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F-HR-type contractions on (α, η)-complete rectangular b-metric spaces

Nawab Hussain[a], Vahid Parvaneh[b,*], Badria A. S. Alamri[a], Zoran Kadelburg[c]

[a] Department of Mathematics, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[b] Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran.
[c] University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Beograd, Serbia.

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Abstract

The aim of this paper is to present some fixed point results for generalized Wardowski-type contractions in the framework of (α, η)-complete rectangular b-metric spaces. We also derive certain fixed point results for generalized F-contractions in rectangular b-metric spaces endowed with a graph or a partial order. Moreover, an illustrative example is presented to support the obtained results.

Keywords: b-metric space, rectangular metric space, Hardy-Rogers condition, F-contraction, admissible mappings, property P.
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1. Introduction and preliminaries

The notion of a b-metric as an extension of usual metric has been introduced first by Bakhtin [1] and extensively applied by Czerwik in [5, 6]. Until now, many interesting results about the existence of fixed points for single-valued and multi-valued mappings in b-metric spaces have been obtained (see, e.g., [3, 12, 22] and the papers cited therein).

Definition 1.1 ([5])**.** Let X be a nonempty set and let $s \geqslant 1$ be a given real number. A function $b : X \times X \to [0, +\infty)$ is a b-metric on X, if for all $x, y, z \in X$, the following assertions hold:

(b_1) $b(x, y) = 0$, if and only if $x = y$;

(b_2) $b(x, y) = b(y, x)$;

(b_3) $b(x, z) \leqslant s[b(x, y) + b(y, z)]$.

In this case, the pair (X, b) is called a b-metric space.

In general, a b-metric space might not be a metric space.

*Corresponding author
Email addresses: nhusain@kau.edu.sa (Nawab Hussain), zam.dalahoo@gmail.com (Vahid Parvaneh), baalamri@kau.edu.sa (Badria A. S. Alamri), kadelbur@matf.bg.ac.rs (Zoran Kadelburg)

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The following definition was given by Branciari in [2].

Definition 1.2. Let X be a nonempty set and let $r : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each of them is distinct from x and y:

(r_1) $r(x, y) = 0$, if and only if $x = y$;

(r_2) $r(x, y) = r(y, x)$;

(r_3) $r(x, y) \leqslant r(x, u) + r(u, v) + r(v, y)$ (rectangular inequality).

Then (X, r) is said to be a generalized metric space (g.m.s.) or a rectangular metric space.

Many researchers (see the references cited in [13]) proved various fixed point results in this framework. The concept of generalized metric space is similar to the notion of metric space. However, its topology may not be Hausdorff, as an example given in [18, 20] shows (see further Example 1.5). Hence, it is quite difficult to treat this concept because a generalized metric space does not necessarily have the topology which is compatible with convergence in r, see [21, Example 7]. So, this concept is very interesting for researchers.

Combining conditions used for definitions of b-metric and g.m. spaces, Roshan et al. [17] announced the following notion (see also [8]).

Definition 1.3 ([17])**.** Let X be a nonempty set, $s \geqslant 1$ be a given real number and let $r_b : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each of them distinct from x and y one has the following conditions:

(rb_1) $r_b(x, y) = 0$, iff $x = y$;

(rb_2) $r_b(x, y) = r_b(y, x)$;

(rb_3) $r_b(x, y) \leqslant s[r_b(x, u) + r_b(u, v) + r_b(v, y)]$ (b-rectangular inequality).

Then (X, r_b) is called a rectangular b-metric space (RbMS) or a generalized b-metric space (b-g.m.s).

The following are easy examples of rectangular b-metric spaces [17].

Example 1.4. Let (X, r) be a g.m.s. and $p \geqslant 1$ be a real number. Let $r_b(x, y) = (r(x, y))^p$. Obviously, from the convexity of the function $f(x) = x^p$ for $x \geqslant 0$ and by Jensen inequality we have

$$(a + b + c)^p \leqslant 3^{p-1}(a^p + b^p + c^p),$$

for nonnegative real numbers a, b, c. So, it is easy to obtain that (X, r_b) is a rectangular b-metric space, where as a parameter any $s \geqslant 3^{p-1}$ can be taken.

Example 1.5. Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$. Define $r : X \times X \to [0, +\infty)$ as follows:

$$r(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y \text{ and } \{x, y\} \subset A \text{ or } \{x, y\} \subset B, \\ y, & x \in A,\ y \in B, \\ x, & x \in B,\ y \in A. \end{cases}$$

Then (X, r) is a complete g.m.s. [18, 20]. Now, taking $r_b(x, y) = r(x, y)^2$, according to Example 1.4, we obtain a rectangular b-metric space (X, r_b) with $s = 3$. It can be shown that:

1. the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ converges to both 0 and 2;

2. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $1 = \lim_{n\to\infty} r_b(\frac{1}{n}, \frac{1}{2}) \neq r_b(0, \frac{1}{2}) = \frac{1}{4}$; hence r_b is not a continuous function.

We will also need the following simple lemma about the convergent sequences in the proof of our main results.

Lemma 1.6 ([17]). *Let (X, r_b) be a rectangular b-metric space.*

(a) *Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have*

$$\frac{1}{s} r_b(x, y) \leqslant \liminf_{n\to\infty} r_b(x_n, y_n) \leqslant \limsup_{n\to\infty} r_b(x_n, y_n) \leqslant s r_b(x, y).$$

(b) *If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for infinitely many $m, n \in \mathbb{N}$, $n \neq m$, converging to $x \neq y$, then*

$$\frac{1}{s} r_b(x, y) \leqslant \liminf_{n\to\infty} r_b(x_n, y) \leqslant \limsup_{n\to\infty} r_b(x_n, y) \leqslant s r_b(x, y),$$

for all $x \in X$.

Using certain mappings $F : \mathbb{R}^+ \to \mathbb{R}$, Wardowski introduced in [24] a new type of contractions called F-contractions and proved a new fixed point theorem for such mappings. Substituting appropriate mappings for F, one can obtain several known contractions from the literature, including the Banach contraction (see, e.g., [4, 23]). Hussain and Salimi [9] generalized the concept of F-contraction and proved certain fixed and common fixed point results.

In this paper, using the idea introduced by Wardowski, we prove some fixed point results for generalized F-contractive mappings in the setup of rectangular b-metric spaces. An example is presented to support our main results. Fixed point results in spaces endowed with a graph or with a partial order are presented at the end as applications of our obtained results.

