, 1)$, $B_i : E \to E$ be λ_i-strictly pseudocontractive mappings and τ_i-strongly accretive mappings with $\lambda_i + \tau_i > 1$, $A_i : C \to E$ be m-accretive mappings and $S_i : C \to C$ be* *nonexpansive mappings, for* $i \in N$. *Suppose* $\{a_i\}$, $\{b_i\}$, $\{\omega_i\}$ *and* $\{c_{n,i}\}$ *are real number sequences in* $(0,1)$, *for* $i \in N$ *and* $n \in N$. *Suppose* $\sum_{i=1}^{\infty} a_i\|f_i\| < +\infty$, $\sum_{i=1}^{\infty} b_i\|B_i\| < +\infty$, $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} \omega_i = 1$ *and* $\bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap Fix(S_i)) \neq \emptyset$. *If for each* $t \in (0,1)$, *defined* $U_t^n : E \to E$ *by*

$$U_t^n x = t\sum_{i=1}^{\infty} a_i f_i(x) + (1-t)(I - \theta_t \sum_{i=1}^{\infty} b_i B_i)(\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x),$$

then U_t^n *has a fixed point* x_t^n, *for each* $t \in (0,1)$ *and* $\theta_t \in (0,1)$. *Moreover, if* $\frac{\theta_t}{t} \to 0$, *then* x_t^n *converges strongly to the unique solution of the following variational inequality, as* $t \to 0$,

$$\langle p_0 - \sum_{i=1}^{\infty} a_i f_i(p_0), J(p_0 - q)\rangle \leqslant 0, \quad \forall q \in \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap Fix(S_i)). \tag{2.1}$$

Proof. We split the proof into five steps.

Step 1. $U_t^n : E \to E$ is a contractive mapping, for $t \in (0,1)$, $\theta_t \in (0,1)$ and $n \in N$.

In fact, for all x, y in E, using Lemma 1.1, we have:

$$\begin{aligned}
\|U_t^n x - U_t^n y\| &\leqslant t\sum_{i=1}^{\infty} a_i\|f_i(x) - f_i(y)\| + (1-t) \\
&\quad \times \|\sum_{i=1}^{\infty} b_i(I - \theta_t B_i)(\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x) \\
&\quad - \sum_{i=1}^{\infty} b_i(I - \theta_t B_i)(\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C y)\| \\
&\leqslant t\sum_{i=1}^{\infty} a_i k_i\|x - y\| + (1-t)\sum_{i=1}^{\infty} b_i[1 - \theta_t(1 - \sqrt{\frac{1-\tau_i}{\lambda_i}})]\|x - y\| \\
&\leqslant [1 - (1 - \sum_{i=1}^{\infty} a_i k_i)t]\|x - y\|,
\end{aligned}$$

which implies that U_t^n is a contractive mapping. By Lemma 1.2, there exists x_t^n such that $U_t^n x_t^n = x_t^n$. That is, $x_t^n = t\sum_{i=1}^{\infty} a_i f_i(x_t^n) + (1-t)(I - \theta_t\sum_{i=1}^{\infty} b_i B_i)(\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n)$.

Step 2. $\{x_t^n\}$ is bounded, for $n \in N$ and $0 < t \leqslant \overline{a}$, where $\overline{a}$ is a sufficiently small positive number.

For all p in $\bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap Fix(S_i))$, we know that

$$\begin{aligned}
\|x_t^n - p\| &\leqslant t\sum_{i=1}^{\infty} a_i k_i\|x_t^n - p\| + t\sum_{i=1}^{\infty} a_i\|f_i(p) - p\| + (1-t)\theta_t\sum_{i=1}^{\infty} b_i\|B_i p\| \\
&\quad + (1-t)\sum_{i=1}^{\infty} b_i\|(I - \theta_t B_i)(\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n \\
&\quad - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C p)\| \\
&\leqslant t\sum_{i=1}^{\infty} a_i\|f_i(p) - p\| + (1 - t + t\sum_{i=1}^{\infty} a_i k_i)\|x_t^n - p\| + (1-t)\theta_t\sum_{i=1}^{\infty} b_i\|B_i\|\|p\|.
\end{aligned}$$

Then

$$\|x_t^n - p\| \leqslant \frac{\sum_{i=1}^{\infty} a_i\|f_i(p) - p\| + \frac{\theta_t}{t}\sum_{i=1}^{\infty} b_i\|B_i\|\|p\|}{1 - \sum_{i=1}^{\infty} a_i k_i}.$$

Since $\lim_{t\to 0}\frac{\theta_t}{t} = 0$, then there exists a sufficiently small positive number $\overline{a}$ such that $0 < \frac{\theta_t}{t} < 1$, for $0 < t \leqslant \overline{a}$. Thus x_t^n is bounded for $n \in N$ and $0 < t \leqslant \overline{a}$. Then both $\{J_{r_{n,i}}^{A_i}Q_C x_t^n\}$ and $\{S_i Q_C x_t^n\}$ are bounded for $i \in N$, $n \in N$ and $0 < t \leqslant \bar{a}$.

Step 3. If $\lim_{t\to 0} \frac{\theta_t}{t} = 0$, then $x_t^n - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n \to 0$, as $t \to 0$, for $n \in N$.

In view of Step 2,

$$\begin{aligned}\|x_t^n - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n\| \leqslant\ & t\sum_{i=1}^{\infty} a_i\|f_i(x_t^n)\| \\ & + t\sum_{i=1}^{\infty} \omega_i\|(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n\| \\ & + (1-t)\theta_t \sum_{i=1}^{\infty} b_i\|B_i(\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} \\ & + c_{n,i}S_i]Q_C x_t^n)\| \to 0,\end{aligned}$$

as $t \to 0$.

Step 4. If the variational inequality (2.1) has solutions, the solution must be unique.

Supposed $p_0 \in \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap Fix(S_i))$ and $q_0 \in \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap Fix(S_i))$ are two solutions of (2.1), then

$$\langle p_0 - \sum_{i=1}^{\infty} a_i f_i(p_0), J(p_0 - q_0)\rangle \leqslant 0, \tag{2.2}$$

and

$$\langle q_0 - \sum_{i=1}^{\infty} a_i f_i(q_0), J(q_0 - p_0)\rangle \leqslant 0. \tag{2.3}$$

Adding up (2.2) and (2.3),

$$\langle p_0 - \sum_{i=1}^{\infty} a_i f_i(p_0) - q_0 + \sum_{i=1}^{\infty} a_i f_i(q_0), J(p_0 - q_0)\rangle \leqslant 0. \tag{2.4}$$

Since

$$\begin{aligned}&\langle p_0 - \sum_{i=1}^{\infty} a_i f_i(p_0) - q_0 + \sum_{i=1}^{\infty} a_i f_i(q_0), J(p_0 - q_0)\rangle \\ &= \|p_0 - q_0\|^2 - \sum_{i=1}^{\infty} a_i\langle f_i(p_0) - f_i(q_0), J(p_0 - q_0)\rangle \\ &\geqslant \|p_0 - q_0\|^2 - \sum_{i=1}^{\infty} a_i k_i\|p_0 - q_0\|^2 \geqslant 0,\end{aligned}$$

then (2.4) implies that $p_0 = q_0$.

Step 5. If $\lim_{t\to 0} \frac{\theta_t}{t} = 0$, then $x_t \to p_0 \in \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap Fix(S_i))$, as $t \to 0$, which solves the variational inequality (2.1).

