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Estimates of initial coefficients for certain subclasses of bi-univalent functions involving quasi-subordination

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Abstract

The object of the present paper is to introduce and investigate new subclasses of the function class Σ of bi-univalent functions defined in the open unit disk $\mathcal{U}$, involving quasi subordination. The coefficients estimate $|a_2|$ and $|a_3|$ for functions in these new subclasses are also obtained.

Keywords: Univalent functions, bi-univalent functions, quasi-subordination, subordination.
2010 MSC: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C}; |z| < 1\}$.

Suppose $\mathcal{S}$ denotes the class of all functions of $\mathcal{A}$ which satisfy normalized condition $f(0) = 0$ and $f'(0) = 1$ which are univalent in $\mathcal{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathcal{U}$. In fact, the Koebe-one quarter theorem [1] ensures that the image of $\mathcal{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $1/4$. Thus, every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z,\ (z \in \mathcal{U})$, and

$$f\left(f^{-1}(w)\right) = w \qquad (|w| < r_0(f),\ r_0(f) \geqslant 1/4).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let Σ denote the class of bi-univalent functions defined in $\mathcal{U}$.

An analytic function $f(z)$ is subordinate to an analytic function $g(z)$ if there exists an analytic function $w(z)$ in $\mathcal{U}$ satisfying $w(0) = 0$ and $|w(z)| < 1\,(z \in \mathcal{U})$ satisfying $f(z) = g(w(z))$. We denote this subordination by, cf. [5, p. 226],

$$f(z) \prec g(z) \quad (z \in \mathcal{U})\,.$$

Further, a function $f(z)$ is said to be quasi-subordinate to $g(z)$ in the open unit disk $\mathcal{U}$ if there exists an analytic function $\varphi(z)$ such that $f(z)/\varphi(z)$ is analytic in $\mathcal{U}$,

$$\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in \mathcal{U})$$

and $|\varphi(z)| \leqslant 1\,(z \in \mathcal{U})$. We also denote this quasi-subordination by

$$f(z) \prec_q g(z) \quad (z \in \mathcal{U})\,. \tag{1.2}$$

Note that the quasi-subordination (1.2) is equivalent to

$$f(z) = \varphi(z)g(w(z)) \quad (z \in \mathcal{U})\,, \tag{1.3}$$

where $|\varphi(z)| \leqslant 1\ (z \in \mathcal{U})$.

In the quasi-subordination (1.3), if $\varphi(z) \equiv 1$, then (1.3) becomes the subordination (1.3).

For analytic functions $f(z)$ and $g(z)$ in $\mathcal{U}$, we say $f(z)$ is majorized by $g(z)$ if there exists an analytic function $\varphi(z)$ in $\mathcal{U}$ satisfying $|\varphi(z)| \leqslant 1$ and $f(z) = \varphi(z)g(z)$ $(z \in \mathcal{U})$. See [7, 11, 12] for works related to quasi-subordination.

Lewin [8] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan et al. [1] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [10], on the other hand, showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. Brannan and Taha [2] obtained initial coefficient bounds for certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. Later, Srivastava et al. [13] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Many researchers (see [3, 4, 6, 13, 14]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates of initial coefficient bounds $|a_2|$ and $|a_3|$.

In the present paper, the coefficient bounds of $|a_2|$ and $|a_3|$ for certain classes involving the quasi-subordination are obtained. The subclasses in this paper are motivated essentially by corresponding subclasses investigated in [7].