2. Results

2.1. Basic notions

We begin with some basic definitions and results which will be applied in the sequel.

As in [24], let Δ_F be the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ verifying the following assertions:

(F_1) F is a continuous and strictly increasing mapping;

(F_2) $\lim_{n\to\infty} t_n = 0$, if and only if $\lim_{n\to\infty} F(t_n) = -\infty$ for all sequence $\{t_n\} \subseteq R^+$.

Example 2.1. The following are some examples of functions belonging to Δ_F: $F_1(t) = \ln t$, $F_2(t) = -\frac{1}{t^p}$, where $p > 0$, $F_3(t) = t - \frac{1}{t}$, $F_4(t) = \frac{1}{e^{-t} - e^t}$, $F_5(t) = \frac{1}{1-e^t}$.

Now we introduce the following new family of functions.

Let Δ_β be the set of all functions $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following condition:

(β_1) $\liminf_{i\to\infty} \beta(t_i) > 0$ for all real sequences (t_i) with $t_i > 0$;

Note that (β_1) implies that:

(β_2) $\sum_{i=0}^{\infty} \beta(t_i) = +\infty$ for each sequence (t_i) with $t_i > 0$.

Definition 2.2. Let (X, r_b) be a rectangular b-metric space with parameter s and let T be a self-mapping on X. Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ are two functions. We say that T is an (α, η)-β-F-contraction, if for all $x, y \in X$ with ($\alpha(x, y) \geqslant 1$ or $\eta(x, y) \leqslant 1$) and $r_b(Tx, Ty) > 0$ we have

$$\beta\big(r_b(x, y)\big) + F\big(s^2 \cdot r_b(Tx, Ty)\big) \leqslant F\big(\alpha_1 r_b(x, y) + \alpha_2 r_b(x, Tx) + \alpha_3 r_b(y, Ty) + \alpha_4 r_b(y, Tx)\big), \tag{2.1}$$

where $\beta \in \Delta_\beta$, $F \in \Delta_F$, $\alpha_i \geqslant 0$ for $i \in \{1, 2, 3, 4\}$, $\sum_{i=1}^{4} \alpha_i = 1$ and $\alpha_3 < \frac{1}{s}$.

In [19], Samet et al. presented the concepts of α-ψ-contractive and α-admissible mappings and obtained various fixed point results for such mappings defined on complete metric spaces. On the other hand, Hussain et al. [7], as well as La Rosa and Vetro [15], extended the notions of α-ψ-contractive and α-admissible mappings and stated some interesting results. Also, Hussain et al. [7] introduced a weaker notion than the concept of completeness and called it α-completeness for a metric space. Motivated by these works, we present the following concepts.

Definition 2.3 ([7])**.** Let T be a self-mapping on X and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an (α, η)-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geqslant 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geqslant 1,$$

and

$$x, y \in X, \quad \eta(x, y) \leqslant 1 \quad \Longrightarrow \quad \eta(Tx, Ty) \leqslant 1.$$

One can easily see that an (α, η)-admissible mapping is an α-admissible and η-sub-admissible mapping simultaneously.

Definition 2.4 ([7])**.** Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$. We say that T is a triangular (α, η)-admissible mapping if

(T1) $\alpha(x, y) \geqslant 1 \Longrightarrow \alpha(Tx, Ty) \geqslant 1$, $x, y \in X$;

(T2) $\eta(x, y) \leqslant 1 \Longrightarrow \eta(Tx, Ty) \leqslant 1$, $x, y \in X$;

(T3) $\begin{cases} \alpha(x, z) \geqslant 1 \\ \alpha(z, y) \geqslant 1 \end{cases} \Longrightarrow \alpha(x, y) \geqslant 1$, for all $x, y, z \in X$;

(T4) $\begin{cases} \eta(x, z) \leqslant 1 \\ \eta(z, y) \leqslant 1 \end{cases} \Longrightarrow \eta(x, y) \leqslant 1$, for all $x, y, z \in X$.

Definition 2.5. Let (X, d) be a metric space or a rectangular b-metric space and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two mappings. The space X is said to be:

a. α-complete, if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, converges in X.

b. η-sub-complete, if every Cauchy sequence $\{x_n\}$ in X with $\eta(x_n, x_{n+1}) \leqslant 1$ for all $n \in \mathbb{N}$, converges in X.

c. (α, η)-complete, if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ or $\eta(x_n, x_{n+1}) \leqslant 1$ for all $n \in \mathbb{N}$, converges in X.

Remark 2.6. If X is a complete space, then X is also an (α, η)-complete space (α-complete space and η-sub-complete space). But, the converse is not true.

Definition 2.7 ([7])**.** Let (X, d) be a metric space or a rectangular b-metric space and let $\alpha, \eta : X \times X \to [0, +\infty)$ and $T : X \to X$ be mappings. We say that:

a. T is an α-continuous mapping on (X, d), if for given point $x \in X$ and sequence $\{x_n\} \subseteq X$,

$$x_n \to x \text{ and } \alpha(x_n, x_{n+1}) \geqslant 1, \text{ for all } n \in \mathbb{N}, \text{ imply that } Tx_n \to Tx.$$

b. T is an η-sub-continuous mapping on (X, d), if for given point $x \in X$ and sequence $\{x_n\} \subseteq X$,

$$x_n \to x \text{ and } \eta(x_n, x_{n+1}) \leqslant 1, \text{ for all } n \in \mathbb{N}, \text{ imply that } Tx_n \to Tx.$$

c. T is an (α, η)-continuous mapping on (X, d), if for given point $x \in X$ and sequence $\{x_n\} \subseteq X$,

$$x_n \to x \text{ and } (\alpha(x_n, x_{n+1}) \geqslant 1 \text{ or } \eta(x_n, x_{n+1}) \leqslant 1, \text{ for all } n \in \mathbb{N}) \text{ imply that } Tx_n \to Tx.$$

Example 2.8. Let $X = [0, +\infty)$ be equipped with the usual metric $d(x, y) = |x - y|$. Assume that $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be defined by

$$Tx = \begin{cases} x^2, & \text{if } x \in [0, 1], \\ \arcsin x + \sinh x + 2, & \text{if } (1, +\infty), \end{cases}$$

$$\alpha(x, y) = \begin{cases} x^4 + y^2 + 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta(x, y) = \begin{cases} \frac{1}{1+x^2+y^6}, & \text{if } x, y \in [0, 1], \\ 2, & \text{otherwise}. \end{cases}$$

Clearly, T is not a continuous mapping, but T is (α, η)-continuous (α-continuous and η-sub-continuous) on (X, d).