Assume $t_m \to 0$. Set $x_m^n := x_{t_m}^n$ and defined $\mu : E \to \mathbb{R}$ by

$$\mu(x) = LIM\|x_m^n - x\|^2, \quad x \in E,$$

where LIM is the Banach limit on l^{∞}. Let

$$K = \{x \in E : \mu(x) = \min_{x\in E} LIM\|x_m^n - x\|^2\}.$$

It is easily seen that K is a nonempty closed convex bounded subset of E. Since

$$x_m^n - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_m^n \to 0,$$

then for $x \in K$,

$$\begin{aligned}\mu(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}}+c_{n,i}S_i]Q_Cx) &= \mathrm{LIM}\|x^n_m-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}}+c_{n,i}S_i]Q_Cx\|^2\\ &\leqslant \mathrm{LIM}\|x^n_m-x\|^2=\mu(x),\end{aligned}$$

it follows that $\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}}+c_{n,i}S_i]Q_C(K)\subset K$, that is, K is invariant under

$$\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}}+c_{n,i}S_i]Q_C.$$

Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, $\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}}+c_{n,i}S_i]Q_C$ has a fixed point, say p_0, in K. That is,

$$\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}}+c_{n,i}S_i]Q_Cp_0=p_0\in C,$$

which ensures from Lemma 1.3 that $p_0\in\bigcap_{i=1}^{\infty}(A_i^{-1}0\bigcap \mathrm{Fix}(S_i))$. Since p_0 is also a minimizer of μ over E, it follows that, for $t\in(0,1)$

$$\begin{aligned}0&\leqslant\frac{\mu(p_0+t\sum_{i=1}^{\infty}a_if_i(p_0)-tp_0)-\mu(p_0)}{t}\\ &=\mathrm{LIM}\frac{\|x^n_m-p_0-t\sum_{i=1}^{\infty}a_if_i(p_0)+tp_0\|^2-\|x^n_m-p_0\|^2}{t}\\ &=\mathrm{LIM}\frac{\langle x^n_m-p_0-t\sum_{i=1}^{\infty}a_if_i(p_0)+tp_0,J(x^n_m-p_0-t\sum_{i=1}^{\infty}a_if_i(p_0)+tp_0)\rangle-\|x^n_m-p_0\|^2}{t}\\ &=\mathrm{LIM}(\langle x^n_m-p_0,J(x^n_m-p_0-t\sum_{i=1}^{\infty}a_if_i(p_0)+tp_0)\rangle\\ &\quad+t\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0),J(x^n_m-p_0-t\sum_{i=1}^{\infty}a_if_i(p_0)+tp_0)\rangle-\|x^n_m-p_0\|^2)/t.\end{aligned}$$

Since E is uniformly smooth, then by letting $t\to0$, we find the two limits above can be interchanged and obtain

$$\mathrm{LIM}\langle\sum_{i=1}^{\infty}a_if_i(p_0)-p_0,J(x^n_m-p_0)\rangle\leqslant0. \tag{2.5}$$

Since

$$\begin{aligned}x^n_m-p_0&=t_m(\sum_{i=1}^{\infty}a_if_i(x^n_m)-p_0)+(1-t_m)\\ &\quad\times\{(I-\theta_{t_m}\sum_{i=1}^{\infty}b_iB_i)(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}}+c_{n,i}S_i]Q_Cx^n_m)-p_0\},\end{aligned}$$

then

$$\begin{aligned}\|x^n_m-p_0\|^2&=\langle x^n_m-p_0,J(x^n_m-p_0)\rangle\\ &\leqslant t_m\langle\sum_{i=1}^{\infty}a_if_i(x^n_m)-\sum_{i=1}^{\infty}a_if_i(p_0),J(x^n_m-p_0)\rangle+t_m\langle\sum_{i=1}^{\infty}a_if_i(p_0)-p_0,J(x^n_m-p_0)\rangle\\ &\quad+(1-t_m)\|\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}}+c_{n,i}S_i]Q_Cx^n_m-p_0\|\|x^n_m-p_0\|\end{aligned}$$

$$
\begin{aligned}
&+(1-t_m)\theta_{t_m}\|\sum_{i=1}^{\infty} b_i B_i(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n)\|\|x_m^n-p_0\| \\
&\leqslant (1-t_m+t_m\sum_{i=1}^{\infty} a_i k_i)\|x_m^n-p_0\|^2+t_m\langle\sum_{i=1}^{\infty} a_i f_i(p_0)-p_0, J(x_m^n-p_0)\rangle \\
&+(1-t_m)\theta_{t_m}\|\sum_{i=1}^{\infty} b_i B_i(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n)\|\|x_m^n-p_0\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x_m^n-p_0\|^2 \leqslant \frac{1}{1-\sum_{i=1}^{\infty} a_i k_i}&[\langle\sum_{i=1}^{\infty} a_i f_i(p_0)-p_0, J(x_m^n-p_0)\rangle \\
&+\frac{\theta_{t_m}}{t_m}\|\sum_{i=1}^{\infty} b_i B_i(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n)\|\|x_m^n-p_0\|].
\end{aligned}
\tag{2.6}
$$

Since $\frac{\theta_{t_m}}{t_m}\to 0$, then from (2.5), (2.6) and the result of Step 2, we have

$$
\mathrm{LIM}\|x_m^n-p_0\|^2 \leqslant 0,
$$

which implies that $\mathrm{LIM}\|x_m^n-p_0\|^2=0$, and then there exists a subsequence which is still denoted by $\{x_m^n\}$ such that $x_m^n \to p_0$.

Next, we shall show that p_0 solves the variational inequality (2.1).

Note that

$$
x_m^n = t_m\sum_{i=1}^{\infty} a_i f_i(x_m^n)+(1-t_m)(I-\theta_{t_m}\sum_{i=1}^{\infty} b_i B_i)(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n),
$$

then for all q in $\bigcap_{i=1}^{\infty}(A_i^{-1}0\bigcap \mathrm{Fix}(S_i))$,

$$
\begin{aligned}
&\langle\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n-\sum_{i=1}^{\infty} a_i f_i(x_m^n), J(x_m^n-q)\rangle \\
&=\frac{1}{t_m}\langle(I-\theta_{t_m}\sum_{i=1}^{\infty} b_i B_i)(\Sigma_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n), J(x_m^n-q)\rangle \\
&\quad-\frac{1}{t_m}\langle x_m^n-t_m\theta_{t_m}\sum_{i=1}^{\infty} b_i B_i(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n), J(x_m^n-q)\rangle \\
&=\frac{1}{t_m}\langle\sum_{i=1}^{\infty} b_i(I-\theta_{t_m}B_i)(\Sigma_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n) \\
&\quad-\sum_{i=1}^{\infty} b_i(I-\theta_{t_m}B_i)(\Sigma_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C q), J(x_m^n-q)\rangle \\
&\quad-\frac{1}{t_m}\|x_m^n-q\|^2-\frac{\theta_{t_m}}{t_m}\langle\sum_{i=1}^{\infty} b_i B_i q, J(x_m^n-q)\rangle \\
&\quad+\theta_{t_m}\langle\sum_{i=1}^{\infty} b_i B_i(\Sigma_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n), J(x_m^n-q)\rangle \\
&\leqslant -\frac{1}{t_m}(1-\sum_{i=1}^{\infty} b_i[1-\theta_{t_m}(1-\sqrt{\frac{1-\tau_i}{\lambda_i}})])\|x_m^n-q\|^2+\frac{\theta_{t_m}}{t_m}\sum_{i=1}^{\infty} b_i\|B_i\|\|q\|\|x_m^n-q\|
\end{aligned}
$$

$$
\begin{aligned}
&+\theta_{t_m}\sum_{i=1}^{\infty} b_i\|B_i(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_m^n)\|\|x_m^n-q\| \\
&\leqslant \frac{\theta_{t_m}}{t_m}\sum_{i=1}^{\infty} b_i\|B_i\|\|q\|\|x_m^n-q\| \\
&+\theta_{t_m}\sum_{i=1}^{\infty} b_i\|B_i(\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} \\
&+c_{n,i}S_i]Q_C x_m^n)\|\|x_m^n-q\|\to 0,
\end{aligned}
$$

as $t_m \to 0$. Since $x_n \to p_0$ and J is uniformly continuous on each bounded subsets of E, then taking the limit on both sides of the above inequality, $\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0), J(p_0-q)\rangle \leqslant 0$, which implies that p_0 satisfies the variational inequality (2.1).

Next, to prove the net $\{x_t^n\}$ converges strongly to p_0, as $t\to 0$, suppose there is another subsequence $\{x_{t_k}^n\}$ of $\{x_t^n\}$ satisfying $x_{t_k}^n \to q_0$ as $t_k \to 0$. Denote $x_{t_k}^n$ by x_k^n. Then result of Step 3 implies that

$$
0=\lim_{t_k\to 0}(x_k^n-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_k^n)=q_0-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C q_0,
$$

which ensures that $q_0\in\bigcap_{i=1}^{\infty}(A_i^{-1}0\bigcap \mathrm{Fix}(S_i))$ in view of Lemma 1.3. Repeating the above proof, we can also know that q_0 solves the variational inequality (2.1). Thus $p_0=q_0$ by using the result of Step 4.