2. Coefficient estimates

Let us assume that there exists a function $\phi(z)$ analytic in the open unit disk $\mathcal{U}$ and $|\varphi(z)| \leqslant 1$ s.t.

$$\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots \quad (|z| < 1). \tag{2.1}$$

Since $\phi(z)$ is analytic and bounded in $\mathcal{U}$, we have [9, page 172]

$$|A_n| \leqslant 1 - |A_0|^2 \leqslant 1 \quad (n > 0). \tag{2.2}$$

Also, let $h(z)$ be an analytic function with positive real part in $\mathcal{U}$, satisfying $h(0) = 1$, $h'(0) > 0$ and $h(\mathcal{U})$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots, B_1 > 0.$$

Suppose that $u(z)$ and $v(z)$ are analytic in $\mathcal{U}$ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \qquad v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n \quad (|z| < 1)\,.$$

It is well-known that

$$|b_1| \leqslant 1, \quad |b_2| \leqslant 1-|b_1|^2, \quad |c_1| \leqslant 1, \quad |c_2| \leqslant 1-|c_1|^2. \tag{2.3}$$

By a simple calculation, we have

$$h(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \cdots, |z| < 1,$$

and

$$h(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \cdots, |w| < 1.$$

We define a subclass of Σ as follows:

Definition 2.1. A function $f \in \Sigma$ is said to be in the class $H_q(\Sigma, h)$ if and only if:

$$f'(z) - 1 \prec_q h(z) - 1, \qquad g'(w) - 1 \prec_q h(w) - 1,$$

or

$$f'(z) - 1 = \phi(z)\left(h(u(z)) - 1\right), \qquad g'(w) - 1 = \phi(w)\left(h(v(w)) - 1\right),$$

where $|\phi(z)| \leqslant 1$ $(z \in \mathcal{U})$ and $g(w) = f^{-1}(w)$.

On taking $\phi(z) \equiv 1$ in Definition 2.1, we get the following subordination class:

$$f'(z) \prec h(z), \qquad g'(w) \prec h(w).$$

We name this class as $H(\Sigma, h)$.

Now, we first derive following coefficient estimates for the subclass $H_q(\Sigma, h)$:

Theorem 2.2. *If* f *given by* (1.1) *is in the subclass* $H_q^{\Sigma}(h)$, *then*

$$|a_2| \leqslant \min\left\{\frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2|}{3}}\right\}, \quad |a_3| \leqslant \min\left\{\frac{B_1^2}{4} + \frac{2}{3}B_1, \frac{|B_2|}{3} + B_1\right\}.$$

Proof. Let $f \in H_q^{\Sigma}(h)$ and $g = f^{-1}$. Then, there are analytic functions $u, v : \mathcal{U} \to \mathcal{U}$ given by (2.2) and a function $\phi(z)$ in $\mathcal{U}$ defined by (2.1) satisfying

$$f'(z) - 1 = \phi(z)\left(h(u(z) - 1\right), \quad g'(w) - 1 = \phi(w)\left(h(v(w) - 1\right).$$

Since

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots, \quad g'(w) = 1 - 2a_2 w + 3\left(2a_2^2 - a_3\right) w^2 + \cdots,$$

and

$$\phi(z)\left(h(u(z) - 1\right) = A_0 B_1 b_1 z + \left[A_1 B_1 b_1 + A_0\left(B_1 b_2 + B_2 b_1^2\right)\right] z^2 + \cdots, \tag{2.4}$$

with

$$\phi(w)\left(h(v(w) - 1\right) = A_0 B_1 c_1 w + \left[A_1 B_1 c_1 + A_0\left(B_1 c_2 + B_2 c_1^2\right)\right] w^2 + \cdots. \tag{2.5}$$

It follows that

$$2a_2 = A_0 B_1 b_1, \tag{2.6}$$

$$3a_3 = A_1 B_1 b_1 + A_0\left(B_1 b_2 + B_2 b_1^2\right), \tag{2.7}$$

$$-2a_2 = A_0 B_1 c_1, \tag{2.8}$$

$$3\left(2a_2^2 - a_3\right) = A_1 B_1 c_1 + A_0\left(B_1 c_2 + B_2 c_1^2\right). \tag{2.9}$$