Definition 2.9. Let (X, d) be a metric space or a rectangular b-metric space and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that

a. (X, d) is α-regular, if $x_n \to x$, where $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, implies $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N}$.

b. (X, d) is η-sub-regular, if $x_n \to x$, where $\eta(x_n, x_{n+1}) \leqslant 1$ for all $n \in \mathbb{N}$, implies $\eta(x_n, x) \leqslant 1$ for all $n \in \mathbb{N}$.

c. (X, d) is (α, η)-regular, if (X, d) is α-regular and (X, d) is η-sub-regular simultaneously.

2.2. *Fixed point results*

Now we state and prove the main results of this section. Note that (X, r_b) will always be a rectangular b-metric space with parameter $s > 1$ and $\mathrm{Fix}(T)$ will denote the set of fixed points of a self-mapping $T : X \to X$.

Theorem 2.10. *Let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions and let the space (X, r_b) be (α, η)-complete rectangular. Let $T : X \to X$ be a self-mapping satisfying the following conditions:*

(i) T *is a triangular (α, η)-admissible mapping;*

(ii) T *is an (α, η)-β-F-contraction;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ or $\eta(x_0, Tx_0) \leqslant 1$;*

(iv) T *is (α, η)-continuous.*

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geqslant 1$ or $\eta(x, y) \leqslant 1$ for all $x, y \in \mathrm{Fix}(T)$.

Proof. Let $x_0 \in X$ be the point given as in (iii). Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$. Since T is a triangular (α, η)-admissible mapping, then $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geqslant 1$ or $\eta(x_1, x_2) = \eta(Tx_0, Tx_1) \leqslant 1$.

Continuing this process we have

$$\alpha(x_{n-1}, x_n) \geqslant 1 \quad \text{or} \quad \eta(x_{n-1}, x_n) \leqslant 1,$$

for all $n \in \mathbb{N}$. By (T3) and (T4), we infer also that

$$\alpha(x_m, x_n) \geqslant 1 \quad \text{or} \quad \eta(x_m, x_n) \leqslant 1, \quad \forall m, n \in \mathbb{N}, \quad m \neq n. \tag{2.2}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$. Then x_{n_0} is a fixed point of T and we have nothing to prove. Hence, we assume that $x_n \neq x_{n+1}$, i.e., $r_b(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$. But then also

$$x_n \neq x_m, \quad \forall m, n \in \mathbb{N}, \quad m \neq n. \tag{2.3}$$

Indeed, suppose that $x_n = x_m$ for some $n = m + k > m$, so we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$. Denote

$$d_n := r_b(x_n, x_{n+1}).$$

Then, (2.1) implies that

$$\begin{aligned} F(d_m) = F(d_n) < F(s^2 d_n) &= F(s^2 r_b(Tx_{n-1}, Tx_n)) \\ &\leqslant F(\alpha_1 d_{n-1} + \alpha_2 d_{n-1} + \alpha_3 d_n + \alpha_4 \cdot 0) - \beta(d_{n-1}) \\ &< F(\alpha_1 d_{n-1} + \alpha_2 d_{n-1} + \alpha_3 d_n). \end{aligned}$$

As F is strictly increasing, so $d_m = d_n < \alpha_1 d_{n-1} + \alpha_2 d_{n-1} + \alpha_3 d_n$, wherefrom $d_n < \frac{\alpha_1+\alpha_2}{1-\alpha_3} d_{n-1} \leqslant d_{n-1}$, since $\alpha_1 + \alpha_2 + \alpha_3 \leqslant 1$. Continuing this process, we can prove that $d_m < d_m$, a contradiction. Thus, in what follows, we can assume that (2.2) and (2.3) hold.

We will show that

$$\lim_{n\to\infty} r_b(x_n, x_{n+1}) = 0, \quad \text{and} \quad \lim_{n\to\infty} r_b(x_n, x_{n+2}) = 0. \tag{2.4}$$

Since T is an (α, η)-β-F-contraction, so we derive

$$\begin{aligned} F(d_n) &= F\big(r_b(Tx_{n-1}, Tx_n)\big) \\ &< \beta\big(r_b(x_{n-1}, x_n)\big) + F\big(s^2 r_b(Tx_{n-1}, Tx_n)\big) \\ &\leqslant F\big(\alpha_1 d_{n-1} + \alpha_2 d_{n-1} + \alpha_3 d_n + \alpha_4 \cdot 0\big). \end{aligned}$$

Since F is strictly increasing, we deduce that $d_n < \alpha_1 d_{n-1} + \alpha_2 d_{n-1} + \alpha_3 d_n$, wherefrom

$$d_n < \frac{\alpha_1 + \alpha_2}{1 - \alpha_3} d_{n-1},$$

where $\frac{\alpha_1+\alpha_2}{1-\alpha_3} \leqslant 1$. It follows that $d_n < d_{n-1}$, which again by (2.1), implies

$$F\big(r_b(x_n, x_{n+1})\big) < F\big(r_b(x_{n-1}, x_n)\big) - \beta\big(r_b(x_{n-1}, x_n)\big).$$

Therefore,

$$\begin{aligned} F\big(r_b(x_n, x_{n+1})\big) &< F\big(r_b(x_{n-1}, x_n)\big) - \beta\big(r_b(x_{n-1}, x_n)\big) \\ &< F\big(r_b(x_{n-2}, x_{n-1})\big) - \beta\big(r_b(x_{n-2}, x_{n-1})\big) - \beta\big(r_b(x_{n-1}, x_n)\big) \\ &\ \ \vdots \\ &< F(r_b(x_0, x_1)) - \sum_{i=1}^{n} \beta\big(r_b(x_{i-1}, x_i)\big). \end{aligned} \tag{2.5}$$

By taking the limit as $n \to \infty$ in (2.5) and using (β_2), we have $\lim_{n\to\infty} F\big(r_b(x_n, x_{n+1})\big) = -\infty$ and since $F \in \Delta_F$, we obtain

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} r_b(x_n, x_{n+1}) = 0. \tag{2.6}$$

