Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 964–974
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Fixed point theorems for (L)-type mappings in complete CAT(0) spaces

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Communicated by M. Eslamian

Abstract

In this paper, fixed point properties for a class of more generalized nonexpansive mappings called (L)-type mappings are studied in geodesic spaces. Existence of fixed point theorem, demiclosed principle, common fixed point theorem of single-valued and set-valued are obtained in the third section. Moreover, in the last section, Δ-convergence and strong convergence theorems for (L)-type mappings are proved. Our results extend the fixed point results of Suzuki's results in 2008 and Llorens-Fuster's results in 2011.

Keywords: (L)-type mappings, geodesic spaces, fixed point theorems, common fixed point theorems, three-step iteration scheme.

2010 MSC: 47H09, 47H10, 54E40.

1. Introduction

Let D be a nonempty subset of a metric space (X, d). A mapping $T : D \to D$ is said to be

1. nonexpansive if $d(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in D$;
2. quasi-nonexpansive if $d(Tx, p) \leqslant d(x, p)$ for all $x \in D$ and $p \in F(T)$, where $F(T) = \{x \in D : Tx = x\}$ denotes the set of fixed points of T.

We can find in the literature research about more general classes of mappings than the nonexpansive ones and quasi-nonexpansive ones. For instance, in 2008, Suzuki [28] defined a class of generalized nonexpansive mappings, which he called (C)-type mappings, whose set-valued version was defined and studied in [1, 2, 26, 30]. In 2011, García-Falset et al. [14] introduced two classes of single-valued generalized nonexpansive mappings called (C_λ)-type mappings and (E_μ)-type mappings, respectively, which both enlarged the family of (C)-type mappings. Again these new classes were generalized to the set-valued case in [3, 9, 12, 17].

Definition 1.1. Let D be a nonempty subset of a metric space (X, d). A mapping $T : D \to D$ is said to

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doi:10.22436/jnsa.010.03.09

Received 2016-11-28

1. satisfy condition (C), (or be a (C)-type mapping) if

$$\frac{1}{2}d(x,Tx) \leqslant d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leqslant d(x,y), \tag{1.1}$$

   for all $x, y \in D$;
2. satisfy condition (C_λ), (or be a (C_λ)-type mapping) if

$$\lambda d(x,Tx) \leqslant d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leqslant d(x,y), \tag{1.2}$$

   for all $x, y \in D$ and $\lambda \in (0,1)$;
3. satisfy condition (E_μ), (or be a (E_μ)-type mapping) if

$$d(x,Ty) \leqslant \mu d(x,Tx) + d(x,y), \tag{1.3}$$

   for all $x, y \in D$ and $\mu \geqslant 1$.

In 2011 [23], fixed point results for a class of single-valued generalized nonexpansive mappings called (L)-type mappings were proved by Llorens-Fuster and Moreno-Gálvez. This class properly contains Suzuki's (C)-type mappings as (1.1) and several of its generalizations such as (C_λ)-type mappings as (1.2) and (E_μ)-type mappings as (1.3) mentioned before. The set-valued case for (L)-type mappings were discussed in [13] and more results in [24]. Their results closely depend upon geometric characteristics of the Banach space under consideration. In this paper, we shall prove the fixed point property for (L)-type mappings in a metric space without notion of a "topology" and "weak topology".

The aim of this paper is to prove fixed point property for (L)-type mappings in a special kind of metric spaces, namely CAT(0) spaces, which will be defined in the next section. Firstly, we prove the existence theorem of fixed point and demiclosed principle for (L)-type mappings in complete CAT(0) spaces. Furthermore, two common fixed point theorems are also obtained. Finally, we prove that a sequence defined by a three-step iteration Δ-converges (even on some condition strongly converges) to a fixed point of these kind of mappings. Our results extend and improve some results in [23] and [13].

2. Preliminaries

Let (X, d) be a metric space and $x, y \in X$ with $d(x, y) = l$. A geodesic path joining x to y is an isometric map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$. The image of c is called a geodesic (or metric) segment joining x and y denoted by $[x, y]$ whenever it is unique. The space (X, d) is said to be a (uniquely) geodesic space if every two points of X are joined by (exactly) one geodesic segment. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\Delta(x_1, x_2, x_3)$ is the triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean space E^2 such that

$$d(x_i, x_j) = d_{E^2}(\bar{x_i}, \bar{x_j}), \quad \forall i, j = 1, 2, 3.$$

A geodesic space is a CAT(0) space, if for each geodesic triangle $\Delta(x_1, x_2, x_3)$ in X and its comparison triangle $\bar{\Delta} := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in E^2, the CAT(0) inequality

$$d(x, y) \leqslant d_{E^2}(\bar{x}, \bar{y}),$$

holds for all $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \bar{\Delta}$.

