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Positive and negative solutions of impulsive functional differential equations

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Abstract

This paper considers the global existence of positive and negative solutions for impulsive functional differential equations (IFDEs). First, we introduce the concept of ε-unstability to IFDEs and establish some sufficient conditions to guarantee the ε-unstability via Lyapunov-Razumikhin method. Based on the obtained results, we present some sufficient conditions for the global existence of positive and negative solutions of IFDEs. An example is also given to demonstrate the effectiveness of the results.

Keywords: Impulsive functional differential equations (IFDEs), global existence, Lyapunov-Razumikhin method, positive solution, negative solution.
2010 MSC: 34K20, 34K45.

1. Introduction

It is known that the theory of impulsive differential equations has become an important area of investigation and attracted many researchers' attention in recent years stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. Many classical problems have been extended to impulsive systems. We refer the reader to some papers and books by Bainov and Simeonov [2, 3], Lakshmikantham et al. [6–8], Gopalsamy and Zhang [5], and [1, 4, 12] among others.

On the other hand, the method of Lyapunov-Razumikhin functions has been widely applied to dynamical analysis of various delay differential equations, especially in the area of stability for IFDEs, see [4, 9–16] and the references therein. The idea of this method originated with Lyapunov (1892) for the ordinary differential equations and was developed by Razumikhin (1956) to delay differential equations. A manifest advantage of this method is that it can exhibit the dynamics of systems and does not require the knowledge of solutions of systems. Since that, one may naturally ask whether it can be applied to the investigation of existence problems of positive and negative solutions? In other words, can we establish some Lyapunov-Razumikhin type conditions to guarantee the existence of positive and negative

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for IFDEs? However, to the best of author's knowledge, so far there is almost no result of Lyapunov-Razumikhin type on the existence of positive and negative solutions for IFDEs and the aim of this paper is to close this gap.

In this paper, we shall develop the Lyapunov-Razumikhin method to study the existence problems of positive and negative solutions for IFDEs. In order to do this, we first introduce the concept of ε-unstability to IFDEs and establish some sufficient conditions to guarantee the ε-unstability via Lyapunov-Razumikhin method. Based on the obtained results, we present some sufficient conditions for the global existence of positive and negative solutions of IFDEs. An example is given to demonstrate the effectiveness of the results.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, $\mathbb{Z}_+$ the set of positive integers, and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $|\bullet|$. $\mathbb{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+)|\ a(0) = 0$ and $a(s) > 0$ for $s > 0$, a is strictly increasing in s and tends to infinite as s tends to infinite$\}$. $C(S, V) = \{\varphi : S \to V$ is continuous$\}$ and $PC(S, V) = \{\varphi : S \to V$ is continuous everywhere except at finite number of points t, at which $\varphi(t^+)$, $\varphi(t)$ exist and $\varphi(t^+) = \varphi(t)\}$. In particular, let $PC_r = PC([-r, 0], \mathbb{R})$. For $\varphi \in PC_r$, the norm of φ is defined by $\|\varphi\|_r = \max_{-r\leqslant\theta\leqslant0} |\varphi(\theta)|$.

Consider the following IFDEs:

$$\begin{cases} x'(t) = f(t, x_t), & t \in [t_{k-1}, t_k), \\ \Delta x|_{t=t_k} = x(t_k) - x(t_k^-) = I_k(t_k, x(t_k^-)), & k \in \mathbb{Z}_+, \\ x_{t_0} = \phi(s), & -r \leqslant s \leqslant 0, \end{cases} \tag{2.1}$$

where $\phi \in PC_r$, the impulse times t_k satisfy $0 \leqslant t_0 < t_1 < \ldots < t_k \to \infty$ as $k \to \infty$ and x' denotes the right-hand derivative of x. For each $t \geqslant t_0$, $x_t, x_{t^-} \in PC_r$ are defined by $x_t(s) = x(t + s)$, $x_{t^-}(s) = x(t^- + s), s \in [-r, 0]$.