From (2.6) and (2.8), we have

$$b_1 = -c_1. \tag{2.10}$$

Squaring and adding (2.6) and (2.8), and then using (2.10), we have

$$4a_2^2 = A_0^2 B_1^2 c_1^2. \tag{2.11}$$

Adding (2.7) and (2.9), we have

$$6a_2^2 = A_0\left[B_1(b_2+c_2)+2B_2c_1^2\right]. \tag{2.12}$$

By using (2.2) and (2.3), we have from (2.11) and (2.12) that

$$|a_2| \leqslant \frac{B_1}{2} \text{ and } |a_2| \leqslant \sqrt{\frac{B_1+|B_2|}{3}}, \tag{2.13}$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.9) from (2.7). Thus, we get

$$6a_3 - 6a_2^2 = A_1B_1(b_1-c_1) + A_0\left[B_1(b_2-c_2)+B_2(b_1^2-c_1^2)\right].$$

Therefore (2.10), (2.11), and (2.13) yield

$$6a_3 = \frac{3}{2}A_0^2B_1^2c_1^2 - 2c_1A_1B_1 + A_0B_1(b_2-c_2).$$

Thus, we find by using (2.2) and (2.3) that

$$|a_3| \leqslant \frac{1}{4}B_1^2 + \frac{2}{3}B_1.$$

Also from (2.10), (2.11), and (2.13), we have

$$6a_3 = A_0\left[B_1(b_2+c_2)+2B_2c_1^2\right] - 2c_1A_1B_1 + A_0B_1(b_2-c_2).$$

Thus, we find by (2.2) and (2.3) that

$$|a_3| \leqslant B_1 + \frac{|B_2|}{3}.$$

This evidently completes the proof of Theorem 2.2. □

Theorem 2.3. *If* f *given by* (1.1) *is in the subclass* $H(\Sigma, h)$ *then*

$$|a_2| \leqslant \min\left\{\frac{B_1}{2}, \sqrt{\frac{B_1+|B_2|}{3}}\right\},$$

and

$$|a_3| \leqslant \min\left\{\frac{B_1^2}{4}+\frac{B_1}{6}, \frac{|B_2|+2B_1}{3}\right\}.$$

Proof. The result is obvious, by taking $\phi(z) \equiv 1 \ (z \in U)$ and using the similar procedure as in Theorem 2.2. □

Definition 2.4. A function $f \in \Sigma$ is said to be in the class $S_q^*(\Sigma, h)$ if and only if:

$$\frac{zf'(z)}{f(z)} - 1 \prec_q h(z) - 1, \qquad \frac{wg'(z)}{g(w)} - 1 \prec_q h(w) - 1,$$

where $|\phi|(z) \leqslant 1$ $(z \in \mathcal{U})$ and $g(w) = f^{-1}(w)$.

On taking $\phi(z) \equiv 1$ in Definition 2.1, we get the following subordination class:

$$\frac{zf'(z)}{f(z)} \prec h(z), \quad \frac{wg'(z)}{g(w)} \prec h(w).$$

We name this class as $S^*(\Sigma, h)$, where $g(w) = f^{-1}(w)$.

Theorem 2.5. *If* f *given by* (1.1) *is in the subclass* $S_q^*(\Sigma, h)$ *then*

$$|a_2| \leqslant \min\left\{B_1, \sqrt{B_1 + |B_2|}\right\} \tag{2.14}$$

and

$$|a_3| \leqslant \min\{B_1(1+B_1), 3B_1\}.$$

Proof. Let $f \in S_q^*(\Sigma, h)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \mathcal{U} \to \mathcal{U}$ given by (2.2) and a function $\phi(z)$ in $\mathcal{U}$ defined by (2.1) satisfying

$$\frac{zf'(z)}{f(z)} - 1 = \phi(z)\left(h(u(z) - 1\right), \quad \frac{wg'(w)}{g(w)} - 1 = \phi(w)\left(h(v(w) - 1\right). \tag{2.15}$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \cdots, \quad \frac{wg'(w)}{g(w)} = 1 - 2a_2 w + \left(3a_2^2 - 2a_3\right) w^2 + \cdots. \tag{2.16}$$