A thorough discussions of these spaces are given in [4]. The following lemma plays an important role in our paper.

Lemma 2.1 ([11]). *Let (X, d) be a* CAT(0) *space.*

1. *For each* $x, y \in X$ *and* $\alpha \in [0,1]$, *there exists a unique point* $z \in [x, y]$ *such that*

$$d(z, x) = \alpha d(x, y), \quad d(z, y) = (1-\alpha) d(x, y).$$

Denote $z = (1-\alpha)x \oplus \alpha y$ *in the above equations conveniently.*

2. *For each* $x, y, z \in X$ *and* $\alpha \in [0,1]$, *we have*

$$d((1-\alpha)x \oplus \alpha y, z) \leqslant (1-\alpha) d(x, z) + \alpha d(y, z).$$

3. *For all* $t \in [0,1]$ *and* x, y, $z \in X$,

$$d^2\big((1-t)x \oplus ty, z\big) \leqslant (1-t)d^2(x, z) + t\, d^2(y, z) - t(1-t)d^2(x, y). \tag{2.1}$$

The inequality (2.1) is also called (CN) inequality. A geodesic space X is a CAT(0) space if and only if (CN) inequality holds.

CAT(0) spaces have a remarkably nice geometric structure. One can see almost immediately from Lemma 2.1 that in such spaces angles exist in a strong sense, the distance function is convex, one has both uniform convexity and orthogonal projection onto convex subsets, etc. Also, because of their generality, CAT(0) spaces arise in a wide variety of contexts. Some examples of CAT(0) spaces are pre-Hilbert spaces (see [4]), R-trees (see [18]), Euclidean buildings (see [6]), the complex Hilbert ball with a hyperbolic metric (see [16]), Hadamard manifolds, and many others.

The following lemma is a consequence of [25, Lemma 2.5].

Lemma 2.2. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a* CAT(0) *space* X *and* $r \in [0,1)$. *Suppose that* $x_{n+1} = ry_n \oplus (1-r)x_n$ *and* $d(y_{n+1}, y_n) \leqslant d(x_{n+1}, x_n)$ *for all* $n \in \mathbb{N}$. *Then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Firstly the definition of (L)-type mappings in the single-valued case will be given in a metric space as follows.

Definition 2.3. Let D be a nonempty bounded closed convex subset of a CAT(0) space X. A mapping $T : D \to D$ is said to satisfy condition (L) (or it is an (L)-type mapping) on D provided that it fulfills the following two conditions.

1. If a set $K \subset D$ is nonempty, closed, convex, and T-invariant, (i.e., $T(K) \subset K$), then there exists an a.f.p.s. for T in K (i.e., $d(x_n, Tx_n) \to 0$ for a sequence $\{x_n\}$ in K).
2. For any a.f.p.s. $\{x_n\}$ of T in D and each $x \in D$,

$$\limsup_{n\to\infty} d(x_n, Tx) \leqslant \limsup_{n\to\infty} d(x_n, x).$$

Proposition 2.4. *Let* D *be a nonempty bounded closed convex subset of a* CAT(0) *space* X *and* $T : D \to D$ *be a mapping satisfying condition (L) with a nonempty fixed point set, then* T *is a quasi-nonexpansive mapping.*

Proof. Let $p \in F(T)$. Taking $x_n = p$ for every positive integer n, it is obvious that $\{x_n\}$ is an a.f.p.s. for T. From condition (L), we have for each $x \in D$,

$$d(p, Tx) = \limsup_{n\to\infty} d(x_n, Tx) \leqslant \limsup_{n\to\infty} d(x_n, x) = d(p, x).$$

In other words, T is a quasi-nonexpansive mapping. □

Next, in order to define the set-valued case for (L)-type mappings, we introduce some elementary concepts. Let D be a nonempty subset of a metric space X. We denote by $B(D)$ the collection of all nonempty bounded closed subsets of D and $C(D)$ the collection of all nonempty compact subsets of D. Suppose H is the Hausdorff metric with respect to d, that is,

$$H(U, V) := \max\left\{\sup_{u\in U} \operatorname{dist}(u, V), \sup_{v\in V} \operatorname{dist}(v, U)\right\}, \quad U, V \in B(X),$$

where $\text{dist}(u, V) = \inf_{v \in V} d(u, v)$ is the distance from the point u to the set V.

