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Multivariate contraction mapping principle with the error estimate formulas in locally convex topological vector spaces and application

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Abstract

The purpose of this paper is to present the concept of multivariate contraction mapping in a locally convex topological vector spaces and to prove the multivariate contraction mapping principle in such spaces. The neighborhood-type error estimate formulas are also established. The results of this paper improve and extend Banach contraction mapping principle in the new idea.

Keywords: Contraction mapping principle, locally convex, topological vector spaces, fixed point, error estimate formula.
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1. Introduction and preliminaries

Banach contraction mapping principle is one of the important tool (or method) in nonlinear analysis and other mathematical field. Weak contractions are generalizations of Banach contraction mappings which have been studied by several authors. Let (X, d) be a metric space and $\phi : [0, +\infty) \to [0, +\infty)$ be a function. We say that $T : X \to X$ is a ϕ-contraction, if

$$d(Tx, Ty) \leqslant \phi(d(x, y)), \quad \forall\, x, y \in X.$$

In 1968, Browder [3] proved that if ϕ is non-decreasing and right continuous and (X, d) is complete, then T has a unique fixed point x^* and $\lim\limits_{n\to\infty} T^n x_0 = x^*$ for any given $x_0 \in X$. Subsequently, this result was extended in 1969 by Boyd and Wong [2] by weakening the hypothesis on ϕ, in the sense that it is sufficient to assume that ϕ is right upper semi-continuous. For a comprehensive study of relations between several such contraction type conditions, see [4, 8, 9, 16].

In 1973, Geraghty [4] introduced the Geraghty-contraction and obtained the fixed point theorem. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Geraghty-contraction, if there exists $\beta \in \Gamma$

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such that for any $x, y \in X$

$$d(Tx, Ty) \leqslant \beta(d(x, y))d(x, y),$$

where the class Γ denotes those functions $\beta : [0, +\infty) \to [0, +\infty)$ satisfying the following condition:

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

On the other hand, in 2015, Su and Yao [18] proved the following generalized contraction mapping principle.

Theorem 1.1. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a mapping such that*

$$\psi(d(Tx, Ty)) \leqslant \phi(d(x, y)), \quad \forall\, x, y \in X,$$

where $\psi, \phi : [0, +\infty) \to [0, +\infty)$ are two functions satisfying the conditions:

(1) $\psi(a) \leqslant \phi(b) \Rightarrow a \leqslant b$;

(2) $\begin{cases} \psi(a_n) \leqslant \phi(b_n) \\ a_n \to \varepsilon,\ b_n \to \varepsilon \end{cases} \Rightarrow \varepsilon = 0.$

Then, T has a unique fixed point and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.

In particular, the study of the fixed points for weak contractions and generalized contractions was extended to partially ordered metric spaces in [1, 5–7, 11, 13–15, 20]. Among them, some results involve altering distance functions. Such functions were introduced by Khan et al. in [10], where some fixed point theorems are presented.

Recently [17], Su et al. presented the concept of multivariate fixed point and proved a multivariate fixed point theorem for the N-variables contraction mappings which further generalizes Banach contraction principle.

Definition 1.2 ([17]). Let (X, d) be a metric space, $T : X^N \to X$ be an N-variables mapping, an element $p \in X$ is called a multivariate fixed point (or a fixed point of order N, see [17]) of T, if

$$p = T(p, p, \cdots, p).$$

The following concept is presented and used in reference [17].

Definition 1.3 ([17]). A multiply metric function $\triangle(a_1, a_2, \cdots, a_N)$ is a continuous N variables non-negative real function with the domain

$$\{(a_1, a_2, \cdots, a_N) \in R^N : a_i \geqslant 0,\ i \in \{1, 2, 3, \cdots, N\}\},$$

which satisfies the following conditions:

(1) $\triangle(a_1, a_2, \cdots, a_N)$ is non-decreasing for each variable a_i, $i \in \{1, 2, 3, \cdots, N\}$;

(2) $\triangle(a_1 + b_1, a_2 + b_2, \cdots, a_N + b_N) \leqslant \triangle(a_1, a_2, \cdots, a_N) + \triangle(b_1, b_2, \cdots, b_N)$;

(3) $\triangle(a, a, \cdots, a) = a$;

(4) $\triangle(a_1, a_2, \cdots, a_N) \to 0 \Leftrightarrow a_i \to 0,\ i \in \{1, 2, 3, \cdots, N\}$,

for all $a_i, b_i, a \in \mathbb{R}$, $i \in \{1, 2, 3, \cdots, N\}$, where $\mathbb{R}$ denotes the set of all real numbers.