Hence $x_t\to p_0$, as $t\to 0$, which is the unique solution of the variational inequality (2.1).

This completes the proof. □

Theorem 2.2. *Let E be a real uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex sunny nonexpansive retract of E and Q_C be the sunny nonexpansive retraction of E onto C. Let $f_i: E\to E$ be contractive mappings with coefficient $k_i\in(0,1)$, $B_i: E\to E$ be λ_i-strictly pseudocontractive mappings and τ_i-strongly accretive mappings with $\lambda_i+\tau_i>1$, $A_i: C\to E$ be m-accretive mappings and $S_i: C\to C$ be nonexpansive mappings, for $i\in N$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\xi_n\}$, $\{\gamma_n\}$, $\{\zeta_n\}$, $\{\mu_n\}$, $\{a_i\}$, $\{b_i\}$, $\{\tau_i\}$, $\{\lambda_i\}$, $\{\omega_i\}$ and $\{c_{n,i}\}$ are real number sequences in $(0,1)$, where $n\in N$ and $i\in N$. Suppose $\{r_{n,i}\}\subset(0,+\infty)$, where $n\in N$ and $i\in N$. $\{e_n'\}\subset E$ and $\{e_n''\}\subset C$ are error sequences. Further suppose $\bigcap_{i=1}^{\infty}(A_i^{-1}0\bigcap \mathrm{Fix}(S_i))\neq\emptyset$. Let $\{x_n\}$ be generated by the following iterative scheme:*

$$
\begin{cases}
x_1\in C,\\
y_n=Q_C[(1-\alpha_n)(x_n+e_n')],\\
z_n=\delta_n y_n+\beta_n\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i](\frac{y_n+z_n}{2})+\xi_n e_n'',\\
x_{n+1}=\gamma_n\sum_{i=1}^{\infty}a_if_i(x_n)+(1-\gamma_n)(I-\zeta_n\mu_n\sum_{i=1}^{\infty}b_iB_i)\sum_{i=1}^{\infty}\omega_iJ_{r_{n,i}}^{A_i}z_n, \quad n\in N.
\end{cases}
\tag{2.7}
$$

Under the following assumptions that

(i) $\delta_n+\beta_n+\xi_n\equiv 1$, *for* $n\in N$;

(ii) $\sum_{i=1}^{\infty}a_i=\sum_{i=1}^{\infty}b_i=\sum_{i=1}^{\infty}\omega_i=1$;

(iii) $\sum_{n=1}^{\infty}\|e_n'\|<+\infty$, $\sum_{n=1}^{\infty}\|e_n''\|<+\infty$, $\sum_{n=1}^{\infty}\alpha_n<+\infty$, $\sum_{n=1}^{\infty}\xi_n<+\infty$, $\lim_{n\to\infty}\sum_{i=1}^{\infty}c_{n,i}=0$;

(iv) $\lim_{n\to\infty}\gamma_n=0$, $\sum_{n=1}^{\infty}\gamma_n=+\infty$;

(v) $\lim_{n\to\infty}\delta_n=0$, $\|e_n'\|+\alpha_n=o(\gamma_n)$, $\xi_n=o(\gamma_n)$, $\zeta_n\mu_n=o(\gamma_n)$, *as* $n\to\infty$;

(vi) $\sum_{i=1}^{\infty} a_i\|f_i\| < +\infty, \quad \sum_{i=1}^{\infty} b_i\|B_i\| < +\infty,$

the iterative sequence $x_n \to p_0 \in \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap \mathrm{Fix}(S_i))$, *which is the unique solution of the variational inequality* (2.1).

Proof. We split the proof into four steps.

Step 1. $\{z_n\}$ is well-defined and so is $\{x_n\}$.

For $s,t \in (0,1)$, define $G_{s,t} : C \to C$ by $G_{s,t}x := su + tG(\frac{u+x}{2}) + (1-s-t)v$, where $G : C \to C$ is nonexpansive for $x \in C$ and $u, v \in C$. Then, for all x, y in C,

$$\|G_{s,t}x - G_{s,t}y\| \leqslant t\|\frac{u+x}{2} - \frac{u+y}{2}\| \leqslant \frac{t}{2}\|x-y\|.$$

Thus $G_{s,t}$ is a contractive mapping, which ensures from Lemma 1.2 that there exists $x_{s,t} \in C$ such that $G_{s,t}x_{s,t} = x_{s,t}$. That is, $x_{s,t} = su + tG(\frac{u+x_{s,t}}{2}) + (1-s-t)v$.

Since $\sum_{i=1}^{\infty}\omega_i = 1$ and both $J_{r_{n,i}}^{A_i}$ and S_i are nonexpansive for $n \in N$ and $i \in N$, then $\{z_n\}$ is well-defined, which implies that $\{x_n\}$ is well-defined.

Step 2. $\{x_n\}$ is bounded.

For all p in $\bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap \mathrm{Fix}(S_i))$, we can easily know that

$$\|y_n - p\| \leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|p\| + (1-\alpha_n)\|e_n'\|,$$

and

$$\begin{aligned}\|z_n - p\| &\leqslant \delta_n\|y_n - p\| + \beta_n\|\frac{y_n + z_n}{2} - p\| + \xi_n\|e_n'' - p\| \\ &\leqslant (\delta_n + \frac{\beta_n}{2})\|y_n - p\| + \frac{\beta_n}{2}\|z_n - p\| + \xi_n\|e_n'' - p\|.\end{aligned}$$

Thus

$$\begin{aligned}\|z_n - p\| &\leqslant (\frac{2\delta_n + \beta_n}{2-\beta_n})\|y_n - p\| + \frac{2\xi_n}{2-\beta_n}\|e_n'' - p\| \\ &\leqslant (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\|e_n'\| + \alpha_n\|p\| + 2\|e_n''\| + \frac{2\xi_n}{2-\beta_n}\|p\|.\end{aligned} \tag{2.8}$$

Using Lemma 1.1 and (2.8), we have for $n \in N$,

$$\begin{aligned}\|x_{n+1} - p\| &\leqslant \gamma_n\|(\sum_{i=1}^{\infty} a_i f_i(x_n) - \sum_{i=1}^{\infty} a_i f_i(p)) \\ &\quad + (\sum_{i=1}^{\infty} a_i f_i(p) - p)\| + (1-\gamma_n)\|(I - \zeta_n\mu_n\sum_{i=1}^{\infty} b_i B_i)\sum_{i=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i} z_n - p\| \\ &\leqslant \gamma_n\sum_{i=1}^{\infty} a_i k_i\|x_n - p\| + \gamma_n\|\sum_{i=1}^{\infty} a_i f_i(p) - p\| \\ &\quad + (1-\gamma_n)\|\zeta_n(I - \mu_n\sum_{i=1}^{\infty} b_i B_i)\sum_{i=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i} z_n - \zeta_n(I - \mu_n\sum_{i=1}^{\infty} b_i B_i)\sum_{i=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i} p\| \\ &\quad + (1-\gamma_n)\zeta_n\mu_n\|\sum_{i=1}^{\infty} b_i B_i p\| + (1-\gamma_n)(1-\zeta_n)\|\sum_{i=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i} z_n - p\| \\ &\leqslant \gamma_n\sum_{i=1}^{\infty} a_i k_i\|x_n - p\| + \gamma_n\|\sum_{i=1}^{\infty} a_i f_i(p) - p\| \\ &\quad + (1-\gamma_n)\zeta_n\|\sum_{i=1}^{\infty} b_i(I - \mu_n B_i)(\sum_{i=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i} z_n - \sum_{i=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i} p)\|\end{aligned}$$

$$
\begin{aligned}
&+(1-\gamma_n)\zeta_n\mu_n\sum_{i=1}^{\infty} b_i\|B_i\|\|p\|+(1-\gamma_n)(1-\zeta_n)\|z_n-p\| \\
&\leqslant \gamma_n\sum_{i=1}^{\infty} a_i k_i\|x_n-p\|+\gamma_n\|\sum_{i=1}^{\infty} a_i f_i(p)-p\| \\
&+(1-\gamma_n)\zeta_n\sum_{i=1}^{\infty} b_i[1-\mu_n(1-\sqrt{\frac{1-\tau_i}{\lambda_i}})]\|z_n-p\| \\
&+(1-\gamma_n)\zeta_n\mu_n\sum_{i=1}^{\infty} b_i\|B_i\|\|p\|+(1-\gamma_n)(1-\zeta_n)\|z_n-p\| \qquad (2.9)\\
&\leqslant \gamma_n\sum_{i=1}^{\infty} a_i k_i\|x_n-p\|+\gamma_n\|\sum_{i=1}^{\infty} a_i f_i(p)-p\| \\
&+(1-\gamma_n)[1-\zeta_n\mu_n(1-\sum_{i=1}^{\infty} b_i\sqrt{\frac{1-\tau_i}{\lambda_i}})][(1-\alpha_n)\|x_n-p\| \\
&+(1-\alpha_n)\|e'_n\|+\alpha_n\|p\|+2\|e''_n\|+\frac{2\xi_n}{2-\beta_n}\|p\|] \\
&+(1-\gamma_n)\zeta_n\mu_n\sum_{i=1}^{\infty} b_i\|B_i\|\|p\|.
\end{aligned}
$$