Let $T : X \to 2^X$ be a set-valued mapping. If an element $x \in X$ satisfies $x \in Tx$, then x is called a fixed point of T. The set of fixed points of T is denoted by $F(T)$. If a sequence $\{x_n\}$ in D satisfies $\text{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$, then $\{x_n\}$ is called an a.f.p.s. for T.

Definition 2.5. Let D be a nonempty bounded closed convex subset of a CAT(0) space X. A set-valued mapping $T : D \to B(D)$ is said to satisfy condition (L), (or it is an (L)-type set-valued mapping), on D provided that it fulfills the following two conditions.

1. If a set $K \subset D$ is nonempty, closed, convex, and T-invariant, then there exists an a.f.p.s. for T in K.
2. For any a.f.p.s. $\{x_n\}$ of T in D and each $x \in D$,

$$\limsup_{n\to\infty} \text{dist}(x_n, Tx) \leqslant \limsup_{n\to\infty} d(x_n, x).$$

Along with Definition 2.3 and the above two lemmas, we can obtain the following propositions which show the inclusion relations between (L)-type mappings and other generalized nonexpansive mappings in CAT(0) spaces.

Proposition 2.6. *Let D be a nonempty, bounded, and convex subset of a CAT(0) space X and $T : D \to D$ be a mapping satisfying condition (C), then T satisfies condition (L).*

Proof. Recall that if $T : D \to D$ is a mapping satisfying condition (C), then there exists an a.f.p.s $\{x_n\}$ for T in D by [25, Lemma 3.6]. Moreover, in view of [25, Lemma 3.5], we have that, for every $x, y \in D$,

$$d(x, Ty) \leqslant 3d(Tx, x) + d(x, y).$$

Hence, for the a.f.p.s. $\{x_n\}$ and each $x \in D$,

$$\limsup_{n\to\infty} d(x_n, Tx) \leqslant \limsup_{n\to\infty}(3d(x_n, Tx_n) + d(x_n, x)) = \limsup_{n\to\infty} d(x_n, x),$$

which means such mappings satisfy condition (L). □

Proposition 2.7. *Let D be a nonempty, bounded, and convex subset of a CAT(0) space X and $T : D \to D$ be a mapping satisfying condition (E_μ) for some $\mu \geqslant 0$, then T satisfies condition (L) provided that it satisfies assumption 1 of Definition 2.3.*

Proof. Replace 3 with μ in the proof of Proposition 2.6. Therefore, the desired conclusion is obtained. □

Proposition 2.8. *Let D be a nonempty, bounded and convex subset of a CAT(0) space X and $T : D \to D$ be a continuous mapping satisfying condition (C_λ) for some $\lambda \in (0, 1)$, then T satisfies condition (L).*

Proof. Define a sequence $\{x_n\}$ in D by taking $x_1 \in D$ and

$$x_{n+1} = rTx_n \oplus (1 - r)x_n,$$

for $n \geqslant 1$ and $r \in [\lambda, 1)$. It follows from Lemma 2.1 (1) that

$$\lambda d(x_n, Tx_n) \leqslant rd(x_n, Tx_n) = d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.$$

By condition (C_λ), we have

$$d(Tx_{n+1}, Tx_n) \leqslant d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.$$

Hence, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ by Lemma 2.2.

Case 1. If for some $x \in D$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging to x. Since $\{x_{n_j}\}$ is an a.f.p.s., then it is obvious that the sequence $\{Tx_{n_j}\}$ has the same limit as $\{x_{n_j}\}$, and therefore by the continuity of T, $x = Tx$. Thus, for $\{x_n\}$ in D and $x \in D$, we have

$$\limsup_{n\to\infty} d(x_n, Tx) \leqslant \limsup_{n\to\infty} d(x_n, x)$$

holds, i.e., T satisfies condition (L).