The following are some basic examples of multiply metric functions:

Example 1.4.

(1) $\triangle_1(a_1, a_2, \cdots, a_N) = \frac{1}{N}\sum_{i=1}^{N} a_i$.

(2) $\triangle_2(a_1, a_2, \cdots, a_N) = \sum_{i=1}^{N} q_i a_i$,

where $q_i \in [0, 1], i \in \{1, \cdots, N\}$ and $\sum_{i=1}^{N} q_i = 1$.

Example 1.5. $\triangle_3(a_1, a_2, \cdots, a_N) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} a_i^2}$.

Example 1.6. $\triangle_4(a_1, a_2, \cdots, a_N) = \max\{a_1, a_2, \cdots, a_N\}$.

In reference [17], Su et al. proved the following multivariate fixed point theorem for the N-variables contraction mappings which further generalizes Banach contraction mapping principle.

Theorem 1.7 ([17])**.** *Let* (X, d) *be a complete metric space,* $T : X^N \to X$ *be a* N*-variables mapping satisfies the following condition:*

$$d(Tx, Ty) \leqslant h\triangle(d(x_1, y_1), d(x_2, y_2), \cdots, d(x_N, y_N)), \quad \forall\, x, y \in X^N,$$

where $\triangle$ *is a multiply metric function,*

$$x = (x_1, x_2, \cdots, x_N) \in X^N, \quad y = (y_1, y_2, \cdots, y_N) \in X^N,$$

and $h \in (0, 1)$ *is a constant. Then,* T *has a unique multivariate fixed point* $p \in X$ *and for any* $p_0 \in X^N$*, the iterative sequence* $\{p_n\} \subset X^N$ *defined by:*

$$\begin{aligned} p_1 &= (Tp_0, Tp_0, \cdots, Tp_0), \\ p_2 &= (Tp_1, Tp_1, \cdots, Tp_1), \\ p_3 &= (Tp_2, Tp_2, \cdots, Tp_2), \\ &\vdots \\ p_{n+1} &= (Tp_n, Tp_n, \cdots, Tp_n), \\ &\vdots \end{aligned}$$

converges, in the multiply metric $\triangle$*, to* $(p, p, \cdots, p) \in X^N$ *and the iterative sequence* $\{Tp_n\} \subset X$ *converges, with respect to* d*, to* $p \in X$*.*

Very recently, Tang et al. [19] presented the concept of contraction mapping in a locally convex topological vector spaces and proved the generalized contraction mapping principle in such spaces.

The purpose of this paper is to present the concept of multivariate contraction mapping in a locally convex topological vector spaces and to prove the multivariate contraction mapping principle in such spaces. The neighborhood-type error estimate formulas are also established. The results of this paper improve and extend Banach contraction mapping principle in the new idea.

2. Generalized contraction mapping principle in locally convex spaces

Let us recall some concepts and results on the topological vector spaces.

Definition 2.1. A Hausdorff topology τ on a real vector space X over R is said to be a vector space topology for X, if addition and scalar-multiplication are continuous, i.e., the mappings

$$(x, y) \mapsto x + y \ \text{ from } \ X \times X \ \text{ into } \ X,$$

and

$$(\alpha, x) \mapsto \alpha x \ \text{ from } \ R \times X \ \text{ into } \ X,$$

are continuous, where $X \times X$ and $R \times X$ are equipped with the respective product topologies. X itself or more precisely (X, τ) is then called a topological vector space.

Remark 2.2. Continuity of addition means: For every neighborhood W of $x_0 + y_0$ there exist neighborhood U of x_0 and V of y_0 such that $U + V \subset W$. Continuity of scalar-multiplication means: For every neighborhood W of $\alpha_0 x_0$ there exist a $\delta > 0$ and a neighborhood U of x_0 such that

$$\alpha U \subset W, \ \ \forall \ |\alpha - \alpha_0| < \delta.$$

Definition 2.3. A topological vector space (X, τ) is said to be locally convex, if there exists a basis of neighborhood of zero Ω such that every $U \in \Omega$ is convex set.