By using the inductive method, we can easily get the following result from (2.9) that:

$$
\begin{aligned}
\|x_{n+1}-p\| \leqslant \max\{\|x_1-p\|, &\frac{\sum_{i=1}^{\infty} b_i\|B_i\|\|p\|}{1-\sum_{i=1}^{\infty} b_i\sqrt{\frac{1-\tau_i}{\lambda_i}}}, \frac{\sum_{i=1}^{\infty} a_i\|f_i(p)-p\|}{1-\sum_{i=1}^{\infty} a_i k_i}\} \\
&+\Sigma_{k=1}^{\infty}(1-\gamma_k)[1-\zeta_k\mu_k(1-\sum_{i=1}^{\infty} b_i\sqrt{\frac{1-\tau_i}{\lambda_i}})] \\
&\times[(1-\alpha_k)\|e'_k\|+\alpha_k\|p\|+2\|e''_k\|+\frac{2\xi_k}{2-\beta_k}\|p\|].
\end{aligned}
$$

Therefore, from assumption (iii), we know that $\{x_n\}$ is bounded.

Step 3. There exists $p_0 \in \bigcap_{i=1}^{\infty}(A_i^{-1}0\bigcap \mathrm{Fix}(S_i))$, which solves the variational inequality (2.1).

Using Theorem 2.1, we know that there exists x_t^n such that

$$
x_t^n = t\Sigma_{k=1}^{\infty} a_i f_i(x_t^n)+(1-t)(I-\theta_t\sum_{i=1}^{\infty} b_i B_i)(\Sigma_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_C x_t^n),
$$

for $t \in (0,1)$. Moreover, under the assumption that $\frac{\theta_t}{t} \to 0$, $x_t^n \to p_0 \in \bigcap_{i=1}^{\infty}(A_i^{-1}0\bigcap \mathrm{Fix}(S_i))$, as $t \to 0$, which is the unique solution of the variational inequality (2.1).

Step 4. $x_n \to p_0$, as $n \to \infty$, where p_0 is the same as that in Step 3.

Set $K_1 := \sup\{2\|(1-\alpha_n)(x_n+e'_n)-p_0\|, 2\|p_0\|\|(1-\alpha_n)(x_n+e'_n)-p_0\| : n \in N\}$, then from Step 2, K_1 is a positive constant.

Using Lemma 1.4, we have

$$
\begin{aligned}
\|y_n-p_0\|^2 &\leqslant (1-\alpha_n)\|x_n-p_0\|^2+2(1-\alpha_n)\langle e'_n, J[(1-\alpha_n)(x_n+e'_n)-p_0]\rangle \\
&\quad -2\alpha_n\langle p_0, J[(1-\alpha_n)(x_n+e'_n)-p_0]\rangle \qquad (2.10)\\
&\leqslant (1-\alpha_n)\|x_n-p_0\|^2+K_1(\|e'_n\|+\alpha_n).
\end{aligned}
$$

Using Lemma1.5, we know that:

$$
\begin{aligned}
\|z_n - p_0\|^2 \leqslant\ & \delta_n\|y_n - p_0\|^2 + \beta_n\Sigma_{i=1}^{\infty}\omega_i\|[(1-c_{n,i})J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2}) \\
& + c_{n,i}S_i(\frac{y_n+z_n}{2})] - p_0\|^2 + \xi_n\|e_n'' - p_0\|^2 \\
\leqslant\ & \delta_n\|y_n - p_0\|^2 + \beta_n\Sigma_{k=1}^{\infty}\omega_i(1-c_{n,i})\|J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2}) \\
& - p_0\|^2 + \beta_n\Sigma_{i=1}^{\infty}\omega_i c_{n,i}\|S_i(\frac{y_n+z_n}{2}) - p_0\|^2 + \xi_n\|e_n'' - p_0\|^2 \\
\leqslant\ & \delta_n\|y_n - p_0\|^2 + \beta_n\Sigma_{i=1}^{\infty}\omega_i(1-c_{n,i})[\|\frac{y_n+z_n}{2} - p_0\|^2 - \varphi(\|\frac{y_n+z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|)] \\
& + \beta_n\Sigma_{k=1}^{\infty}\omega_i c_{n,i}\|\frac{y_n+z_n}{2} - p_0\|^2 + \xi_n\|e_n'' - p_0\|^2 \\
\leqslant\ & (\delta_n + \frac{\beta_n}{2})\|y_n - p_0\|^2 + \frac{\beta_n}{2}\|z_n - p_0\|^2 \\
& - \beta_n\Sigma_{k=1}^{\infty}\omega_i(1-c_{n,i})\varphi(\|\frac{y_n+z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|) + \xi_n\|e_n'' - p_0\|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|z_n - p_0\|^2 \leqslant\ & \frac{2\delta_n + \beta_n}{2-\beta_n}\|y_n - p_0\|^2 \\
& - \frac{2\beta_n}{2-\beta_n}\Sigma_{i=1}^{\infty}\omega_i(1-c_{n,i})\varphi(\|\frac{y_n+z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|) + \frac{2\xi_n}{2-\beta_n}\|e_n'' - p_0\|^2.
\end{aligned}
\tag{2.11}
$$

Now, from (2.10), (2.11) and Lemma 1.4, we know that for $n \in N$,

$$
\begin{aligned}
\|x_{n+1} - p_0\|^2 =\ & \|\gamma_n(\Sigma_{i=1}^{\infty}a_i f_i(x_n) - p_0) + (1-\gamma_n)(\Sigma_{k=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i}z_n - p_0) \\
& - (1-\gamma_n)\zeta_n\mu_n\Sigma_{i=1}^{\infty}b_iB_i(\Sigma_{k=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i}z_n)\|^2 \\
\leqslant\ & (1-\gamma_n)^2\|z_n - p_0\|^2 + 2\gamma_n\langle\Sigma_{i=1}^{\infty}a_i f_i(x_n) \\
& - \Sigma_{i=1}^{\infty}a_i f_i(p_0), J(x_{n+1} - p_0)\rangle + 2\gamma_n\langle\Sigma_{i=1}^{\infty}a_i f_i(p_0) - p_0, J(x_{n+1} - p_0)\rangle \\
& - 2(1-\gamma_n)\zeta_n\mu_n\langle\Sigma_{i=1}^{\infty}b_iB_i(\Sigma_{k=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i}z_n), J(x_{n+1} - p_0)\rangle \\
\leqslant\ & (1-\gamma_n)\|x_n - p_0\|^2 + K_1(\|e_n'\| + \alpha_n) + \frac{2\xi_n}{2-\beta_n}\|e_n'' - p_0\|^2 \\
& + \gamma_n\Sigma_{k=1}^{\infty}a_ik_i\|x_n - p_0\|^2 + \gamma_n\Sigma_{k=1}^{\infty}a_ik_i\|x_{n+1} - p_0\|^2 \\
& - (1-\gamma_n)^2\frac{2\beta_n}{2-\beta_n}\sum_{i=1}^{\infty}\omega_i(1-c_{n,i})\varphi(\|\frac{y_n+z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|) \\
& + 2\gamma_n\langle\Sigma_{i=1}^{\infty}a_i f_i(p_0) - p_0, J(x_{n+1} - p_0)\rangle \\
& - 2(1-\gamma_n)\zeta_n\mu_n\langle\Sigma_{i=1}^{\infty}b_iB_i(\Sigma_{i=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i}z_n), J(x_{n+1} - p_0)\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - p_0\|^2 \leqslant\ & \frac{1-\gamma_n(1-\Sigma_{i=1}^{\infty}a_ik_i)}{1-\gamma_n\Sigma_{i=1}^{\infty}a_ik_i}\|x_n - p_0\|^2 + \frac{1}{1-\gamma_n\Sigma_{i=1}^{\infty}a_ik_i}(K_1\|e_n'\| + K_1\alpha_n) \\
& + \frac{1}{1-\gamma_n\Sigma_{i=1}^{\infty}a_ik_i}(\frac{2\xi_n}{2-\beta_n}\|e_n'' - p_0\|^2 + 2\gamma_n\langle\Sigma_{i=1}^{\infty}a_i f_i(p_0) - p_0, J(x_{n+1} - p_0)\rangle \\
& + 2(1-\gamma_n)\zeta_n\mu_n\Sigma_{i=1}^{\infty}b_i\|B_i(\Sigma_{k=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i}z_n)\|\|x_{n+1} - p_0\|) \\
& - (1-\gamma_n)^2\frac{2\beta_n}{2-\beta_n}\frac{1}{1-\gamma_n\Sigma_{i=1}^{\infty}a_ik_i}\sum_{i=1}^{\infty}\omega_i(1-c_{n,i})\varphi(\|\frac{y_n+z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|).
\end{aligned}
$$