On the other hand,

$$\begin{aligned}F(s^2 r_b(x_n,x_{n+2})) &< \beta\big(r_b(x_{n-1},x_{n+1})\big) + F\big(s^2 r_b(Tx_{n-1},Tx_{n+1})\big)\\ &\leqslant F\big(\alpha_1 r_b(x_{n-1},x_{n+1}) + \alpha_2 r_b(x_{n-1},x_n) + \alpha_3 r_b(x_{n+1},x_{n+2}) + \alpha_4 r_b(x_{n+1},x_n)\big)\\ &\leqslant F\big(s\alpha_1 r_b(x_{n-1},x_{n+2}) + s\alpha_1 r_b(x_{n+2},x_n) + s\alpha_1 r_b(x_n,x_{n+1})\\ &\quad + \alpha_2 r_b(x_{n-1},x_n) + \alpha_3 r_b(x_{n+1},x_{n+2}) + \alpha_4 r_b(x_{n+1},x_n)\big)\\ &\leqslant F\big(s^2\alpha_1 r_b(x_{n-1},x_n) + s^2\alpha_1 r_b(x_n,x_{n+1}) + s^2\alpha_1 r_b(x_{n+1},x_{n+2})\\ &\quad + s\alpha_1 r_b(x_{n+2},x_n) + s\alpha_1 r_b(x_n,x_{n+1}) + \alpha_2 r_b(x_{n-1},x_n)\\ &\quad + \alpha_3 r_b(x_{n+1},x_{n+2}) + \alpha_4 r_b(x_{n+1},x_n)\big),\end{aligned}$$

which implies that

$$(s^2 - s\alpha_1) r_b(x_n,x_{n+2}) \leqslant s^2\alpha_1 d_{n-1} + +s^2\alpha_1 d_n + s^2\alpha_1 d_{n+1} + s\alpha_1 d_n + \alpha_2 d_{n-1} + \alpha_3 d_{n+1} + \alpha_4 d_n.$$

Taking the limit as $n \to \infty$ in the above and using (2.6), since $s > 1$, we have

$$\lim_{n\to\infty} r_b(x_n,x_{n+2}) = 0,$$

hence (2.4) is proved.

Next, we show that $\{x_n\}$ is an r_b-Cauchy sequence in X.

Suppose to the contrary that $\{x_n\}$ is not an r_b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \quad \text{and} \quad r_b(x_{m_i},x_{n_i}) \geqslant \varepsilon. \tag{2.7}$$

This means that

$$r_b(x_{m_i},x_{n_i-1}) < \varepsilon. \tag{2.8}$$

From (2.7) and using the rectangular inequality, we get

$$\varepsilon \leqslant r_b(x_{m_i},x_{n_i}) \leqslant s r_b(x_{m_i},x_{m_i+1}) + s r_b(x_{m_i+1},x_{n_i+1}) + s r_b(x_{n_i+1},x_{n_i}).$$

Taking the upper limit as $i \to \infty$, and using (2.4), we get

$$\frac{\varepsilon}{s} \leqslant \limsup_{i\to\infty} r_b(x_{m_i+1},x_{n_i+1}). \tag{2.9}$$

Also,

$$r_b(x_{m_i},x_{n_i}) \leqslant s r_b(x_{m_i},x_{n_i-1}) + s r_b(x_{n_i-1},x_{n_i+1}) + s r_b(x_{n_i+1},x_{n_i}).$$

Then, from (2.8) and (2.4),

$$\limsup_{i\to\infty} r_b(x_{m_i},x_{n_i}) \leqslant s\varepsilon. \tag{2.10}$$

Also,

$$r_b(x_{n_i},x_{m_i+1}) \leqslant s r_b(x_{n_i},x_{n_i-1}) + s r_b(x_{n_i-1},x_{m_i}) + s r_b(x_{m_i},x_{m_i+1}).$$

Then, from (2.8) and (2.4),

$$\limsup_{i\to\infty} r_b(x_{n_i},x_{m_i+1}) \leqslant s\varepsilon. \tag{2.11}$$

Because of (2.2) and (2.3), we can apply (2.1) to conclude that

$$\begin{aligned}F(s^2\cdot r_b(x_{m_i+1},x_{n_i+1})) &= F(s^2\cdot r_b(Tx_{m_i},Tx_{n_i}))\\ &\leqslant F(\alpha_1 r_b(x_{m_i},x_{n_i}) + \alpha_2 r_b(x_{m_i},x_{m_i+1}) + \alpha_3 r_b(x_{n_i},x_{n_i+1})\\ &\quad + \alpha_4 r_b(x_{n_i},x_{m_i+1})) - \beta(r_b(x_{m_i},x_{n_i})).\end{aligned} \tag{2.12}$$

Now, taking the upper limit as $i \to \infty$ in (2.12) and using (F1) and (2.9), (2.10), (2.11), we have

$$\begin{aligned}
F\left(s^2 \cdot \frac{\varepsilon}{s}\right) &\leqslant F(s^2 \cdot \limsup_{i\to\infty} r_b(x_{m_i+1}, x_{n_i+1})) \\
&\leqslant F(\limsup_{i\to\infty}[\alpha_1 r_b(x_{m_i}, x_{n_i}) + \alpha_4 r_b(x_{n_i}, x_{m_i+1})]) - \liminf_{i\to\infty} \beta(r_b(x_{m_i}, x_{n_i})) \\
&\leqslant F\big([\alpha_1 + \alpha_4] s\varepsilon\big) - \liminf_{i\to\infty} \beta(r_b(x_{m_i}, x_{n_i})),
\end{aligned}$$

which further implies that

$$\liminf_{i\to\infty} \beta(r_b(x_{m_i}, x_{n_i})) = 0,$$

which is a contradiction with (2.7) because of the property (β_1).

Thus, we have proved that $\{x_n\}$ is an r_b-Cauchy sequence in the rectangular b-metric space (X, r_b). Since (X, r_b) is (α, η)-complete and

$$\alpha(x_{n-1}, x_n) \geqslant 1 \quad \text{or} \quad \eta(x_{n-1}, x_n) \leqslant 1,$$

for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ r_b-converges to some $z \in X$, that is, $\lim_{n\to\infty} r_b(x_n, z) = 0$.