Conclusion 2.4. *Let (X, τ) be a locally convex topological vector space. For any convex neighborhood of zero $U \in \Omega$, there exists a balanced convex neighborhood of zero V such that $V \subset U$.*

Proof. For any convex neighborhood of zero $U \in \Omega$, there exists a balanced neighborhood of zero W such that $W \subset U$. Let

$$A = \bigcap_{|\alpha|=1} \alpha U,$$

then A and A^0 are convex. Since W is balanced, we have

$$W = \alpha W \subset \alpha U, \quad \forall \, |\alpha| = 1,$$

which implies

$$W \subset A, \quad W \subset A^0.$$

Hence A^0 is a neighborhood of zero. Next, we show A^0 is balanced. In fact that, for any $|\lambda| \leqslant 1$, we have

$$\lambda A = \bigcap_{|\alpha|=1} \lambda \alpha U = \bigcap_{|\alpha|=1} |\lambda| \alpha U \subset \bigcap_{|\alpha|=1} \alpha U = A,$$

which implies A is balanced, so is A^0. Let $V = A^0$, we have V is a balanced convex neighborhood of zero such that $V \subset U$. This completes the proof. $\square$

From Conclusion 2.4, we can get the following result.

Conclusion 2.5. *Let (X, τ) be a locally convex topological vector space. Then there exists a basis of balanced convex neighborhood of zero Ω. Furthermore, each $U \in \Omega$ is absorbing, balanced and convex.*

Definition 2.6 ([19])**.** Let (X, τ) be a locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω.

(1) A mapping $T : X \to X$ is said to be contractive, if there exists a constant $h \in (0, 1)$ such that for any $U \in \Omega$ and any $x, y \in X$

$$x - y \in tU \ \text{ implies } \ Tx - Ty \in htU,$$

for any $t > 0$.

(2) A mapping $T : X \to X$ is said to be (ψ, ϕ)- contractive, if there exist two functions $\psi : [0, +\infty) \to [0, +\infty)$, $\phi : [0, +\infty) \to [0, +\infty)$ such that for any $U \in \Omega$ and any $x, y \in X$

$$x - y \in \phi(t)U \ \text{ implies } \ Tx - Ty \in \psi(t)U,$$

for any $t > 0$.

Definition 2.7. Let (X, τ) be a topological vector space with a basis of balanced convex neighborhood of zero Ω, a net $\{x_\lambda\}_{\lambda \in I} \subset X$ is said to be Cauchy, if for any $U \in \Omega$, there exists a $\lambda_0 \in I$ such that

$$x_{\lambda_1} - y_{\lambda_2} \in U, \quad \forall\, \lambda_1, \lambda_2 \geqslant \lambda_0.$$

The topological vector space (X, τ) is said to be complete, if every Cauchy net is convergent.

The following results are well-known in the theory of topological vector space.

Conclusion 2.8. *Let (X, τ) be a locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. For any $U \in \Omega$, the Minkowski functional of U is defined by*

$$M_U(x) = \inf\{t > 0 : x \in tU\}, \quad \forall\, x \in X.$$

Then the following hold:

(1) $M_U(x) \geqslant 0$, *for any* $x \in X$ *and* $x = 0$ *implies* $M_U(x) = 0$.

(2) $M_U(\lambda x) = |\lambda| M_U(x)$ *for any* $x \in X$, $\lambda \in R$.

(3) $M_U(x + y) \leqslant M_U(x) + M_U(y)$ *for any* $x, y \in X$.

(4) *net* $\{x_\lambda\}_{\lambda \in I} \subset X$ *converges to* $x_0 \in X$, *if and only if* $\lim_{\lambda \in I} M_U(x_\lambda - x_0) = 0$.

(5) *net* $\{x_\lambda\}_{\lambda \in I} \subset X$ *is a Cauchy net, if and only if for any* $U \in \Omega$

$$\lim_{\lambda_1, \lambda_2 \in I} M_U(x_{\lambda_1} - x_{\lambda_2}) = 0.$$

Remark 2.9. In fact that, for any $U \in \Omega$, the Minkowski functional $M_U(\cdot)$ is a semi-norm on the X.

The following results have been proved in reference [19] by Tang et al.

Theorem 2.10 ([19] Generalized contraction mapping principle)**.** *Let (X, τ) be a complete locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. Let $T : X \to X$ be a (ψ, ϕ)-contractive mapping satisfying the following conditions:*

(1) $\psi(t), \phi(t)$ *are continuous and strictly increasing;*

(2) $\psi(0) = \phi(0)$ *and* $\psi(t) < \phi(t)$ *for all* $t > 0$.

Then T has a unique fixed point and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.