In this paper, we make the following assumptions:

- (H_1) $f : [t_{k-1}, t_k) \times PC_r \to \mathbb{R}, k \in \mathbb{Z}_+$ is continuous and $f(t, 0) = 0$. For any $\varphi \in PC_r, k \in \mathbb{Z}_+$, the limit $\lim_{(t,\theta)\to(t_k^-,\varphi)} f(t, \theta) = f(t_k^-, \varphi)$ exists.
- (H_2) $f(t, \varphi)$ is Lipschitzian in φ in each compact set in PC_r.
- (H_3) $I_k(t, x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, k \in \mathbb{Z}_+$ is continuous and $I_k(t, 0) = 0$. For any $\rho > 0$, there exists a $\rho_1 \in (0, \rho)$ such that $x \in S(\rho_1)$ implies that $x + I_k(t_k, x) \in S(\rho)$, where $S(\rho) = \{x : |x| < \rho, x \in \mathbb{R}\}$.
- (H_4) For $\varphi \in PC_r, \|\phi\|_{r0} \doteq \min_{-r\leqslant\theta\leqslant0} |\varphi(\theta)| > 0$ holds.
- (H_5) For any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $x[x + I_k(t, x)] \geqslant 0$ holds for all $k \in \mathbb{Z}_+$.

Remark 2.1. Under the assumptions (H_1)-(H_3), the initial value problem (2.1) exists with a unique solution which can be written in the form $x(t, t_0, \phi)$, see [4, 13] for details. Assumptions (H_4) and (H_5) are given for later use.

Definition 2.2. The function $V : [-r, \infty) \times PC_r \to \mathbb{R}_+$ belongs to class ν_0 if

- (H_1) V is continuous on each of the sets $[t_{k-1}, t_k) \times PC_r$ and $\lim_{(t,\varphi_1)\to(t_k^-,\varphi_2)} V(t, \varphi_1) = V(t_k^-, \varphi_2)$ exists;
- (H_2) $V(t, \varphi)$ is locally Lipschitzian in φ and $V(t, 0) \equiv 0$.

Definition 2.3. Let $V \in \nu_0$, for any $(t, \psi) \in [t_{k-1}, t_k) \times PC_r$, the upper right-hand Dini derivative of V along the solution of system (2.1) is defined by

$$D^+V(t, \psi(0)) = \limsup_{h\to0^+} \frac{1}{h}\{V(t + h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))\}.$$

Definition 2.4. Assume that $x(t) = x(t, t_0, \phi)$ is the solution of system (2.1) through (t_0, ϕ). Then the trivial solution of system (2.1) is said to be

1. ε-unstable, if for any $\varepsilon > 0$ and $t_0 \geqslant 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|\phi\|_{r0} \geqslant \delta$ implies $|x(t)| \geqslant \varepsilon, t \geqslant t_0$;
2. uniformly ε-unstable, if δ is independent of t_0.

3. ε-unstability results

In this section, we shall establish some sufficient conditions to guarantee the ε-unstability of the trivial solution of system (2.1) by using Lyapunov-Razumikhin method and some analysis techniques.

Theorem 3.1. *Assume that* (H_1)-(H_4) *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}, c \in C(\mathbb{R}_+, \mathbb{R}_+), p \in PC(\mathbb{R}_+, \mathbb{R}_+), V \in \nu_0$, *and constants* $q > 1, \sigma > 0,\ \beta_k \in [0, 1),\ k \in \mathbb{Z}_+$ *such that*

(i) $w_1(|x|) \leqslant V(t, x) \leqslant w_2(|x|),\ (t, x) \in [t_0 - r, \infty) \times \mathbb{R}$;

(ii) $D^+V(t, \psi(0)) \geqslant -p(t)c(V(t, \psi(0)))$, *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$ *whenever* $qV(t+\theta, \psi(\theta)) \geqslant V(t, \psi(0))$, *for* $\theta \in [-r, 0]$ *and* $\psi \in PC_r$;