Suppose that $z \neq Tz$. Then, from Lemma 1.6 (all of its assumptions are fulfilled), as T is (α, η)-continuous,

$$\frac{1}{s} r_b(z, Tz) \leqslant \liminf_{n\to\infty} r_b(x_n, Tx_n) \limsup_{n\to\infty} r_b(x_n, Tx_n) = \limsup_{n\to\infty} r_b(x_n, x_{n+1}) = 0.$$

Hence, we have $r_b(Tz, z) = 0$ and so $Tz = z$. Thus, z is a fixed point of T.

Let $x, y \in \mathrm{Fix}(T)$ where $x \neq y$ and $\alpha(x, y) \geqslant 1$ or $\eta(x, y) \leqslant 1$. Then from

$$\beta\big(r_b(x, y)\big) + F\big(r_b(Tx, Ty)\big) \leqslant F\big(\alpha_1 r_b(x, y) + \alpha_2 r_b(x, Tx) + \alpha_3 r_b(y, Ty) + \alpha_4 r_b(y, Tx)\big),$$

we get

$$\beta\big(r_b(x, y)\big) + F\big(r_b(x, y)\big) \leqslant F\big(r_b(x, y)\big),$$

which is a contradiction. Hence, $x = y$. Therefore, T has a unique fixed point. $\square$

It is clear that, if u is a fixed point of T, then u is also a fixed point of T^n for every $n \in \mathbb{N}$, i.e., $\mathrm{Fix}(T) \subseteq \mathrm{Fix}(T^n)$. However, the converse is not true. Recall [11] that if a mapping T satisfies $\mathrm{Fix}(T) = \mathrm{Fix}(T^n)$ for each $n \in \mathbb{N}$, then it is said that T has property P or that T has no periodic points.

Theorem 2.11. *Let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions and let (X, r_b) be an (α, η)-complete rectangular b-metric space. Let $T : X \to X$ be a self-mapping satisfying the following conditions:*

(i) *T is a triangular (α, η)-admissible mapping;*

(ii) *T is an (α, η)-β-F-contraction;*

(iii) *$\alpha(u, Tu) \geqslant 1$ or $\eta(u, Tu) \leqslant 1$, for all $u \in F(T)$.*

Then T has the property P.

Proof. Let $u \in F(T^n)$ for some fixed $n > 1$. As $\alpha(u, Tu) \geqslant 1$ or $\eta(u, Tu) \leqslant 1$ and since T is a triangular (α, η)-admissible mapping, then $\alpha(Tu, T^2u) \geqslant 1$ or $\eta(Tu, T^2u) \leqslant 1$. Continuing this process we have

$$\alpha(T^{n-1}u, T^n u) \geqslant 1 \quad \text{or} \quad \eta(T^{n-1}u, T^n u) \leqslant 1,$$

for all $n \in \mathbb{N}$. By (T3), we infer that also

$$\alpha(T^m u, T^n u) \geqslant 1 \quad \text{or} \quad \eta(T^m u, T^n u) \leqslant 1, \quad \forall m, n \in \mathbb{N}, \quad m \neq n.$$

Suppose that $u \notin \mathrm{Fix}(T)$, i.e., $r_b(u, Tu) > 0$. Then, as F is strictly increasing, it follows from (2.1) that

$$\begin{aligned} r_b(u, Tu) &= r_b(T^n u, Tu) = r_b(TT^{n-1}u, Tu) \\ &< \alpha_1 r_b(T^{n-1}u, u) + \alpha_2 r_b(T^{n-1}u, T^n u) + \alpha_3 r_b(u, Tu) + \alpha_4 r_b(u, T^n u) \\ &= \alpha_1 r_b(T^{n-1}u, T^n u) + \alpha_2 r_b(T^{n-1}u, T^n u) + \alpha_3 r_b(u, Tu). \end{aligned}$$

Therefore,

$$(1-\alpha_3) r_b(u, Tu) < (\alpha_1 + \alpha_2) r_b(T^{n-1}u, T^n u).$$

So, we have

$$r_b(u, Tu) < \frac{\alpha_1 + \alpha_2}{1-\alpha_3} r_b(T^{n-1}u, T^n u) \leqslant r_b(T^{n-1}u, T^n u),$$

a contradiction as $r_b(T^{n-1}u, T^n u) \to 0$ and $r_b(u, Tu) > 0$. □

Theorem 2.12. *Let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions and let (X, r_b) be an (α, η)-complete rectangular b-metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *T is a triangular (α, η)-admissible mapping;*

(ii) *T is an (α, η)-β-F-contraction;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ or $\eta(x_0, Tx_0) \leqslant 1$;*

(iv) *(X, r_b) is an (α, η)-regular rectangular b-metric space.*

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geqslant 1$ or $\eta(x, y) \leqslant 1$ for all $x, y \in \mathrm{Fix}(T)$.

Proof. Let $x_0 \in X$ satisfy condition (iii). As in the proof of Theorem 2.10 we can conclude that

$$(\alpha(x_n, x_{n+1}) \geqslant 1 \quad \text{or} \quad \eta(x_n, x_{n+1}) \leqslant 1), \quad \text{and} \quad x_n \to z \text{ as } n \to \infty,$$

where $x_{n+1} = Tx_n$. So, from (iv)

$$\alpha(x_{n+1}, z) \geqslant 1 \quad \text{or} \quad \eta(x_{n+1}, z) \leqslant 1$$

holds for all $n \in \mathbb{N}$.

Suppose that, eventually, $Tz = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$. From the proof of Theorem 2.10 we know that the members of the sequence $\{x_n\}$ are distinct. Hence, we have $Tz \neq Tx_n$, i.e., $r_b(Tx_n, Tz) > 0$ for all $n > n_0$. Thus, we can apply (2.1) to x_n and z for all $n > n_0$ to get

$$\beta\big(r_b(x_n, z)\big) + F\big(r_b(Tx_n, Tz)\big) \leqslant F\big(\alpha_1 r_b(x_n, z) + \alpha_2 r_b(x_n, Tx_n) + \alpha_3 r_b(z, Tz) + \alpha_4 r_b(z, Tx_n)\big),$$

which implies

$$F\big(r_b(x_{n+1}, Tz)\big) < F\big(\alpha_1 r_b(x_n, z) + \alpha_2 r_b(x_n, x_{n+1}) + \alpha_3 r_b(z, Tz) + \alpha_4 r_b(z, x_{n+1})\big).$$

From (F1) we have

$$r_b(x_{n+1}, Tz) < \alpha_1 r_b(x_n, z) + \alpha_2 r_b(x_n, x_{n+1}) + \alpha_3 r_b(z, Tz) + \alpha_4 r_b(z, x_{n+1}).$$

Suppose that $z \neq Tz$. Then, from Lemma 1.6,

$$\frac{1}{s} r_b(z, Tz) \leqslant \liminf_{n \to \infty} r_b(x_{n+1}, Tz) \leqslant \limsup_{n \to \infty} r_b(x_{n+1}, Tz) \leqslant \alpha_3 r_b(z, Tz).$$

Since, by assumption $\alpha_3 < \frac{1}{s}$, we have $r_b(z, Tz) = 0$ and so $z = Tz$. Thus, z is a fixed point of T.