Theorem 2.11 ([19] Contraction mapping principle and the error estimate formula)**.** *Let (X, τ) be a complete locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. Let $T : X \to X$ be a contractive mapping. Then*

(1) T *has a unique fixed point* x^* *and for any given* $x_0 \in X$, *the iterative sequence* $T^n x_0$ *converges to this fixed point;*

(2) *the following error estimate formula holds: for any* $U \in \Omega$, *if take sufficiently large* n *such that* $x_1 - x_0 \in \frac{1-h}{h^n} U$, *then* $x_n - x^* \in U$.

3. Multivariate contraction mapping principle in locally convex spaces

Definition 3.1. Let (X, τ) be a locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. An N-variables mapping $T : X^N \to X$ is said to be Δ-contractive, if for any $U \in \Omega$ and any

$$t_1, t_2, t_3, \cdots, t_N > 0, \quad x_i - y_i \in t_i U, \quad i = 1, 2, 3, \cdots, N$$

where $x_i, y_i \in X$, there exists a real number t_0 such that

$$0 < t_0 \leqslant h\Delta(t_1, t_2, t_3, \cdots, t_N),$$

and

$$Tx - Ty \in t_0 U,$$

where Δ is a multiply metric function, $0 < h < 1$ is a constant and

$$x = (x_1, x_2, x_3, \cdots, x_N),\ y = (y_1, y_2, y_3, \cdots, y_N) \in X^N.$$

Lemma 3.2. *Let (X, τ) be a locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. Let $T : X^N \to X$ be a Δ-contractive mapping. Then for any $U \in \Omega$, the following inequality holds:*

$$M_U(Tx - Ty) \leqslant h\Delta(M_U(x_1 - y_1), M_U(x_2 - y_2), \cdots, M_U(x_N - y_N)),$$

for all

$$x = (x_1, x_2, x_3, \cdots, x_N),\ y = (y_1, y_2, y_3, \cdots, y_N) \in X^N,$$

where $M_U(x)$ is Minkowski functional of U defined by

$$M_U(x) = \inf\{t > 0 : x \in tU\}, \quad \forall\, x \in X.$$

Proof. From the definition of Δ-contraction mapping, we have that

$$\inf\{t > 0 : Tx - Ty \in tU\} \leqslant h\Delta(t_1, t_2, t_3, \cdots, t_N), \quad x_i - y_i \in t_i U,$$

for all $i = 1, 2, 3, \cdots, N$. Since $\Delta(t_1, t_2, t_3, \cdots, t_N)$ is continuous, the above inequality implies that

$$M_U(Tx - Ty) \leqslant h\Delta(M_U(x_1 - y_1), M_U(x_2 - y_2), \cdots, M_U(x_N - y_N)),$$

for all

$$x = (x_1, x_2, x_3, \cdots, x_N), \quad y = (y_1, y_2, y_3, \cdots, y_N) \in X^N.$$

This completes the proof. □

Lemma 3.3. *Let (X, τ) be a locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. For any $U \in \Omega$, let*

$$M^*_U(x) = \Delta(M_U(x_1), M_U(x_2), M_U(x_3), \cdots, M_U(x_N)),$$

where

$$x = (x_1, x_2, x_3, \cdots, x_N) \in X^N.$$

Then M^*_U is a semi-norm on the linear space X^N.

Proof. From the definition of Δ, we have that

(1) $x = (x_1, x_2, x_3, \cdots, x_N) = 0 \Rightarrow M^*_U(x) = \Delta(M_U(x_1), M_U(x_2), M_U(x_3), \cdots, M_U(x_N)) = 0;$

(2)
$$\begin{aligned} M^*_U(\lambda x) &= \Delta(M_U(\lambda x_1), M_U(\lambda x_2), M_U(\lambda x_3), \cdots, M_U(\lambda x_N)) \\ &= \Delta(|\lambda| M_U(x_1), |\lambda| M_U(x_2), |\lambda| M_U(x_3), \cdots, |\lambda| M_U(x_N)) \\ &= |\lambda| \Delta(M_U(x_1), M_U(x_2), M_U(x_3), \cdots, M_U(x_N)) \\ &= |\lambda| M^*_U(x); \end{aligned}$$