(iii) $V(t_k, \psi(0) + I_k(t_k, \psi)) \geqslant q(1 - \beta_k)V(t_k^-, \psi(0))$ *for all* $(t_k, \psi) \in \mathbb{R}_+ \times PC_r$, *and* $\inf_{m\in\mathbb{Z}_+} \prod_{k=1}^{m}(1 - \beta_k) > 0$;

(iv)
$$\inf_{s>0} \int_s^{qs} \frac{du}{c(u)} - \int_{t_{k-1}}^{t_k} p(s)ds > 0 \text{ for all } k \in \mathbb{Z}_+.$$

Then the trivial solution of system (2.1) *is uniformly* ε*-unstable.*

Proof. Let $x(t) = x(t, t_0, \phi)$ be the solution of system (2.1) through (t_0, ϕ). For any $\varepsilon > 0$, choose $\beta = \beta(\varepsilon) > 0$ and $\delta = \delta(\varepsilon) > 0$ such that

$$w_2(\varepsilon) \leqslant \sigma w_1(\beta) \leqslant w_1(\beta) < qw_1(\beta) \leqslant w_1(\delta), \tag{3.1}$$

where

$$\sigma \doteq \inf_{m\in\mathbb{Z}_+} \prod_{k=1}^{m}(1 - \beta_k) > 0.$$

For the sake of brevity, let $V(t) = V(t, x(t))$. Next we show that for any $\phi \in PC_r$, $\|\phi\|_{r0} \geqslant \delta$ implies

$$V(t) \geqslant \prod_{k=0}^{m-1}(1 - \beta_k)w_1(\beta),\ t \in [t_{m-1}, t_m),\ m \in \mathbb{Z}_+,$$

where $\beta_0 = 0$. First, we show that $V(t) \geqslant w_1(\beta), t \in [t_0, t_1)$. Suppose not, then there exists some $t \in [t_0, t_1)$ such that $V(t) < w_1(\beta)$. Let $\bar{t} = \inf\{t \in [t_0, t_1),\ V(t) < w_1(\beta)\}$. It follows from (3.1) that $V(t_0) \geqslant w_1(\delta) > w_1(\beta)$. Then it is obvious that $\bar{t} > t_0,\ V(\bar{t}) = w_1(\beta)$, and $V(t) \geqslant w_1(\beta), t \in [t_0, \bar{t}]$. In view of $\|\phi\|_{r0} \geqslant \delta$, we get

$$V(t) \geqslant w_1(\beta), t \in [t_0 - r, \bar{t}]. \tag{3.2}$$

Since $V(t_0) \geqslant qw_1(\beta)$, we further define $\underline{t} = \sup\{t \in [t_0, \bar{t}],\ V(t) \geqslant qw_1(\beta)\}$. Obviously, $\underline{t} < \bar{t},\ V(\underline{t}) = qw_1(\beta)$. Together with (3.2), we get $w_1(\beta) \leqslant V(t) \leqslant qw_1(\beta), t \in [\underline{t}, \bar{t}]$. Thus it can be deduced that $qV(t+\theta) \geqslant qw_1(\beta) \geqslant V(t), \theta \in [-r, 0],\ t \in [\underline{t}, \bar{t}]$, which implies that $D^+V(t, \psi(0)) \geqslant -p(t)c(V(t, \psi(0)))$ for $t \in [\underline{t}, \bar{t}]$. Hence, we get

$$\inf_{s>0} \int_s^{qs} \frac{du}{c(u)} \leqslant \int_{w_1(\beta)}^{qw_1(\beta)} \frac{du}{c(u)} = \int_{V(\bar{t})}^{V(\underline{t})} \frac{du}{c(u)} \leqslant \int_{\underline{t}}^{\bar{t}} p(u)du \leqslant \int_{t_0}^{t_1} p(u)du,$$

which is a contradiction with condition (iv). Thus we obtain $V(t) \geqslant w_1(\beta),\ t \in [t_0, t_1)$. Also, we have $V(t_1) \geqslant q(1 - \beta_1)V(t_1^-) \geqslant q(1 - \beta_1)w_1(\beta)$.