From Step 2, if we set $K_2 = \sup\{\Sigma_{i=1}^{\infty}b_i\|B_i(\Sigma_{i=1}^{\infty}\omega_i J_{r_{n,i}}^{A_i}z_n)\|, \|x_n - p_0\| : n \in N\}$, then K_2 is a positive constant.

Let

$$\varepsilon_n^{(1)} = \frac{\gamma_n(1-2\Sigma_{i=1}^{\infty}a_ik_i)}{1-\gamma_n\Sigma_{i=1}^{\infty}a_ik_i},$$

$$\varepsilon_n^{(2)} = \frac{1}{\gamma_n(1-2\Sigma_{i=1}^{\infty}a_ik_i)}(K_1\|e_n'\| + K_1\alpha_n + \frac{2\xi_n}{2-\beta_n}\|e_n''-p_0\|^2 + 2\gamma_n\langle\Sigma_{i=1}^{\infty}a_if_i(p_0)-p_0, J(x_{n+1}-p_0)\rangle + 2(1-\gamma_n)\zeta_n\mu_nK_2^2),$$

and

$$\varepsilon_n^{(3)} = \frac{2\beta_n(1-\gamma_n)^2}{(2-\beta_n)(1-\gamma_n\sum_{i=1}^{\infty}a_ik_i)}\sum_{i=1}^{\infty}\omega_i(1-c_{n,i})\varphi(\|\frac{y_n+z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|).$$

Then

$$\|x_{n+1}-p_0\|^2 \leqslant (1-\varepsilon_n^{(1)})\|x_n-p_0\|^2 + \varepsilon_n^{(1)}\varepsilon_n^{(2)} - \varepsilon_n^{(3)}. \tag{2.12}$$

Our next discussion will be divided into two cases:

Case 1. $\{\|x_n-p_0\|\}$ is decreasing.

If $\{\|x_n-p_0\|\}$ is decreasing, we know from the result of Step 4, (2.12) and assumptions (iv) and (v) that

$$0 \leqslant \varepsilon_n^{(3)} \leqslant \varepsilon_n^{(1)}(\varepsilon_n^{(2)} - \|x_n-p_0\|^2) + (\|x_n-p_0\|^2 - \|x_{n+1}-p_0\|^2) \to 0,$$

which ensures that $\sum_{i=1}^{\infty}\omega_i\varphi(\|\frac{y_n+z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\|) \to 0$, as $n \to +\infty$. Then from the property of φ, we know that $\sum_{i=1}^{\infty}\omega_i\|\frac{y_n+z_n}{2} - J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2})\| \to 0$, as $n \to +\infty$.

Now, our purpose is to show that $\limsup_{n\to\infty}\varepsilon_n^{(2)} \leqslant 0$, which reduces to show that

$$\limsup_{n\to\infty}\langle\Sigma_{i=1}^{\infty}a_if_i(p_0)-p_0, J(x_{n+1}-p_0)\rangle \leqslant 0.$$

Since

$$\begin{aligned}\|y_n-z_n\| &\leqslant \beta_n\sum_{i=1}^{\infty}\omega_i\|[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i](\frac{y_n+z_n}{2}) - y_n\| + \xi_n\|e_n''-y_n\| \\ &\leqslant \beta_n\sum_{i=1}^{\infty}\omega_i\|J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2}) - \frac{y_n+z_n}{2}\| \\ &\quad + \beta_n\|\frac{y_n+z_n}{2} - y_n\| + \beta_n\sum_{i=1}^{\infty}\omega_ic_{n,i}\|S_i(\frac{y_n+z_n}{2}) \\ &\quad - y_n\| + \xi_n\|e_n''-y_n\|,\end{aligned}$$

then

$$\begin{aligned}\|y_n-z_n\| &\leqslant \frac{2\beta_n}{2-\beta_n}\sum_{i=1}^{\infty}\omega_i\|J_{r_{n,i}}^{A_i}(\frac{y_n+z_n}{2}) - \frac{y_n+z_n}{2}\| \\ &\quad + \frac{2\beta_n}{2-\beta_n}\sum_{i=1}^{\infty}\omega_ic_{n,i}\|S_i(\frac{y_n+z_n}{2}) - y_n\| + \frac{2\xi_n}{2-\beta_n}\|e_n''-y_n\| \to 0,\end{aligned}$$

as $n \to +\infty$.

Let x_t^n be the same as that in Step 3. Since $\|x_t^n\| \leqslant \|x_t^n - p_0\| + \|p_0\|$, then $\{x_t^n\}$ is bounded, as $t \to 0$. Using Lemma 1.4, we have

$$\begin{aligned}\|x_t^n - y_n\|^2 = \|x_t^n &- \sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_Cy_n \\ &+ \sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_Cy_n - y_n\|^2\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|x_t^n - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C y_n\|^2 \\
&\quad + 2\langle \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]y_n - y_n, J(x_t^n - y_n)\rangle \\
&= \|t\sum_{i=1}^{\infty} a_i f_i(x_t^n) + (1-t)(I-\theta_t\sum_{i=1}^{\infty} b_i B_i)(\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n) \\
&\quad - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C y_n\|^2 \\
&\quad + 2\langle \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]y_n - y_n, J(x_t^n - y_n)\rangle \\
&\leqslant \|x_t^n - y_n\|^2 + 2t\langle \sum_{i=1}^{\infty} a_i f_i(x_t^n) - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n \\
&\quad - \frac{\theta_t}{t}(1-t)\sum_{i=1}^{\infty} b_i B_i \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} \\
&\quad + c_{n,i}S_i]Q_C x_t^n, J(x_t^n - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]y_n)\rangle \\
&\quad + 2\langle \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]y_n - y_n, J(x_t^n - y_n)\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&t\langle \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n \\
&\quad - \sum_{i=1}^{\infty} a_i f_i(x_t^n) + (1-t)\frac{\theta_t}{t}\sum_{i=1}^{\infty} b_i B_i \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n, \\
&\quad J(x_t^n - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]y_n)\rangle \leqslant \|\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]y_n - y_n\|\|x_t^n - y_n\|.
\end{aligned}
$$

So,

$$
\begin{aligned}
\lim_{t\to 0}\limsup_{n\to+\infty} &\langle \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n \\
&- \sum_{i=1}^{\infty} a_i f_i(x_t^n) + (1-t)\frac{\theta_t}{t}\sum_{i=1}^{\infty} b_i B_i \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n, \\
&J(x_t^n - \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]y_n)\rangle \leqslant 0.
\end{aligned}
$$

Since $x_t^n \to p_0$, then

$$
\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C x_t^n \to \sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}S_i]Q_C p_0 = p_0,
$$

as $t \to 0$.