(3)
$$\begin{aligned} M^*_U(x + y) &= \Delta(M_U(x_1 + y_1), M_U(x_2 + y_2), M_U(x_3 + y_3), \cdots, M_U(x_N + y_N)) \\ &\leqslant \Delta(M_U(x_1) + M_U(y_1), M_U(x_2) + M_U(y_2), \cdots, M_U(x_N) + M_U(y_N)) \\ &\leqslant \Delta(M_U(x_1), M_U(x_2), \cdots, M_U(x_N)) + \Delta(M_U(y_1), M_U(y_2), \cdots, M_U(y_N)) \\ &= M^*_U(x) + M^*_U(y), \end{aligned}$$

where

$$x = (x_1, x_2, x_3, \cdots, x_N),\ y = (y_1, y_2, y_3, \cdots, y_N) \in X^N,$$

and λ is a real number. From above (1)-(3), we see that, M^*_U is a semi-norm on the linear space X^N. This completes the proof. □

Let (X, τ) be a locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. It is easy to see that, $\{M^*_U : U \in \Omega\}$ is a family of semi-norms satisfying the following condition:

for any nonzero element $x \in X^N$, there exists a semi-norm M^*_U such that $M^*_U(x) \neq 0$. On the other hand, if (X, τ) is a complete locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω, then $(X^N, \{M^*_U\}_{U\in\Omega})$ is also a complete locally convex topological vector space with a basis of balanced convex neighborhood of zero $\{U^N : U \in \Omega\}$. Let $T : X^N \to X$ be a Δ-contractive mapping, we define a mapping $T^* : X^N \to X^N$ as follows:

$$T^* : x \longmapsto (Tx, Tx, Tx, \cdots, Tx).$$

Then T^* is a contraction mapping from $(X^N, \{M^*_U\}_{U\in\Omega})$ into itself, that is,

$$M^*_U(T^*x - T^*y) \leqslant hM^*_U(x - y), \quad \forall\, U \in \Omega, \ \ \forall\, x, y \in X^N.$$

In fact, for any $U \in \Omega$, $x, y \in X^N$, we have

$$\begin{aligned} M^*_U(T^*x - T^*y) &= \Delta(M_U(Tx - Ty), M_U(Tx - Ty), \cdots, M_U(Tx - Ty)) \\ &= M_U(Tx - Ty) \\ &\leqslant h\Delta(M_U(x_1 - y_1), M_U(x_2 - y_2), \cdots, M_U(x_N - y_N)) \\ &= hM^*_U(x - y). \end{aligned}$$

Theorem 3.4 (Multivariate contraction mapping principle)**.** *Let (X, τ) be a complete locally convex topological vector space with a basis of balanced convex neighborhood of zero Ω. Let $T : X^N \to X$ be a Δ-contractive mapping. Then, T has a unique multivariate fixed point $p \in X$ and for any $p_0 \in X^N$, the iterative sequence $\{p_n\} \subset X^N$ defined by:*

$$\begin{aligned} p_1 &= (Tp_0, Tp_0, \cdots, Tp_0), \\ p_2 &= (Tp_1, Tp_1, \cdots, Tp_1), \\ p_3 &= (Tp_2, Tp_2, \cdots, Tp_2), \\ &\ \vdots \\ p_{n+1} &= (Tp_n, Tp_n, \cdots, Tp_n), \\ &\ \vdots \end{aligned}$$

converges, in the topological space (X^N, τ^N) to $(p, p, \cdots, p) \in X^N$ and the iterative sequence $\{Tp_n\} \subset X$ converges, in the topological space (X, τ), to $p \in X$.

Proof. For any $p_0 \in X^N$, we define an iterative sequence $\{p_n\}$ by the following scheme:

$$p_{n+1} = T^*p_n, \quad n = 0, 1, 2, \cdots.$$

For any $U \in \Omega$, we get the following inequalities

$$M^*_U(p_{n+1} - p_n) \leqslant h^n M^*_U(p_1 - p_0), \quad n = 0, 1, 2, \cdots.$$

Therefore

$$\begin{aligned} M^*_U(p_{n+m} - p_n) &\leqslant \sum_{i=n}^{m+n-1} M^*_U(p_{i+1} - p_i) \\ &\leqslant \sum_{i=n}^{m+n-1} h^i M^*_U(p_1 - p_0) \\ &\leqslant M^*_U(p_1 - p_0) \sum_{i=n}^{m+n-1} h^i \end{aligned}$$

$$\leqslant M^*_U(p_1 - p_0)\sum_{i=n}^{\infty} h^i$$
$$\leqslant M^*_U(p_1 - p_0)\frac{h^n}{1-h},$$

which implies that

$$\lim_{n\to\infty} M^*_U(p_{n+m} - x_n) = 0.$$

From the arbitrariness of $U \in \Omega$, we know that the iterative sequence $\{p_n\}$ is a Cauchy sequence. The completeness of (X, τ) implies the completeness of (X^N, τ^N) and hence there exists a point $p^* \in X^N$ such that $p_n \to p^*$ in the topology τ^N. That is