Case 2. Suppose that for every $x \in D$, the sequence $\{x_n\}$ does not have any subsequence converging to x. Noticing that $\{x_n\}$ is an a.f.p.s., for any $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that $d(x_n, Tx_n) < \varepsilon$ for all $n \geqslant n_0$. Since $\{x_n\}$ does not converge to x, we can put $\varepsilon := \frac{1}{2} \liminf_n d(x_n, x) > 0$. Therefore,

$$\lambda d(x_n, Tx_n) \leqslant d(x_n, Tx_n) < \varepsilon < d(x_n, x).$$

By condition (C_λ), we have

$$d(Tx_n, Tx) \leqslant d(x_n, x),$$

which implies

$$\limsup_{n\to\infty} d(x_n, Tx) \leqslant \limsup_{n\to\infty}(d(x_n, Tx_n) + d(Tx_n, Tx)) \leqslant \limsup_{n\to\infty} d(x_n, x).$$

So T satisfies condition (L). □

We now give the notion of Δ-convergence and collect some of its basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $z \in X$, we set

$$r(z, \{x_n\}) = \limsup_{n\to\infty} d(z, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(z, \{x_n\}) : z \in X\}.$$

The asymptotic radius $r_D(\{x_n\})$ of $\{x_n\}$ with respect to $D \subset X$ is given by

$$r_D(\{x_n\}) = \inf\{r(z, \{x_n\}) : z \in D\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{z \in X : r(z, \{x_n\}) = r(\{x_n\})\}.$$

And the asymptotic center $A_D(\{x_n\})$ of $\{x_n\}$ with respect to $D \subset X$ is the set

$$A_D(\{x_n\}) = \{z \in D : r(z, \{x_n\}) = r(\{x_n\})\}.$$

It follows from [10, Proposition 7]) that $A(\{x_n\})$ consists of exactly one point in a CAT(0) space. In 1976, Lim [21] introduced the concept of Δ-convergence in a general metric space. In 2008, Kirk and Panyanak [19] brought in Δ-convergence to CAT(0) spaces and proved that there is an analogy between Δ-convergence and weak convergence.

Definition 2.9 ([19]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.10 ([19]). *If D is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic centerasymptotic center of $\{x_n\}$ is in D.*

Lemma 2.11 ([19]). *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.12 ([11]). *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{p\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ is convergent, then $p = u$.*

3. Fixed point theorems

Theorem 3.1. *Let* D *be a nonempty bounded closed convex subset of a complete* CAT(0) *space* X. *Suppose* $T : D \to D$ *is a mapping satisfying condition* (L). *Then* T *has a fixed point in* D.

Proof. Since T satisfies condition (L), there exists an a.f.p.s. for T in D, say $\{x_n\}$. By Proposition 7 of [10] we let $A(\{x_n\}) = \{z\}$. It follows from Lemma 2.10 that $z \in D$. By condition (L), we get

$$\limsup_{n\to\infty} d(x_n, Tz) \leqslant \limsup_{n\to\infty} d(x_n, z),$$

which means

$$r(Tz, \{x_n\}) \leqslant r(z, \{x_n\}).$$

By the uniqueness of asymptotic centers, we have $z = Tz$. □

Theorem 3.1 extends [23, Theorem 4.2]. By using this theorem along with Proposition 2.4 and [8, Theorem 1.3], we can obtain the following corollary.

Corollary 3.2. *Let* D *be a nonempty bounded closed convex subset of a complete* CAT(0) *space* X. *Suppose* $T : D \to D$ *is a mapping satisfying condition* (L). *Then* $F(T)$ *is nonempty closed, convex and hence contractible.*

In 1968, Browder proved demiclosedness principle [5] for nonexpansive mappings which has been one of the fundamental and celebrated results in fixed point theory. Demiclosedness principle states that if D is a nonempty closed convex subset of a uniformly convex Banach space X, and $T : D \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at 0, that is, for any sequence $\{x_n\}$ in D, if $\{x_n\}$ weakly converges to x and $(I - T)x_n$ strongly converges to 0, then $x = Tx$ (here I is the identity operator of X into itself). The principle is also valid in a space satisfying Opial's condition. It has been known that the demiclosedness principle plays a key role in studying the asymptotic and ergodic behavior of nonexpansive mapping, see for example [15, 22].