Now we suppose that

$$\left.\begin{aligned} V(t) &\geqslant \prod_{k=0}^{m-1}(1-\beta_k)w_1(\beta),\ t\in[t_{m-1},t_m),\\ V(t_m) &\geqslant q\prod_{k=0}^{m}(1-\beta_k)w_1(\beta), \end{aligned}\right\} \text{for } 1\leqslant m\leqslant N,\ N\in\mathbb{Z}_+. \tag{3.3}$$

Next we show that

$$V(t)\geqslant \prod_{k=0}^{N}(1-\beta_k)w_1(\beta),\ t\in[t_N,t_{N+1}). \tag{3.4}$$

For the sake of brevity, we define

$$\mathbb{B}=\prod_{k=0}^{N-1}(1-\beta_k)w_1(\beta).$$

It follows from (3.3) that

$$\begin{cases} V(t)\geqslant \mathbb{B},\ t\in[t_0-r,t_N),\\ V(t_N)\geqslant q(1-\beta_N)\mathbb{B}. \end{cases} \tag{3.5}$$

Then we only need prove that $V(t)\geqslant(1-\beta_N)\mathbb{B},\ t\in[t_N,t_{N+1})$. Suppose not, then there exists some $t\in[t_N,t_{N+1})$ such that $V(t)<(1-\beta_N)\mathbb{B}$. Let $t^\star=\inf\{t\in[t_N,t_{N+1}),\ V(t)<(1-\beta_N)\mathbb{B}\}$, then $t^\star>t_N$, $V(t^\star)=(1-\beta_N)\mathbb{B}$ and $V(t)\geqslant(1-\beta_N)\mathbb{B},\ t\in[t_N,t^\star]$, which, together with (3.5), yields

$$V(t)\geqslant(1-\beta_N)\mathbb{B},\ t\in[t_0-r,t^\star]. \tag{3.6}$$

Note that $V(t_N)\geqslant q(1-\beta_N)\mathbb{B}$, we further define $t^*=\sup\{t\in[t_N,t^\star],\ V(t)\geqslant q(1-\beta_N)\mathbb{B}\}$, then $t^*<t^\star$, $V(t^*)=q(1-\beta_N)\mathbb{B}$ and $(1-\beta_N)\mathbb{B}\leqslant V(t)\leqslant q(1-\beta_N)\mathbb{B},\ t\in[t^*,t^\star]$. It follows from (3.6) and the above inequality that $qV(t+\theta)\geqslant q(1-\beta_N)\mathbb{B}\geqslant V(t), \theta\in[-r,0],\ t\in[t^*,t^\star]$, which implies that $D^+V(t,\psi(0))\geqslant -p(t)c(V(t,\psi(0)))$ for $t\in[t^*,t^\star]$. Hence, we get

$$\inf_{s>0}\int_s^{qs}\frac{du}{c(u)}\leqslant\int_{(1-\beta_N)\mathbb{B}}^{q(1-\beta_N)\mathbb{B}}\frac{du}{c(u)}=\int_{V(t^\star)}^{V(t^*)}\frac{du}{c(u)}\leqslant\int_{t^*}^{t^\star}p(u)du\leqslant\int_{t_N}^{t_{N+1}}p(u)du,$$

which is a contradiction and we have proven (3.4) holds.