Noticing that

$$\begin{aligned}
&\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0),J(p_0-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\rangle\\
&\quad=\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0),J(p_0-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\\
&\qquad-J(x_t^n-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\rangle\\
&\qquad+\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0),J(x_t^n-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\rangle\\
&\quad=\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0),J(p_0-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\\
&\qquad-J(x_t^n-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\rangle\\
&\qquad+\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0)-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_Cx_t^n+\sum_{i=1}^{\infty}a_if_i(x_t^n)\\
&\qquad+\frac{\theta_t}{t}(1-t)\sum_{i=1}^{\infty}b_iB_i\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}\\
&\qquad+c_{n,i}S_i]Q_Cx_t^n,J(x_t^n-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\rangle\\
&\qquad+\langle\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_Cx_t^n-\sum_{i=1}^{\infty}a_if_i(x_t^n)\\
&\qquad-\frac{\theta_t}{t}(1-t)\sum_{i=1}^{\infty}b_iB_i\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]Q_Cx_t^n,\\
&\qquad J(x_t^n-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\rangle,
\end{aligned}$$

then we have $\limsup_{n\to+\infty}\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0),J(p_0-\sum_{i=1}^{\infty}\omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i}+c_{n,i}S_i]y_n)\rangle\leqslant 0$.

Since $y_n-z_n\to 0$, $x_{n+1}-\sum_{i=1}^{\infty}\omega_iJ_{r_{n,i}}^{A_i}z_n\to 0$ and $\lim_{n\to\infty}\sum_{i=1}^{\infty}c_{n,i}=0$, then

$$\limsup_{n\to+\infty}\langle p_0-\sum_{i=1}^{\infty}a_if_i(p_0),J(p_0-x_{n+1})\rangle\leqslant 0.$$

Thus $\limsup_{n\to\infty}\varepsilon_n^{(2)}\leqslant 0$.

Employing (2.12) again, we have

$$\|x_n-p_0\|^2\leqslant\frac{\|x_n-p_0\|^2-\|x_{n+1}-p_0\|^2}{\varepsilon_n^{(1)}}+\varepsilon_n^{(2)}.$$

Assumption (iv) implies that $\liminf_{n\to\infty}\frac{\|x_n-p_0\|^2-\|x_{n+1}-p_0\|^2}{\varepsilon_n^{(1)}}=0$. Then

$$\lim_{n\to\infty}\|x_n-p_0\|^2\leqslant\liminf_{n\to\infty}\frac{\|x_n-p_0\|^2-\|x_{n+1}-p_0\|^2}{\varepsilon_n^{(1)}}+\limsup_{n\to\infty}\varepsilon_n^{(2)}\leqslant 0.$$

Then the result that $x_n \to p_0$ follows.

Case 2. If $\{\|x_n - p_0\|\}$ is not eventually decreasing, then we can find a subsequence $\{\|x_{n_k} - p_0\|\}$ so that $\|x_{n_k} - p_0\| \leqslant \|x_{n_{k+1}} - p_0\|$ for all $k \geqslant 1$. From Lemma 1.6, we can define a subsequence $\{\|x_{\tau(n)} - p_0\|\}$ so that $\max\{\|x_{\tau(n)} - p_0\|, \|x_n - p_0\|\} \leqslant \|x_{\tau(n)+1} - p_0\|$ for all $n > n_1$. This enables us to deduce that (similar to Case 1)

$$0 \leqslant \varepsilon^{(3)}_{\tau(n)} \leqslant \varepsilon^{(1)}_{\tau(n)}(\varepsilon^{(2)}_{\tau(n)} - \|x_{\tau(n)} - p_0\|^2) + (\|x_{\tau(n)} - p_0\|^2 - \|x_{\tau(n)+1} - p_0\|^2) \to 0,$$

and then copy Case 1, we have $\lim_{n\to\infty} \|x_{\tau(n)} - p_0\| = 0$. Thus $0 \leqslant \|x_n - p_0\| \leqslant \|x_{\tau(n)+1} - p_0\| \to 0$, as $n \to \infty$.

This completes the proof. □

Corollary 2.3. *If in Theorem 2.2, $e'_n \equiv 0$ and $e''_n \equiv 0$, then iterative scheme (2.7) becomes to the accurate one:*

$$\begin{cases} x_1 \in C, \\ y_n = Q_C[(1-\alpha_n)x_n], \\ z_n = \delta_n y_n + \beta_n \displaystyle\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}} + c_{n,i}S_i](\frac{y_n + z_n}{2}), \\ x_{n+1} = \gamma_n \displaystyle\sum_{i=1}^{\infty} a_i f_i(x_n) + (1-\gamma_n)(I - \zeta_n \mu_n \sum_{i=1}^{\infty} b_i B_i) \sum_{i=1}^{\infty} \omega_i J^{A_i}_{r_{n,i}} z_n, \quad n \in N. \end{cases}$$

Corollary 2.4. *If in Corollary 2.3, $\zeta_n = 0$ or $\mu_n = 0$, then it becomes to the case without perturbed operators:*

$$\begin{cases} x_1 \in C, \\ y_n = Q_C[(1-\alpha_n)x_n], \\ z_n = \delta_n y_n + \beta_n \displaystyle\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J^{A_i}_{r_{n,i}} + c_{n,i}S_i](\frac{y_n + z_n}{2}), \\ x_{n+1} = \gamma_n \displaystyle\sum_{i=1}^{\infty} a_i f_i(x_n) + (1-\gamma_n) \sum_{i=1}^{\infty} \omega_i J^{A_i}_{r_{n,i}} z_n, \quad n \in N. \end{cases}$$

Theorem 2.5. *Set $w_{n+1} := \frac{\sum_{i=1}^{n+1} d_i x_i}{\sum_{i=1}^{n+1} d_i}$, where x_{n+1} is defined by (2.7), for $n \in N$. Suppose $\sum_{i=1}^{n} d_i \to \infty$, as $n \to +\infty$, then under the assumptions of Theorem 2.2, we can obtain the ergodic convergence in the sense that $w_n \to p_0$ which solves the variational inequality (2.1).*

Proof. The proof is similar to the proof of Step 5 in [16, Theorem 2.2]. □

Lemma 2.6 ([18])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1-b_n)a_n + b_n c_n, \quad n \in N,$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} b_n = +\infty$;

(ii) *either* $\limsup_{n\to\infty} c_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |b_n c_n| = +\infty$.

Then $\lim_{n\to\infty} a_n$.

Lemma 2.7 ([3])**.** *For $\lambda, \mu > 0$, there holds the following identity:*

$$J^A_\lambda x = J^A_\mu(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J^A_\lambda x), \quad x \in E,$$

where $A : E \to E$ is m-accretive.

Lemma 2.8 ([19]). *Assume* $c_2 \geqslant c_1 > 0$. *Then*

$$\|J_{c_1}^A x - x\| \leqslant 2\|J_{c_2}^A x - x\|, \quad x \in E,$$

where $A : E \to E$ *is m-accretive.*

Remark 2.9. Our differences from the main references are:

(i) the normalized duality mapping $J : E \to E^*$ is no longer required to be weakly sequentially continuous at zero as that in [16];

(ii) the parameter $\{r_{n,i}\}$ in the resolvent $J_{r_{n,i}}^{A_i}$ does not need satisfying the condition "$\sum_{n=1}^{\infty} |r_{n+1,i} - r_{n,i}| < +\infty$ and $r_{n,i} \geqslant \varepsilon > 0$ for $i \in N$ and some $\varepsilon > 0$" as that in [16] or [5];

(iii) Lemmas 2.6, 2.7 and 2.8 (see above) and Lemma 1.2 are the main tools to prove the strong convergence of the iterative sequence in [16] or [5]. However, Lemmas 1.4, 1.5 and 1.6 are main tools in our paper. The proof techniques are different, which lead to different restrictions on the parameters.

Remark 2.10. Theorem 2.2 is reasonable, if we suppose $E = C = (-\infty, +\infty)$ and take $\alpha_n = \delta_n = \xi_n = e'_n = e''_n = \frac{1}{n^2}$, $\beta_n = 1 - \frac{2}{n^2}$, $\gamma_n = \mu_n = \zeta_n = \frac{1}{n}$, $a_i = b_i = \tau_i = \omega_i = k_i = \frac{1}{2^i}$, $c_{n,i} = \frac{1}{2^{n+i}}$, $\tau_i = \frac{1}{2^{i+2}}$, $\lambda_i = \frac{2^{i+1} - \frac{3}{2} + \frac{1}{2^{i+1}}}{2^{i+1} - 1}$, $r_{n,i} = (2^{n+i-1} - 1)2^i$, $f_i(x) = \frac{x}{4^i}$, $A_i x = S_i x = \frac{x}{2^i}$, $B_i x = \frac{1}{2^{i+1}} x$, for $n \in N$ and $i \in N$.