Remark 3.3. Let D be a closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in D. We need the following notation:

$$\{x_n\} \rightharpoonup \omega \quad \text{if and only if} \quad \Phi(\omega) = \inf_{x\in C} \Phi(x),$$

where $\Phi(x) := \limsup_{n\to\infty} d(x_n, x)$.

Theorem 3.4 in the following takes use of the notion defined above to prove demiclosedness principle for (L)-type mappings which extend [23, Theorem 4.6] to CAT(0) spaces.

Theorem 3.4 (Demiclosed principle)**.** *Suppose* D *is a bounded closed convex subset of a complete* CAT(0) *space* X *and* $T : D \to D$ *is a mapping satisfying condition* (L). *If* $\{x_n\} \subset D$ *is an a.f.p.s. for* T *such that* $\{x_n\} \rightharpoonup p$, *then* $Tp = p$.

Proof. By the definition, $\{x_n\} \rightharpoonup p$ if and only if $A_D(\{x_n\}) = \{p\}$. We have $A(\{x_n\}) = \{p\}$ from Lemma 2.10 and Lemma 2.11. Since $\{x_n\}$ is an a.f.p.s. for T, we have

$$\Phi(x) := \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(Tx_n, x). \tag{3.1}$$

Taking $x = Tp$ in (3.1), we have

$$\Phi(Tp) = \limsup_{n\to\infty} d(x_n, Tp) \leqslant \limsup_{n\to\infty} d(x_n, p) = \Phi(p).$$

Furthermore, for any $n \geqslant 1$, it follows from (CN) inequality with $t = \frac{1}{2}$ that

$$d^2\left(x_n, \frac{p \oplus Tp}{2}\right) \leqslant \frac{1}{2} d^2(x_n, p) + \frac{1}{2} d^2(x_n, Tp) - \frac{1}{4} d^2(p, Tp).$$

Letting $n \to \infty$ and taking superior limit on the both sides of the above inequality, we get

$$\Phi\left(\frac{p \oplus Tp}{2}\right) \leqslant \frac{1}{2}\Phi(p) + \frac{1}{2}\Phi(Tp) - \frac{1}{4}d^2(p, Tp).$$

Since $A(\{x_n\}) = \{p\}$, we have

$$\Phi(p) \leqslant \Phi\left(\frac{p \oplus Tp}{2}\right) \leqslant \frac{1}{2}\Phi(p) + \frac{1}{2}\Phi(Tp) - \frac{1}{4}d^2(p, Tp),$$

which implies that

$$d(p, Tp) = 0,$$

i.e., $p = Tp$. □

Lemma 3.5 (cf. [20, 27]). *Let X be a complete CAT(0) space, then the intersection of any decreasing sequence of nonempty bounded closed convex subsets of X is nonempty.*

Together with Theorem 3.1 and Lemma 3.5, we have a common fixed point theorem of a countable family of mappings which satisfy condition (L).

Theorem 3.6. *Let D be a nonempty bounded closed and convex subset of a complete CAT(0) space X. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of commuting mappings on D satisfying condition (L). Then $\{T_i\}_{i=1}^{\infty}$ has a common fixed point.*

Proof. Let $C_n := \bigcap_{i=1}^{n} F(T_i)$ for each n. From Corollary 3.2, $C_1 = F(T_1)$ is nonempty bounded closed and convex subset of X. Now we assume that C_{k-1} is nonempty bounded closed and convex for $k \in \mathbb{N}$. We are going to show that C_k is also nonempty bounded closed and convex. Let $p \in C_{k-1}$ and $i \in \mathbb{N}$ with $1 \leqslant i < k$. Since T_k and T_i commute, we have

$$T_k p = T_k \circ T_i p = T_i \circ T_k p.$$

Thus $T_k p$ is a fixed point of T_i, which implies that $T_k p \in C_{k-1}$. Hence we get $T_k(C_{k-1}) \subset C_{k-1}$. By Theorem 3.1, T_k has a fixed point in C_{k-1}, that is,

$$C_k = C_{k-1} \bigcap F(T_k) \neq \emptyset.$$

Also, it is closed and convex by Corollary 3.2. By induction, C_n is nonempty bounded closed and convex for all $n \in \mathbb{N}$. Since $C_n \subset C_{n-1}$ for all $n \in \mathbb{N}$, by Lemma 3.5 we have

$$\bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$

This completes the proof. □

Theorem 3.7. *Let $t : D \to D$ and $T : D \to C(D)$ be a single-valued mapping and a set-valued mapping, respectively. If both t and T satisfy the condition (L) and in the meantime, they have common a.f.p.s., then they have a common fixed point, that is, there exists a point $z \in D$ such that $z = tz \in Tz$.*