By the method of induction, in general, we get

$$V(t)\geqslant\prod_{k=0}^{m}(1-\beta_k)w_1(\beta),\ t\in[t_m,t_{m+1}),m\in\mathbb{Z}_+,$$

which implies that

$$w_2(|x(t)|)\geqslant V(t)\geqslant\prod_{k=0}^{m}(1-\beta_k)w_1(\beta)\geqslant\sigma w_1(\beta)\geqslant w_2(\varepsilon),\ t\geqslant t_0,$$

i.e.,

$$|x(t)|\geqslant\varepsilon,\quad t\geqslant t_0.$$

Thus the trivial solution of system (2.1) is uniformly ε-unstable. The proof of Theorem 3.1 is therefore complete. □

Corollary 3.2. *Assume that* (H_1)-(H_4) *hold. If there exist some functions* $w_1,w_2\in\mathbb{K}, V\in\nu_0$, *and some constants* $p>0, q>1, \sigma>0,\ \beta_k\in[0,1),\ k\in\mathbb{Z}_+$ *such that*

(i) $w_1(|x|) \leqslant V(t,x) \leqslant w_2(|x|)$, $(t,x) \in [t_0 - r, \infty) \times \mathbb{R}$;
(ii) $D^+V(t,\psi(0)) \geqslant -pV(t,\psi(0))$, *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$ *whenever* $qV(t+\theta,\psi(\theta)) \geqslant V(t,\psi(0))$, *for* $\theta \in [-r, 0]$ *and* $\psi \in PC_r$;
(iii) $V(t_k, \psi(0) + I_k(t_k,\psi)) \geqslant q(1-\beta_k)V(t_k^-, \psi(0))$ *for all* $(t_k,\psi) \in \mathbb{R}_+ \times PC_r$, *and* $\inf_{m\in\mathbb{Z}_+} \prod_{k=1}^m (1-\beta_k) > 0$;
(iv) $t_k - t_{k-1} < \frac{\ln q}{p}$, *for all* $k \in \mathbb{Z}_+$.

Then the trivial solution of system (2.1) *is uniformly* ε*-unstable.*

Remark 3.3. Theorem 3.1 presents some sufficient conditions from the view of impulsive control to ensure the uniform ε-unstability. In fact, the ε-unstability can also be derived from the view of impulsive perturbation. Next we shall give the main result and its proof is similar to Theorem 3.1 and thus omitted here.

Theorem 3.4. *Assume that* (H_1)-(H_4) *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}, V \in \nu_0$, *and some constants* $\sigma > 0$, $\beta_k \in [0,1)$, $k \in \mathbb{Z}_+$ *such that*

(i) $w_1(|x|) \leqslant V(t,x) \leqslant w_2(|x|)$, $(t,x) \in [t_0 - r, \infty) \times \mathbb{R}$;
(ii) $D^+V(t,\psi(0)) \geqslant 0$, *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$ *whenever* $V(t+\theta,\psi(\theta)) \geqslant V(t,\psi(0))$, *for* $\theta \in [-r,0]$ *and* $\psi \in PC_r$;
(iii) $V(t_k, \psi(0) + I_k(t_k,\psi)) \geqslant (1-\beta_k)V(t_k^-, \psi(0))$ *for all* $(t_k,\psi) \in \mathbb{R}_+ \times PC_r$, *and* $\inf_{m\in\mathbb{Z}_+} \prod_{k=1}^m (1-\beta_k) > 0$.

Then the trivial solution of system (2.1) *is uniformly* ε*-unstable.*

From Theorem 3.4, we can obtain the uniform ε-unstability result for system (2.1) without impulsive effects.

Corollary 3.5. *Assume that* (H_1)-(H_4) *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}$ *such that*

(i) $w_1(|x|) \leqslant V(t,x) \leqslant w_2(|x|)$, $(t,x) \in [t_0 - r, \infty) \times \mathbb{R}$;
(ii) $D^+V(t,\psi(0)) \geqslant 0$ *for all* $t \geqslant t_0$, *whenever* $V(t+\theta,\psi(\theta)) \geqslant V(t,\psi(0))$ *for* $\theta \in [-r,0]$.

Then the trivial solution of system (2.1) *without impulsive effects is uniformly* ε*-unstable.*

4. Global existence of positive and negative solutions

Based on the obtained results in Section 3, we next study the global existence of positive and negative solutions for system (2.1).