By using (2.4), (2.5), (2.15), and (2.16), we have

$$a_2 = A_0 B_1 b_1, \tag{2.17}$$

$$2a_3 - a_2^2 = A_1 B_1 b_1 + A_0\left(B_1 b_2 + B_2 b_1^2\right), \tag{2.18}$$

$$-a_2 = A_0 B_1 c_1, \tag{2.19}$$

$$3a_2^2 - 2a_3 = A_1 B_1 c_1 + A_0\left(B_1 c_2 + B_2 c_1^2\right). \tag{2.20}$$

From (2.17) and (2.19), we have

$$b_1 = -c_1. \tag{2.21}$$

Squaring and adding (2.17) and (2.19), and then using (2.21), we have

$$a_2^2 = A_0^2 B_1^2 c_1^2. \tag{2.22}$$

Adding (2.18) and (2.20), and then using (2.21), we have

$$2a_2^2 = A_0\left[B_1(b_2 + c_2) + 2B_2 c_1^2\right]. \tag{2.23}$$

By using (2.2) and (2.3), we have from (2.22) and (2.23) that

$$|a_2| \leqslant B_1 \quad \text{and} \quad |a_2| \leqslant \sqrt{B_1 + |B_2|},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.14).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.20) from (2.18). Thus, we get

$$4a_3 - 4a_2^2 = A_1 B_1(b_1 - c_1) + A_0\left[B_1(b_2 - c_2) + B_2(b_1^2 - c_1^2)\right]. \tag{2.24}$$

Therefore, (2.21), (2.22), and (2.24) yield

$$4a_3 = 4A_0^2 B_1^2 c_1^2 - 2c_1 A_1 B_1 + A_0 B_1(b_2 - c_2).$$

Hence, we find by using (2.2) and (2.3) that

$$|a_3| \leqslant B_1(1+B_1).$$

Also from (2.21), (2.22), and (2.24), we have

$$4a_3 = 2A_0\left[B_1(b_2+c_2)+2B_2c_1^2\right] - 2c_1A_1B_1 + A_0B_1(b_2-c_2).$$

Thus, we find by (2.2) and (2.3) that

$$|a_3| \leqslant 3B_1.$$

This evidently completes the proof of Theorem 2.5. □

Theorem 2.6. *If* f *given by* (1.1) *is in the subclass* $S^*(\Sigma, h)$, *then*

$$|a_2| \leqslant \min\left\{B_1, \sqrt{B_1+|B_2|}\right\}$$

and

$$|a_3| \leqslant \min\left\{B_1^2+\frac{B_1}{2}, \frac{3}{2}B_1+|B_2|\right\}.$$

Proof. The result is obvious, by taking $\phi(z) \equiv 1 \;\; (z \in \mathcal{U})$ and using the similar procedure as in Theorem 2.5. □

Definition 2.7. A function $f \in \Sigma$ is said to be in the class $C_q(\Sigma, h)$ if and only if:

$$\frac{zf''(z)}{f'(z)} \prec_q h(z)-1, \qquad \frac{wg''(z)}{g'(w)} \prec_q h(w)-1,$$

where $|\phi(z)| \leqslant 1$ $(z \in \mathcal{U})$ and $g(w) = f^{-1}(w)$.

On taking $\phi(z) \equiv 1$ in Definition 2.1, we get the following subordination class:

$$1+\frac{zf''(z)}{f'(z)} \prec h(z), \qquad 1+\frac{wg''(z)}{g'(w)} \prec h(w).$$

We name this class as $C(\Sigma, h)$.