Proof. By Theorem 3.1 and Corollary 3.2, we know that the mapping t has a fixed point set $F(t)$ which is a nonempty closed convex subset of X. Let $p \in F(t)$. Since Tp is a bounded closed convex subset of X, we can obtain that t has a fixed point in Tp for $p \in F(t)$. From the assumption, let $\{u_n\}$ be the common a.f.p.s. and $A(\{u_n\}) = \{z\}$. By the proof of Theorem 3.1, we have that $z \in F(t)$. Since Tz is a compact set, there exists $v_n \in Tz$ such that

$$d(u_n, v_n) = \operatorname{dist}(u_n, Tz).$$

Again from the compactness of Tz, we may assume that $v_n \to z' \in Tz$. Since T satisfies condition (L),

$$\limsup_{n\to\infty} d(u_n, z') \leqslant \limsup_{n\to\infty} d(u_n, v_n) + \limsup_{n\to\infty} d(v_n, z') = \limsup_{n\to\infty} \operatorname{dist}(u_n, Tz) \leqslant \limsup_{n\to\infty} \operatorname{dist}(u_n, z).$$

This implies that

$$r(z', \{u_n\}) \leqslant r(z, \{u_n\}).$$

By the uniqueness of asymptotic centers, we have $z = z' \in Tz$. Hence $z = tz \in Tz$. □

4. Convergence theorems

In this section, we shall prove Δ and strong convergence theorems for (L)-type mappings of a three-step iteration scheme introduced by Thakur et al. in [29] which not only converges faster than the known iterations but also is stable. Give $x_1 \in D$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} x_1 \in D, \\ x_{n+1} = Ty_n, \\ y_n = T((1-\alpha_n)x_n \oplus \alpha_n z_n), \\ z_n = (1-\beta_n)x_n \oplus \beta_n Tx_n, \end{cases} \tag{4.1}$$

for all $n \geqslant 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences with $0 < a \leqslant \alpha_n, \beta_n \leqslant b < 1$.

We now establish the following useful lemma.

Lemma 4.1. *Let D be a nonempty bounded closed convex subset of a complete CAT(0) space X and let $T : D \to D$ be a mapping satisfying condition (L). For arbitrary chosen $x_1 \in D$ and $\{x_n\}$ generated by (4.1), $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

Proof. By Theorem 3.1, $F(T)$ is nonempty. Given $p \in F(T)$, by Lemma 2.1 (2) and Proposition 2.4 we have

$$\begin{aligned} d(z_n, p) &= d((1-\beta_n)x_n \oplus \beta_n Tx_n, p)) \\ &\leqslant (1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, p) \\ &\leqslant (1-\beta_n)d(x_n, p) + \beta_n d(x_n, Tp) \\ &\leqslant (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p), \end{aligned} \tag{4.2}$$

and from (4.2),

$$\begin{aligned} d(y_n, p) &= d(T((1-\alpha_n)x_n \oplus \alpha_n z_n), p) \\ &\leqslant d((1-\alpha_n)x_n \oplus \alpha_n z_n), p) \\ &\leqslant (1-\alpha_n)d(x_n, p) + \alpha_n d(z_n, p) \\ &\leqslant (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{4.3}$$

By (4.3) we can obtain that

$$d(x_{n+1}, p) = d(Ty_n, p) \leqslant d(y_n, p) \leqslant d(x_n, p). \tag{4.4}$$

Thus, $\{d(x_n, p)\}$ is bounded and decreasing for all $p \in F(T)$, i.e., $\lim_{n\to\infty} d(x_n, p)$ exists. □

Lemma 4.2 ([7, Lemma 3.2]). *Let X be a CAT(0) space, $x \in X$ be a given point, and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \leqslant r, \quad \limsup_{n\to\infty} d(y_n, x) \leqslant r, \quad \textit{and} \quad \lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r,$$

for some $r \geqslant 0$. Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Theorem 4.3. *Let D be a nonempty bounded closed convex subset of a complete CAT(0) space X. Suppose $T : D \to D$ is a mapping satisfying condition (L). For arbitrary chosen $x_1 \in D$ and $\{x_n\}$ generated by (4.1), $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. First we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