Theorem 4.1. *Assume that* (H_1)-(H_5) *hold. If there exist some functions* $w_1, w_2 \in \mathbb{K}, c \in C(\mathbb{R}_+, \mathbb{R}_+), p \in PC(\mathbb{R}_+, \mathbb{R}_+), V \in \nu_0$, *and constants* $q > 1, \sigma > 0$, $\beta_k \in [0,1)$, $k \in \mathbb{Z}_+$ *such that*

(i) $w_1(|x|) \leqslant V(t,x) \leqslant w_2(|x|)$, $(t,x) \in [t_0 - r, \infty) \times \mathbb{R}$;
(ii) $D^+V(t,x(t)) \geqslant -p(t)c(V(t,x(t)))$ *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$, *whenever* $qV(t+\theta, x(t+\theta)) \geqslant V(t,x(t))$ *for* $\theta \in [-r,0]$;
(iii) $V(t_k, x(t_k^-) + I_k(t_k, x(t_k^-))) \geqslant q(1-\beta_k)V(t_k^-, x(t_k^-))$, *and* $\inf_{m\in\mathbb{Z}_+} \prod_{k=1}^m (1-\beta_k) > 0$;
(iv)
$$\inf_{s>0} \int_s^{qs} \frac{du}{c(u)} - \int_{t_{k-1}}^{t_k} p(s)ds > 0 \text{ for all } k \in \mathbb{Z}_+,$$

where $x(t) = x(t, t_0, \phi)$ *is a solution of system* (2.1) *with* $\|\phi\|_{r0} > 0$ *and* $\phi(0) > 0$. *Then* $x(t)$ *is a global positive solution of system* (2.1).

Proof. From Theorem 3.1, we know that the trivial solution of system (2.1) is uniformly ε-unstable. So for any $\varepsilon > 0$, one may choose $\delta = w_1^{-1}\left(\frac{q}{\sigma} w_2(\varepsilon)\right)$ such that $\|\phi\|_{r0} \geqslant \delta$ implies $|x(t)| \geqslant \varepsilon, t \geqslant t_0$, where

$\sigma \doteq \inf_{m\in\mathbb{Z}_+} \prod_{k=1}^{m}(1-\beta_k) > 0$. Note that $w_1, w_2 \in \mathbb{K}$ and

$$\lim_{s\to 0} w_1^{-1}\left(\frac{q}{\sigma}w_2(s)\right) = 0.$$

Thus we can analyze it from another point of view. Since $x(t) = x(t, t_0, \phi)$ is a solution of system (2.1) with $\|\phi\|_{r0} > 0$, we define

$$\delta_\phi = \|\phi\|_{r0} \text{ and } \varepsilon_\phi = w_2^{-1}\left(\frac{\sigma}{q}w_1(\delta_\phi)\right).$$

Obviously, for $\varepsilon_\phi > 0$, we have $|x(t)| \geqslant \varepsilon_\phi$, $t \geqslant t_0$. Then considering the continuity of $x(t)$ on $[t_0, t_1)$ and $\phi(0) > 0$, we get $x(t) > 0, t \in [t_0, t_1)$. From ($H_5$), it is clear that $x(t_1^-) > 0$ implies $x(t_1) > 0$. Similarly, we get $x(t) > 0, t \in [t_1, t_2)$ in view of the continuity of $x(t)$ on $[t_1, t_2)$. In this way, we can deduce that $x(t) > 0, t \geqslant t_0$. Thus the proof is complete. □

Remark 4.2. It should be noted that in the proof of Theorem 4.1, we are interested in the existence of positive constant ε rather than its concrete value. Moreover, one may find that assumption (H_5) plays an important role in guaranteeing the global existence of positive (negative) solution.