Theorem 2.8. *If* f *given by* (1.1) *is in the subclass* $C_q(\Sigma, h)$, *then*

$$|a_2| \leqslant \min\left\{\frac{B_1}{2}, \frac{\sqrt{B_1+|B_2|}}{2}\right\}, \tag{2.25}$$

and

$$|a_3| \leqslant \min\left\{\frac{1}{4}B_1^2+\frac{1}{3}B_1, \frac{5}{6}B_1+\frac{1}{2}|B_2|\right\}.$$

Proof. Let $f \in K_q^{\Sigma}(h)$ and $g = f^{-1}$. Then, there are analytic functions $u, v : \mathcal{U} \to \mathcal{U}$ given by (2.2) and a function $\phi(z)$ in $\mathcal{U}$ defined by (2.1) satisfying

$$\frac{zf''(z)}{f'(z)} = \phi(z)\left(h(u(z)-1\right), \quad \frac{wg''(w)}{g'(w)} = \phi(w)\left(h(v(w)-1\right). \tag{2.26}$$

Since

$$\frac{zf''(z)}{f'(z)} = 2a_2z+\left(6a_3-4a_2^2\right)z^2+\cdots, \quad \frac{wg''(w)}{g'(w)} = -2a_2w+\left(8a_2^2-6a_3\right)w^2+\cdots. \tag{2.27}$$

By using (2.4), (2.5), (2.26), and (2.27), we have

$$2a_2 = A_0B_1b_1, \tag{2.28}$$

$$6a_3 - 4a_2^2 = A_1B_1b_1 + A_0\left(B_1b_2 + B_2b_1^2\right), \tag{2.29}$$

$$-2a_2 = A_0B_1c_1, \tag{2.30}$$

$$8a_2^2 - 6a_3 = A_1B_1c_1 + A_0\left(B_1c_2 + B_2c_1^2\right). \tag{2.31}$$

From (2.28) and (2.30), we have

$$b_1 = -c_1. \tag{2.32}$$

Squaring and adding (2.28) and (2.30), and then using (2.32), we have

$$4a_2^2 = A_0^2B_1^2c_1^2. \tag{2.33}$$

Adding (2.29) and (2.31), and then using (2.32), we have

$$4a_2^2 = A_0\left[B_1(b_2 + c_2) + 2B_2c_1^2\right]. \tag{2.34}$$

By using (2.2) and (2.3), we have from (2.33) and (2.34) that

$$|a_2| \leqslant \frac{B_1}{2} \quad \text{and} \quad |a_2| \leqslant \frac{\sqrt{B_1 + |B_2|}}{2},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.25).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.31) from (2.29). Thus, we get

$$12a_3 - 12a_2^2 = A_1B_1(b_1 - c_1) + A_0\left[B_1(b_2 - c_2) + B_2(b_1^2 - c_1^2)\right]. \tag{2.35}$$

Therefore the equations (2.32), (2.33), and (2.34) yield

$$12a_3 = 3A_0^2B_1^2c_1^2 - 2c_1A_1B_1 + A_0B_1(b_2 - c_2).$$

Thus, we find by using (2.2) and (2.3) that

$$|a_3| \leqslant \frac{1}{4}B_1^2 + \frac{1}{3}B_1.$$

Also from (2.32), (2.34), and (2.35), we have

$$12a_3 = 3A_0\left[B_1(b_2 + c_2) + 2B_2c_1^2\right] - 2c_1A_1B_1 + A_0B_1(b_2 - c_2).$$

Hence, we find by (2.2) and (2.3) that

$$|a_3| \leqslant \frac{5}{6}B_1 + \frac{1}{2}|B_2|.$$

This evidently completes the proof of Theorem 2.8. □

Theorem 2.9. *If* f *given by* (1.1) *is in the subclass* $\mathcal{C}(\Sigma, h)$, *then*

$$|a_2| \leqslant \min\left\{\frac{B_1}{2}, \sqrt{\frac{B_1 + |B_2|}{2}}\right\},$$

and

$$|a_3| \leqslant \min\left\{\frac{B_1^2}{4} + \frac{B_1}{6}, \frac{2}{3}B_1 + \frac{|B_2|}{2}\right\}.$$

Proof. The result is obvious, by taking $\phi(z) \equiv 1 \ (z \in \mathcal{U})$ and using the similar procedure as in Theorem 2.8. □

3. Conclusion

In the paper, classes of analytic bi-univalent are introduced with the help of quasi-subordination. Further, coefficient estimates of initial Maclaurin coefficients are also obtained.

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