The uniqueness follows similarly as in Theorem 2.10. □

2.3. Some special cases

We obtain the following extension of Wardowski's result [24, Theorem 2.1] to the setup of rectangular b-metric spaces if in the above theorems we take $\beta(t) = \tau$, for some fixed $\tau > 0$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$.

Corollary 2.13. *Let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions, (X, r_b) be an (α, η)-complete rectangular b-metric space and let $T : X \to X$ be a self-mapping. Suppose that for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$ or $\eta(x, y) \leqslant 1$ and $r_b(Tx, Ty) > 0$ we have*

$$\tau + F\big(s^2 \cdot r_b(Tx, Ty)\big) \leqslant F\big(r_b(x, y)\big),$$

where $\tau > 0$ and $F \in \Delta_F$. Then T has a fixed point, if

(i) T *is a triangular (α, η)-admissible mapping;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ or $\eta(x_0, Tx_0) \leqslant 1$;*

and

(iii) T *is (α, η)-continuous; or*

(iii') (X, r_b) *is an (α, η)-regular rectangular b-metric space.*

If in Theorems 2.10 and 2.12 we take $F(t) = \ln t$, for all $t > 0$, then we deduce the following corollary.

Corollary 2.14. *Let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions, (X, r_b) be an (α, η)-complete rectangular b-metric space and let $T : X \to X$ be a self-mapping. Suppose that for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$ or $1 \geqslant \eta(x, y)$ and $r_b(Tx, Ty) > 0$ we have*

$$s^2 r_b(Tx, Ty) \leqslant e^{-\beta(r_b(x,y))}[\alpha_1 r_b(x, y) + \alpha_2 r_b(x, Tx) + \alpha_3 r_b(y, Ty) + \alpha_4 r_b(y, Tx)],$$

where $\beta \in \Delta_\beta$, $\alpha_i \geqslant 0$, for $i \in \{1, 2, 3, 4\}$, $\sum_{i=1}^4 \alpha_i = 1$ and $\alpha_3 < \frac{1}{s}$. Then T has a fixed point, if

(i) T *is a triangular (α, η)-admissible mapping;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ or $\eta(x_0, Tx_0) \leqslant 1$;*

and

(iii) T *is (α, η)-continuous; or*

(iii') (X, r_b) *is an (α, η)-regular rectangular b-metric space.*

Many consequences can be obtained by using other forms of functions $F \in \Delta_F$ mentioned in [24] and/or some other concrete choices of $\beta \in \Delta_\beta$.

2.4. An illustrative example

Example 2.15. Consider the set $X = \{1, 2, 3, 4, 5\}$ and choose $u, v > 0$ such that $u + \frac{289}{30} v < \log \frac{281}{270}$ (i.e., $\frac{270}{281} < e^{-(u + \frac{289}{30} v)}$). It is easy to check that the mapping $r_b : X \times X \to [0, +\infty)$ given by

$$\begin{aligned}
&r_b(1, 3) = r_b(1, 5) = r_b(2, 3) = r_b(3, 5) = v,\\
&r_b(2, 4) = r_b(2, 5) = r_b(4, 5) = 4v,\\
&r_b(1, 2) = 9v,\\
&r_b(1, 4) = r_b(3, 4) = 10v,
\end{aligned}$$

$r_b(x, x) = 0$ and $r_b(x, y) = r_b(y, x)$ for all $x, y \in X$, is a rectangular b-metric on X with parameter $s = 3$.

Define mappings $T : X \to X$ and $\alpha, \eta : X \times X \to \mathbb{R}$ by

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 3 & 1 & 2 \end{pmatrix}, \qquad \alpha(x,y) = \begin{cases} 1 + \cosh(x+y), & x,y \in \{1,3,4\}, \\ \frac{1}{1+e^{x+y}}, & \text{otherwise}, \end{cases}$$

$$\eta(x,y) = \begin{cases} \tanh(x+y), & x,y \in \{1,3,4\}, \\ 2 + e^{-(x+y)}, & \text{otherwise}. \end{cases}$$

Then, T is an (α,η)-continuous triangular (α,η)-admissible mapping. We will show that the contractive condition (2.1) of Theorem 2.10 is satisfied with $\beta \in \Delta_\beta$ and $F \in \Delta_F$ given by $\beta(t) = u + t$ and $F(t) = t + \ln t$, respectively and taking $\alpha_1 = \frac{9}{10}$, $\alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{30}$. Evidently, the only cases when $(\alpha(x,y) \geqslant 1$ or $1 \geqslant \eta(x,y))$ and $r_b(Tx,Ty) > 0$ are when $\{x,y\} = \{1,4\}$ or $\{x,y\} = \{3,4\}$. Consider the following four possibilities:

1° $x = 1$, $y = 4$. Then $r_b(Tx,Ty) = \nu$, $r_b(x,y) = 10\nu$, $r_b(x,Tx) = \nu$, $r_b(y,Tx) = 10\nu$, $r_b(y,Ty) = 10\nu$.
2° $x = 4$, $y = 1$. Then $r_b(Tx,Ty) = \nu$, $r_b(x,y) = 10\nu$, $r_b(x,Tx) = 10\nu$, $r_b(y,Tx) = 0$, $r_b(y,Ty) = \nu$.
3° $x = 3$, $y = 4$. Then $r_b(Tx,Ty) = \nu$, $r_b(x,y) = 10\nu$, $r_b(x,Tx) = 0$, $r_b(y,Tx) = 10\nu$, $r_b(y,Ty) = 10\nu$.
4° $x = 4$, $y = 3$. Then $r_b(Tx,Ty) = \nu$, $r_b(x,y) = 10\nu$, $r_b(x,Tx) = 10\nu$, $r_b(y,Tx) = \nu$, $r_b(y,Ty) = 0$.