Corollary 4.3. *Under the conditions in Theorem 4.1, assume that $x(t) = x(t, t_0, \phi)$ is a solution of system (2.1) with $\|\phi\|_{r0} > 0$ and $\phi(0) < 0$. Then the solution $x(t)$ is a global negative solution of system (2.1).*

Theorem 4.4. *Assume that (H_1)-(H_5) hold. If there exist some functions $w_1, w_2 \in \mathbb{K}, V \in \nu_0$, and constants $\sigma > 0$, $\beta_k \in [0, 1)$, $k \in \mathbb{Z}_+$ such that*

(i) $w_1(|x|) \leqslant V(t, x) \leqslant w_2(|x|)$, $(t, x) \in [t_0, \infty) \times \mathbb{R}$;
(ii) $D^+V(t, x(t)) \geqslant 0$ *for all* $t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+$, *whenever* $V(t+\theta, x(t+\theta)) \geqslant V(t, x(t))$ *for* $\theta \in [-r, 0]$;
(iii) $V(t_k, x(t_k^-) + I_k(t_k, x(t_k^-))) \geqslant (1-\beta_k)V(t_k^-, x(t_k^-))$, *and* $\inf_{m\in\mathbb{Z}_+} \prod_{k=1}^{m}(1-\beta_k) > 0$,

where $x(t) = x(t, t_0, \phi)$ is a solution of system (2.1) with $\|\phi\|_{r0} > 0$ and $\phi(0) > 0$. Then $x(t)$ is a global positive solution of system (2.1).

Corollary 4.5. *Under the conditions in Theorem 4.4, assume that $x(t) = x(t, t_0, \phi)$ is a solution of system (2.1) with $\|\phi\|_{r0} > 0$ and $\phi(0) < 0$. Then the solution $x(t)$ is a global negative solution of system (2.1).*

Example 4.6. Consider the following IFDEs

$$\begin{cases} x'(t) = a(t)x(t) + b(t)\displaystyle\int_{-r}^{0} |x(t+u)|\text{sign}(x(t))du, \ t \geqslant 0, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \ k \in \mathbb{Z}_+, \\ x(s) = \varphi(s), \ s \in [-r, 0], \end{cases} \tag{4.1}$$

where $\varphi \in PC_r, a \in C(\mathbb{R}_+, \mathbb{R}), b \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $r > 0$ is a constant. Here we consider the following two cases:

$$\left.\begin{array}{ll} \textbf{(I)} & I_k(s) = (\lambda - 1 - \lambda\beta_k)s, \lambda > 1, \\ \textbf{(II)} & I_k(s) = -\beta_k s, \end{array}\right\}, \text{ where } \beta_k \in [0, 1), \inf_{m\in\mathbb{Z}_+}\prod_{k=1}^{m}(1-\beta_k) > 0.$$

Property 4.7. Case(I). *Assume that there exist constants $q \in (1, \lambda]$ and $p > 0$ such that*

(P_1) $-a(t) - \dfrac{r}{q}b(t) \leqslant p, \ t \geqslant 0$;
(P_2) $t_k - t_{k-1} < \dfrac{\ln q}{p}, k \in \mathbb{Z}_+$;
(P_3) $\min_{-r\leqslant\theta\leqslant 0} |\varphi(\theta)| > 0$.

Then $x(t) = x(t, 0, \varphi)$ *is a global positive (negative) solution of system* (4.1) *if* $\varphi(0) > 0$ (< 0).

Property 4.8. **Case**(**II**). *Assume that*

(Q_1) $a(t) + rb(t) \geqslant 0,\ t \geqslant 0;$
(Q_2) $\min_{-r\leqslant\theta\leqslant 0} |\varphi(\theta)| > 0.$

Then $x(t) = x(t, \varphi)$ *is a global positive (negative) solution of system* (4.1) *if* $\varphi(0) > 0$ (< 0).

Remark 4.9. Let $V(t) = |x(t)|$, then the results in Properties 4.7 and 4.8 can be easily obtained by Theorems 4.1 and 4.4, respectively. From among, one may observe that Properties 4.7 and 4.8 present the global existence of positive (negative) solutions of system (4.1) from the point of view of the impulsive control and impulsive perturbation, respectively.

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