In fact, it follows from Lemma 4.1 that for each given $p \in F(T)$, $\lim_{n\to\infty} d(x_n, p)$ exists, without loss of generality, let

$$\lim_{n\to\infty} d(x_n, p) = r \geqslant 0. \tag{4.5}$$

By Proposition 2.4, we have

$$\limsup_{n\to\infty} d(Tx_n, p) \leqslant \limsup_{n\to\infty} d(x_n, p) = r. \tag{4.6}$$

Since $\{\alpha_n\}$ is a sequence with $0 < a \leqslant \alpha_n \leqslant b < 1$, we can assume that $\lim_{n\to\infty} \alpha_n = \alpha \in [a, b]$. By using (4.3)-(4.5), we get

$$\begin{aligned} r = \lim_{n\to\infty} d(x_{n+1}, p) &= \lim_{n\to\infty} d(Ty_n, p) \\ &\leqslant \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n z_n), p) \\ &\leqslant \lim_{n\to\infty} (1-\alpha_n) d(x_n, p) + \lim_{n\to\infty} \alpha_n d(z_n, p) \\ &= (1-\alpha)r + \alpha \lim_{n\to\infty} d(z_n, p), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(z_n, p) \geqslant r. \tag{4.7}$$

On the other hand, it follows from (4.2) and (4.5) that

$$\lim_{n\to\infty} d(z_n, p) \leqslant \lim_{n\to\infty} d(x_n, p) = r. \tag{4.8}$$

Hence, together with (4.7) and (4.8), we have

$$r \leqslant \lim_{n\to\infty} d(z_n, p) = \lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n Tx_n, p) \leqslant r,$$

which implies that

$$\lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n Tx_n, p) = r, \tag{4.9}$$

where $0 < a \leqslant \beta_n \leqslant b < 1$. By (4.5), (4.6), (4.9), as well as Lemma 4.2, it gets that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0, \tag{4.10}$$

i.e., $\{x_n\}$ is an a.f.p.s. of T in D.

Now we prove that

$$\omega_w(x_n) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\}) \subset F(T), \tag{4.11}$$

and $\omega_w(x_n)$ consists of exactly one point.

In fact, $u \in \omega_w(x_n)$, then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.10 and Lemma 2.11, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in D$. In view of (4.10) and Theorem 3.4, we have $v \in F(T)$. Furthermore, $u = v$ by Lemma 2.12. This implies that $\omega_w(x_n) \subset F(T)$. Next we claim that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, from Lemma 4.1 we know that $\{d(x_n, u)\}$ is convergent. In view of Lemma 2.12, we have $x = u$.

Finally we prove that $\{x_n\}$ Δ-converges to a fixed point of T.

In fact, by Lemma 4.1 we know that $\{d(x_n, p)\}$ is convergent for each $p \in F(T)$. By (4.10), $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By (4.11), $\omega_w(x_n) \subset F(T)$ and $\omega_w(x_n)$ consists of exactly one point. This shows that $\{x_n\}$ Δ-converges to a point of $F(T)$. This completes the proof. □

Theorem 4.4. *Suppose that $X, T, \{x_n\}$ are as in Theorem 4.3 and D is a nonempty bounded closed convex compact subset of X. Then $\{x_n\}$ strongly converges to a fixed point of T.*

Proof. In view of the proof of Theorem 4.3, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ strongly converges to some $z \in D$. By condition (L), we have

$$\limsup_{k\to\infty} d(x_{n_k}, Tz) \leqslant \limsup_{k\to\infty} d(x_{n_k}, z) \text{ for all } k \in \mathbb{N}.$$

Thus we have $\{x_{n_k}\}$ converges to Tz. This implies $z = Tz$, i.e., $z \in F(T)$. By Lemma 4.1, we have $\lim_{n\to\infty} d(x_n, z)$ exists, thus z is the strong limit of the sequence $\{x_n\}$ itself. □

Acknowledgment

The authors would like to appreciate the anonymous referee for some valuable comments and useful suggestions. Besides, the paper is supported by NFS of HeiLongjiang Provience (A2015018).

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