Remark 2.11. Choosing Remark 2.10 as the example of Theorem 2.2, we know that $\bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap Fix(S_i)) = \{0\}$. By using codes of Visual Basic Six, we get Table 1 and Figures 1 and 2 below. From Table 1 and Figure 1 we can see the convergence of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$, and, from Figure 2, we can see the convergence of $\{x_n\}$ under different initial values x_1 varying from $[-4, 4]$.

Table 1: Numerical results of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ with initial $x_1 = -1.0$.

n	y_n	z_n	x_n
1	0.00000000	0.96551726	−1.0000000
2	0.08035714	0.09386720	−0.1428571
3	0.09635855	0.02828697	−0.0027077
4	0.05991887	0.00923581	0.00141346
5	0.03872265	0.00365402	0.00033609
6	0.02707432	0.00169872	0.00007010
7	0.02000768	0.00088913	0.00001635
8	0.01538508	0.00050917	0.00000429
9	0.01219448	0.00031270	0.00000123
10	0.00990038	0.00020295	0.00000038
11	0.00819628	0.00013767	0.00000012
12	0.00689626	0.00009680	0.00000004

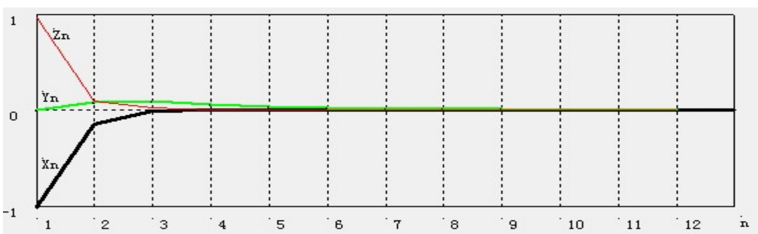

Figure 1: Convergence of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$.

Figure 2: Convergence of $\{x_n\}$ with different initial values.

3. One kind parabolic systems

Our example of parabolic systems are based on the parabolic equation discussed in [15].

In this section, unless otherwise stated, we shall assume that $N \geqslant 1, \frac{2N}{N+1} < p_i < +\infty, 1 \leqslant r_i \leqslant \min\{p_i, p_i'\}, \frac{1}{p_i} + \frac{1}{p_i'} = 1$, for $i \in N$ and assume that Green's Formula is available.

Now, we examine the following nonlinear parabolic systems:

$$\begin{cases} \dfrac{\partial u^{(i)}(x,t)}{\partial t} - div[\alpha(|\nabla u^{(i)}|^{p_i})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)}] + \varepsilon_i |u^{(i)}|^{r_i-2}u^{(i)} \\ \qquad + g(x, u^{(i)}, \dfrac{\partial u^{(i)}}{\partial t}, \nabla u^{(i)}) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ -< \vartheta, \alpha(|\nabla u^{(i)}|^{p_i})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)} > \in \beta_x(u^{(i)}(x,t)), \quad (x,t) \in \Gamma \times (0,T) \\ u^{(i)}(x,0) = u^{(i)}(x,T), \quad x \in \Omega, \quad i \in N, \end{cases} \tag{3.1}$$

where $\alpha : R^+ \cup \{0\} \to R^+$ is a continuous nonlinear mapping such that $p_i t\alpha'(t) + (p_i - 1)\alpha(t) > 0$, $\alpha(t) \leqslant k_1$, $\lim_{t \to +\infty} \alpha(t) = k_2 > 0$, where k_1 and k_2 are positive constants, for $i \in N$.

In (3.1), Ω is a bounded conical domain of a Euclidean space $R^N (N \geqslant 1)$, Γ is the boundary of Ω with $\Gamma \in C^1$ [15] and ϑ denotes the exterior normal derivative to Γ. $< \cdot, \cdot >$ and $|\cdot|$ denote the Euclidean inner-product and Euclidean norm in R^N, respectively. T is a positive constant. $\nabla u^{(i)} = (\frac{\partial u^{(i)}}{\partial x_1}, \frac{\partial u^{(i)}}{\partial x_2}, \cdots, \frac{\partial u^{(i)}}{\partial x_N})$ and $x = (x_1, x_2, \cdots, x_N) \in \Omega$. ε_i is nonnegative constant, for each $i \in N$.

Let $\varphi : \Gamma \times R \to R$ be a given function such that for each $x \in \Gamma$, $\varphi_x = \varphi(x, \cdot) : R \to R$ is a proper, convex and lower-semicontinuous [3] function with $\varphi_x(0) = 0$. Let β_x be the subdifferential [3] of φ_x, i.e., $\beta_x \equiv \partial\varphi_x$. Suppose that $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Suppose that $g : \Omega \times R^{N+2} \to R$ is a given function satisfying the following conditions, which can be seen in [15]:

(a) Carathéodory's conditions.

$$x \to g(x, r) \text{ is measurable on } \Omega, \quad \forall r \in R^{N+2},$$

$$r \to g(x, r) \text{ is continuous on } R^{N+2}, \quad \text{for almost all } x \in \Omega.$$

(b) Growth condition.

$$g(x, s_1, \cdots, s_{N+2}) \leqslant h(x) + k_3 |s_1|^{\min\{p_i/p_i', 1\}},$$

where $(s_1, s_2, \cdots, s_{N+2}) \in R^{N+2}$, $h(x) \in L^2(\Omega) \cap L^{p_i'}(\Omega)$ and k_3 is a positive constant for $i \in N$.

(c) Monotone condition.
g is monotone with respect to r_1, i.e.,

$$(g(x, s_1, \cdots, s_{N+2}) - g(x, t_1, \cdots, t_{N+2}))(s_1 - t_1) \geqslant 0,$$

for all $x \in \Omega$ and $(s_1, \cdots, s_{N+2}), (t_1, \cdots, t_{N+2}) \in R^{N+2}$.

(d) Coercive condition.
$$g(x, s_1, \cdots, s_{N+2})s_1 \geqslant k_4 s_1^2,$$
where k_4 is a fixed positive constant.

Imitating [15], we have the following definitions or results:

Definition 3.1. An operator $B : E \to 2^{E^*}$ is called monotone, if
$$\langle u_1 - u_2, v_1 - v_2 \rangle \geqslant 0, \quad \forall u_i \in D(B), \quad v_i \in Bu_i, \quad i = 1, 2.$$
The monotone operator B is said to be maximal monotone if $R(B + \lambda J) = E^*$, for all $\lambda > 0$.

Lemma 3.2 ([15])**.** *The mapping* $B_i : L^{p_i}(0, T; W^{1,p_i}(\Omega)) \to L^{p_i'}(0, T; (W^{1,p_i}(\Omega))^*)$ *defined by*
$$\langle w, B_i u\rangle = \int_0^T \int_\Omega \langle \alpha(|\nabla u|^{p_i})|\nabla u|^{p_i - 2}\nabla u, \nabla w\rangle dxdt + \varepsilon_i \int_0^T \int_\Omega |u|^{r_i - 2} uwdxdt,$$
for any $u, w \in L^{p_i}(0, T; W^{1,p_i}(\Omega))$, *where* $i \in N$, *is maximal monotone.*

Lemma 3.3 ([15])**.** *Define the function* $\Phi_i : L^{p_i}(0, T; W^{1,p_i}(\Omega)) \to R$ *by*
$$\Phi_i(u) = \int_0^T \int_\Gamma \varphi_x(u|_\Gamma(x, t))d\Gamma(x)dt,$$
for $u(x, t) \in L^{p_i}(0, T; W^{1,p_i}(\Omega))$.

Then the subdifferential of Φ_i, $\partial\Phi_i$, *is maximal monotone, for* $i \in N$.

Lemma 3.4 ([15])**.** *The mapping*
$$S_i : D(S_i) = \{u(x, t) \in L^{p_i}(0, T; W^{1,p_i}(\Omega)) : \frac{\partial u}{\partial t} \in L^{p_i'}(0, T; (W^{1,p_i}(\Omega))^*),$$
$$u(x, 0) = u(x, T)\} \to L^{p_i'}(0, T; (W^{1,p_i}(\Omega))^*),$$
defined by:
$$S_i u = \frac{\partial u}{\partial t}, \quad i \in N,$$
is linear maximal monotone.