In all the cases, it is $\alpha_1 r_b(x,y) + \alpha_2 r_b(x,Tx) + \alpha_3 r_b(y,Ty) + \alpha_4 r_b(y,Tx) \geqslant \frac{281}{30} t$ and hence

$$\begin{aligned} \beta(r_b(x,y)) + F(s^2 r_b(Tx,Ty)) &= u + 10\nu + 9\nu + \log(9\nu) \\ &\leqslant \frac{281}{30}\nu + \log\left(\frac{281}{30}\nu\right) \quad \left(\text{since, } u + \frac{289}{30}\nu < \log\frac{281}{270}\right) \\ &\leqslant F(\alpha_1 r_b(x,y) + \alpha_2 r_b(x,Tx) + \alpha_3 r_b(y,Ty) + \alpha_4 r_b(y,Tx)), \end{aligned}$$

i.e., the condition (2.1) is satisfied.

We see that T has a unique fixed point ($z = 3$).

3. Applications

3.1. β-F-*contractions on rectangular b-metric spaces endowed with a graph*

Let (X, r_b) be a rectangular b-metric space and Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, but as the graph G is directed, the edges (x,y) and (y,x) are not the same. We identify G with the pair $(V(G), E(G))$.

Recently, some results have been presented in the setting of metric spaces endowed with a graph. The first result in this direction was given by Jachymski [10]. Recall that the mapping $T : X \to X$, where X is endowed with a graph G, is said to preserve edges, if for each $x, y \in X$ with $(x,y) \in E(G)$, we have $(Tx, Ty) \in E(G)$.

Definition 3.1. Let (X, r_b) be a rectangular b-metric space endowed with a graph and let $T : X \to X$ be a mapping.

1. (X, r_b) is said to be G-complete, if every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ or $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$, converges in X.
2. (X, r_b) is said to be G-regular, if for each sequence $\{x_n\}$ in X, $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ or $(x_{n+1}, x_n) \in E(G)$ imply that $(x_n, x) \in E(G)$ or $(x, x_n) \in E(G)$ for all $n \in \mathbb{N}$, respectively.
3. T is said to be G-continuous, if for given $x \in X$ and sequence $\{x_n\}$ with $(x_n, x_{n+1}) \in E(G)$ or $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$,
$$x_n \to x \Longrightarrow Tx_n \to Tx.$$
4. T is said to be a G-β-F-contraction, if for all $x, y \in X$ with $((x,y) \in E(G)$ or $(y,x) \in E(G))$ and $r_b(Tx,Ty) > 0$ we have
$$\beta\big(r_b(x,y)\big) + F\big(s^2 \cdot r_b(Tx,Ty)\big) \leqslant F\big(\alpha_1 r_b(x,y) + \alpha_2 r_b(x,Tx) + \alpha_3 r_b(y,Ty) + \alpha_4 r_b(y,Tx)\big),$$
where $\beta \in \Delta_\beta$, $F \in \Delta_F$, $\alpha_i \geqslant 0$ for $i \in \{1,2,3,4\}$, $\sum_{i=1}^4 \alpha_i = 1$ and $\alpha_3 < \frac{1}{s}$.

Let (X, d) be a rectangular b-metric space endowed with a graph and let

$$\alpha(x,y) = \begin{cases} 1, & (x,y) \in E(G), \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \eta(x,y) = \begin{cases} 1, & (y,x) \in E(G), \\ 2, & \text{otherwise}. \end{cases}$$

With these assumptions, we see that the above definitions are special cases of the definition of (α,η)-admissibility, (α,η)-completeness, (α,η)-regularity and (α,η)-continuity.

As a consequence of the results of previous section, we obtain:

Theorem 3.2. *Let (X, r_b) be a G-complete rectangular b-metric space such that for all $(x,y) \in E(G)$ and $(y,z) \in E(G)$, we have $(x,z) \in E(G)$. Let $T : X \to X$ be a self-mapping satisfying the following conditions*

(i) T *preserves edges;*

(ii) T *is a G-β-F-contraction;*

(iii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ or $(Tx_0, x_0) \in E(G)$;*

and

(iv) T *is G-continuous; or*

(iv′) (X, r_b) *is a G-regular rectangular b-metric space.*

Then T has a fixed point. Moreover, T has a unique fixed point when $(x,y) \in E(G)$ or $(y,x) \in E(G)$ for all $x, y \in Fix(T)$.

3.2. A result in ordered spaces

Fixed point theorems for monotone operators in ordered metric spaces have been widely investigated and have had various applications in differential and integral equations and other branches (see, e.g., [7, 14, 16] and the references therein). From Theorems 2.10 and 2.12, we derive the following new results in partially ordered rectangular b-metric spaces, i.e., spaces of the type $(X, r_b, \preceq)$ where (X, r_b) is a rectangular b-metric spaces and $\preceq$ is a partial order on X. Recall that $T : X \to X$ is nondecreasing if for all $x, y \in X$, $x \preceq y$ implies $T(x) \preceq T(y)$.

Motivated by [14] we introduce the following concepts in an ordered rectangular b-metric space.

Definition 3.3. Let $(X, r_b, \preceq)$ be an ordered rectangular b-metric space and let $T : X \to X$ be a mapping.

1. (X, r_b) is said to be O-complete, if every Cauchy sequence $\{x_n\}$ in X with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ or $x_n \succeq x_{n+1}$ for all $n \in \mathbb{N}$, converges in X.
2. (X, r_b) is said to be O-regular, if for each sequence $\{x_n\}$ in X, $x_n \to x$ and $x_n \preceq x_{n+1}$ or $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$ imply that $x_n \preceq x$ or $x \preceq x_n$ for all $n \in \mathbb{N}$, respectively.
3. T is said to be O-continuous, if for given $x \in X$ and sequence $\{x_n\}$ with $x_n \preceq x_{n+1}$ or $x_n \succeq x_{n+1}$ for all $n \in \mathbb{N}$,
$$x_n \to x \Longrightarrow Tx_n \to Tx.$$
4. T is said to be an ordered β-F-contraction, if for all $x, y \in X$ with ($x \preceq y$ or $x \succeq y$) and $r_b(Tx, Ty) > 0$ we have
$$\beta\big(r_b(x,y)\big) + F\big(s^2 \cdot r_b(Tx,Ty)\big) \leqslant F\big(\alpha_1 r_b(x,y) + \alpha_2 r_b(x,Tx) + \alpha_3 r_b(y,Ty) + \alpha_4 r_b(y,Tx)\big),$$
where $\beta \in \Delta_\beta$, $F \in \Delta_F$, $\alpha_i \geqslant 0$ for $i \in \{1,2,3,4\}$, $\sum_{i=1}^4 \alpha_i = 1$ and $\alpha_3 < \frac{1}{s}$.