Definition 3.5 ([15])**.** For $i \in N$, define a mapping $A_i : L^2(0, T; L^2(\Omega)) \to L^2(0, T; L^2(\Omega))$ as follows:
$$D(A_i) = \{u \in L^2(0, T; L^2(\Omega))| \text{ there exists an } f \in L^2(0, T; L^2(\Omega)) \text{ such that } f \in B_i u + \partial\Phi_i(u) + S_i u\}.$$
For $u \in D(A_i)$, set $A_i u = \{f \in L^2(0, T; L^2(\Omega)) \mid f \in B_i u + \partial\Phi_i(u) + S_i u\}$.

Theorem 3.6 ([15])**.** *The mapping* $A_i : L^2(0, T; L^2(\Omega)) \to L^2(0, T; L^2(\Omega))$ *is m-accretive, for* $i \in N$.

Definition 3.7 ([15])**.** Define the mapping $F_i : L^{p_i}(0, T; W^{1,p_i}(\Omega)) \to L^{p_i'}(0, T; (W^{1,p_i}(\Omega))^*)$ by
$$\langle v, F_i u\rangle = \int_0^T \int_\Omega g(x, u, \frac{\partial u}{\partial t}, \nabla u)v(x, t)dxdt,$$
for $u(x, t)$, $v(x, t) \in L^{p_i}(0, T; W^{1,p_i}(\Omega))$, where $i \in N$.

Definition 3.8 ([15])**.** Define the mapping $H_i : D(H_i) = \{u(x, t) \in L^2(0, T; L^2(\Omega)) \mid$ there exists $v(x, t) \in L^2(0, T; L^2(\Omega))$ such that $v(x, t) = F_i u(x, t)\} \to L^2(0, T; L^2(\Omega))$ by
$$H_i u(x, t) = \{v(x, t) \in L^2(0, T; L^2(\Omega)) \mid v(x) = F_i u(x, t)\},$$
for $u \in D(H_i)$, where F_i is the same as that in Definition 3.7, for $i \in N$.

Theorem 3.9 ([15]). *The mapping* $H_i : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *is bounded, coercive, hemi-continuous and accretive.*

If, further suppose that $g : \Omega \times R^{N+2} \to R$ satisfies that

$$|g(x,s_1',s_2',\cdots,s_{N+2}') - g(x,s_1'',s_2'',\cdots,s_{N+2}'')| \leqslant |s_1' - s_1''|,$$

where $(s_1',s_2',\cdots,s_{N+2}'), (s_1'',s_2'',\cdots,s_{N+2}'') \in R^{N+2}$, then H_i is nonexpansive, for $i \in N$.

Theorem 3.10 ([15]). *For* $f(x,t) \in L^2(0,T;L^2(\Omega))$, *nonlinear parabolic systems* (3.1) *have a unique solution* $u^{(i)}(x,t) \in L^2(0,T;L^2(\Omega))$, *for* $i \in N$.

Theorem 3.11. *If* $g(x,r_1,\cdots,r_{N+1}) \equiv r_1$, $\varepsilon_i \equiv 0$ *and* $f(x,t) \equiv$ *Constant, for* $(x,t) \in \Omega \times (0,T)$, *then*

(i) $u(x,t) \equiv$ *Constant, for* $(x,t) \in \Omega \times (0,T)$, *satisfies parabolic systems* (3.1);

(ii) $\{u(x,t) \in L^2(0,T;L^2(\Omega)) | \, u(x,t) \equiv$ *Constant, for* $(x,t) \in \Omega \times (0,T)\} = \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap \mathrm{Fix}(H_i))$.

Proof.

(i) It is easy to check that $u(x,t) \equiv$ Constant, for $(x,t) \in \Omega \times (0,T)$, satisfies (3.1) in view of Theorem 3.10.

(ii) The result $\{u(x,t) \in L^2(0,T;L^2(\Omega)) | u(x,t) \equiv$ Constant, for $(x,t) \in \Omega \times (0,T)\} \subset \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap \mathrm{Fix}(H_i))$ follows from the definitions of A_i and H_i.

For $u(x,t) \in \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap \mathrm{Fix}(H_i))$, we know that $A_i u = 0$, which implies that $B_i u + \partial\Phi_i(u) + S_i u = 0$, for $i \in N$.

Since $0 = B_i 0 + \partial\Phi_i(0) + S_i 0$, then from the monotonicity of B_i, $\partial\Phi_i$ and S_i, we have

$$0 = \langle B_i u + \partial\Phi_i(u) + S_i u, u\rangle = \langle B_i u, u\rangle + \langle \partial\Phi_i(u), u\rangle + \langle S_i u, u\rangle \geqslant 0.$$

Then $\langle B_i u, u\rangle = 0$ and $\langle S_i u, u\rangle = 0$. From the definitions of B_i and S_i, we know that $u(x,t) =$ Constant, for $(x,t) \in \Omega \times (0,T)$. Thus $\bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap \mathrm{Fix}(H_i)) \subset \{u(x,t) \in L^2(0,T;L^2(\Omega)) | u(x,t) \equiv \text{Constant}\}$.

This completes the proof. □

In view of Theorem 2.2 and Theorem 3.11, we have the following result:

Theorem 3.12. *Suppose* A_i *and* H_i *are the same as those in Definition* 3.5 *and Definition* 3.8, *respectively. Let* X *be the nonempty closed convex sunny nonexpansive retract of* $\bigcap_{i=1}^{\infty}(D(A_i) \bigcap D(H_i))$. *Let*

$$f_i : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$$

be contractive mappings with coefficient $k_i \in (0,1)$, $B_i : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *be* λ_i*-strictly pseudo-contractive mappings and* τ_i*-strongly accretive mappings with* $\lambda_i + \tau_i > 1$. *Suppose* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\xi_n\}$, $\{\gamma_n\}$, $\{\zeta_n\}$, $\{\mu_n\}$, $\{a_i\}$, $\{b_i\}$, $\{\tau_i\}$, $\{\lambda_i\}$, $\{\omega_i\}$, $\{c_{n,i}\}$, $\{r_{n,i}\}$, $\{e_n'\}$ *and* $\{e_n''\}$ *satisfy the same restrictions as those in Theorem* 2.2.

Let $\{u_n\}$ *be generated by the following iterative scheme:*

$$\begin{cases} u_0(x,t) \in X \subset L^2(0,T;L^2(\Omega)), \\ v_n(x,t) = Q_X[(1-\alpha_n)(u_n(x,t) + e_n')], \\ w_n(x,t) = \delta_n v_n(x,t) + \beta_n \displaystyle\sum_{i=1}^{\infty} \omega_i[(1-c_{n,i})J_{r_{n,i}}^{A_i} + c_{n,i}H_i](\frac{v_n + w_n}{2}) + \xi_n e_n'', \\ u_{n+1}(x,t) = \gamma_n \displaystyle\sum_{i=1}^{\infty} a_i f_i(u_n) + (1-\gamma_n)(I - \zeta_n\mu_n \sum_{i=1}^{\infty} b_i B_i) \sum_{i=1}^{\infty} \omega_i J_{r_{n,i}}^{A_i} w_n(x,t), \quad n \in N. \end{cases}$$

Then, under the special case of $g(x, r_1, \cdots, r_{N+1}) \equiv r_1$, $\varepsilon_i \equiv 0$ *and* $f(x,t) \equiv$ *Constant, the iterative sequence* $\{u_n(x,t)\}$ *converges strongly to* $\widetilde{u}(x,t) \equiv$ *Constant* $\in \bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap \mathrm{Fix}(H_i))$, *which is the solution of the parabolic systems* (3.1) *and which also satisfies the following variational inequality:*

$$\langle (I - \sum_{i=1}^{\infty} a_i f_i)\widetilde{u}(x,t), \widetilde{u}(x,t) - y\rangle \leqslant 0,$$

for all y *in* $\bigcap_{i=1}^{\infty}(A_i^{-1}0 \bigcap \mathrm{Fix}(H_i))$.

Remark 3.13. From the above discussion, we can see the connection among parabolic systems, variational inequalities and iterative schemes. This may emphasize the significance of the work in this paper.

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