Theorem 3.4. *Let $(X, r_b, \preceq)$ be an O-complete partially ordered rectangular b-metric space. Let $T : X \to X$ be a self-mapping satisfying the following conditions*

(i) T *is monotone and is an ordered* β*-F-contraction;*

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *or* $x_0 \succeq Tx_0$*;*

(iii) *either* T *is* O*-continuous; or*

(iii') (X, r_b) *is* O*-regular.*

Then T *has a fixed point.*

Proof. This is obtained as a consequence of Theorems 2.10 and 2.12 if one takes

$$\alpha(x,y) = \begin{cases} 1, & x \preceq y, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta(x,y) = \begin{cases} 1, & y \preceq x, \\ 2, & \text{otherwise.} \end{cases}$$

□

Corollary 3.5. *Let* $(X, r_b, \preceq)$ *be an* O*-complete partially ordered rectangular b-metric space. Let* $T : X \to X$ *be a self-mapping satisfying the following assertions:*

(i) T *is monotone and the following inequality holds for all* $x, y \in X$ *with* ($x \preceq y$ *or* $x \succeq y$) *and* $r_b(Tx, Ty) > 0$*:*

$$s^2 r_b(Tx, Ty) \leqslant e^{-\beta(r_b(x,y))}[\alpha_1 r_b(x,y) + \alpha_2 r_b(x, Tx) + \alpha_3 r_b(y, Ty) + \alpha_4 r_b(y, Tx)],$$

where $\beta \in \Delta_\beta$, $F \in \Delta_F$, $\alpha_i \geqslant 0$ *for* $i \in \{1,2,3,4\}$, $\sum_{i=1}^{4} \alpha_i = 1$ *and* $\alpha_3 < \frac{1}{s}$*;*

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *or* $x_0 \succeq Tx_0$*;*

(iii) *either* T *is* O*-continuous; or*

(iii') (X, r_b) *is* O*-regular.*

Then T *has a fixed point.*

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References

[1] I. A. Bakhtin, *The contraction mapping principle in almost metric space*, (Russian) Functional analysis, Ulyanovsk. Gos. Ped. Inst., Ulyanovsk, (1989), 26–37. 1

[2] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, **57** (2000), 31–37. 1

[3] M. Cosentino, M. Jleli, B. Samet, C. Vetro, *Solvability of integrodifferential problems via fixed point theory in* b*-metric spaces*, Fixed Point Theory Appl., **2015** (2015), 15 pages. 1

[4] M. Cosentino, P. Vetro, *Fixed point results for* F*-contractive mappings of Hardy-Rogers-type*, Filomat, **28** (2014), 715–722. 1

[5] S. Czerwik, *Contraction mappings in* b*-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5–11. 1, 1.1

[6] S. Czerwik, *Nonlinear set-valued contraction mappings in* b*-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 263–276. 1

[7] N. Hussain, M. A. Kutbi, P. Salimi, *Fixed point theory in α-complete metric spaces with applications*, Abstr. Appl. Anal., **2014** (2014), 11 pages. 2.1, 2.3, 2.4, 2.7, 3.2

[8] N. Hussain, V. Parvaneh, J. R. Roshan, *Fixed point results for G-α-contractive maps with applications to boundary value problems*, Scientific World J., **2014** (2014), 14 pages. 1

[9] N. Hussain, P. Salimi, *Suzuki-Wardowski type fixed point theorems for α-GF-contractions*, Taiwanese J. Math., **18** (2014), 1879–1895. 1

[10] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373. 3.1

[11] G. S. Jeong, B. E. Rhoades, *Maps for which* $F(T) = F(T^n)$, Fixed Point Theory applications, Vol. **6**, 71–105, Nova Sci. Publ., New York, (2007). 2.2

[12] M. Jleli, B. Samet, C. Vetro, F. Vetro, *Fixed points for multivalued mappings in b-metric spaces*, Abstr. Appl. Anal., **2015** (2015), 7 pages. 1

[13] Z. Kadelburg, S. Radenović, *On generalized metric spaces: a survey*, TWMS J. Pure Appl. Math., **5** (2014), 3–13. 1

[14] M. A. Kutbi, A. Alam, M. Imdad, *Sharpening some core theorems of Nieto and Rodrguez-López with application to boundary value problem*, Fixed Point Theory Appl., **2015** (2015), 15 pages. 3.2

[15] V. La Rosa, P. Vetro, *Common fixed points for α-ψ-ϕ-contractions in generalized metric spaces*, Nonlinear Anal. Model. Control, **19** (2014), 43–54. 2.1

[16] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–229. 3.2

[17] J. R. Roshan, V. Parvaneh, Z. Kadelburg, N. Hussain, *New fixed point results in b-rectangular metric spaces*, Nonlinear Anal. Model. Control, **21** (2016), 614–634. 1, 1.3, 1, 1.6

[18] B. Samet, *Discussion on "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces" by A. Branciari*, Publ. Math. Debrecen, **76** (2010), 493–494. 1, 1.5

[19] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha\psi$-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 2.1

[20] I. R. Sarma, J. M. Rao, S. S. Rao, *Contractions over generalized metric spaces*, J. Nonlinear Sci. Appl., **2** (2009), 180–182. 1, 1.5

[21] T. Suzuki, *Generalized metric spaces do not have the compatible topology*, Abstr. Appl. Anal., **2014** (2014), 5 pages. 1

[22] T. Van An, N. Van Dung, Z. Kadelburg, S. Radenović, *Various generalizations of metric spaces and fixed point theorems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, **109** (2015), 175–198. 1

[23] F. Vetro, *F-contractions of Hardy-Rogers type and application to multistage decision processes*, Nonlinear Anal. Model. Control, **21** (2016), 531–546. 1

[24] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012** (2012), 6 pages. 1, 2.1, 2.3, 2.3