$$\lim_{n\to\infty} M^*_U(p_n - p^*) = 0.$$

By the triangle inequality, we have that

$$\begin{aligned} M^*_U(T^*p - p^*) &\leqslant M^*_U(T^*p - x_{n+1}) + M^*_U(x_{n+1} - p^*) \\ &= M^*_U(T^*p^* - T^*x_n) + M^*_U(x_{n+1} - p^*) \\ &= hM^*_U({}^*p - x_n) + M^*_U(x_{n+1} - p^*) \to 0, \end{aligned}$$

as $n \to \infty$, so that $M^*_U(T^*p^* - p^*) = 0$. From the arbitrariness of $U \in \Omega$, we know that, $T^*p^* = p^*$. Let $p^* = (p_1, p_2, p_3, \cdots, p_N)$, from the definition of T^*, we have that

$$\begin{aligned} (p_1, p_2, p_3, \cdots, p_N) &= T^*(p_1, p_2, p_3, \cdots, p_N) \\ &= (T(p_1, p_2, p_3, \cdots, p_N), T(p_1, p_2, p_3, \cdots, p_N), \cdots, T(p_1, p_2, p_3, \cdots, p_N)), \end{aligned}$$

which implies that

$$p_1 = p_2 = p_3 = \cdots = p_N = T(p_1, p_2, p_3, \cdots, p_N).$$

Therefore, there exists a point $p \in X$ such that

$$p = T(p, p, p, \cdots, p).$$

The element p is namely the multivariate fixed point of T. Next we prove the uniqueness of the multivariate fixed point of T. Assume there exists a point $q \in X$ such that

$$q = T(q, q, q, \cdots, q).$$

Then we have

$$\begin{aligned} M_U(q - p) &\leqslant h\Delta(M_U(q - p), M_U(q - p), \cdots, M_U(q - p)) \\ &= hM_U(q - p) \\ &= 0, \end{aligned}$$

which implies $q = p$.

From the definition of iterative sequence $\{p_n\}$, we know that

$$\begin{aligned} p_1 &= T^*p_0 = (Tp_0, Tp_0, \cdots, Tp_0), \\ p_2 &= T^*p_1 = (Tp_1, Tp_1, \cdots, Tp_1), \\ p_3 &= T^*p_2 = (Tp_2, Tp_2, \cdots, Tp_2), \\ &\vdots \\ p_{n+1} &= T^*p_n = (Tp_n, Tp_n, \cdots, Tp_n), \\ &\vdots \end{aligned}$$

converges, in the topological space (X^N, τ^N) to $p^* = (p, p, \cdots, p) \in X^N$ and the iterative sequence $\{Tp_n\} \subset X$ converges, in the topological space (X, τ), to $p \in X$. This completes the proof. □

4. The error estimate formulas

Under the condition of Theorem 3.4, we can get the following error estimate formulas. Let $m \to \infty$ in the following inequality

$$\begin{aligned} M_U^*(p_{n+m} - p_n) &\leqslant \sum_{i=n}^{m+n-1} M_U^*(p_{i+1} - p_i) \\ &\leqslant \sum_{i=n}^{m+n-1} h^i M_U^*(p_1 - p_0) \\ &\leqslant M_U^*(p_1 - p_0) \sum_{i=n}^{m+n-1} h^i \\ &\leqslant M_U^*(p_1 - p_0) \sum_{i=n}^{\infty} h^i \\ &\leqslant M_U^*(p_1 - p_0) \frac{h^n}{1-h}, \end{aligned}$$

we can get

$$M_U^*(p_n - p^*) \leqslant M_U^*(p_1 - p_0) \frac{h^n}{1-h}.$$

That is

$$\begin{aligned} &\Delta(M_U(Tp_{n-1} - p), M_U(Tp_{n-1} - p), \cdots, M_U(Tp_{n-1} - p)) \\ &\qquad \leqslant \Delta(M_U(Tp_0 - p_{0,1}), M_U(Tp_0 - p_{0,2}), \cdots, M_U(Tp_0 - p_{0,N})) \frac{h^n}{1-h}, \end{aligned}$$

where $p_0 = (p_{0,1}, p_{0,2}, \cdots, p_{0,N}) \in X^N$. From the definition of the multiply function Δ and above inequality, we have

$$M_U(Tp_{n-1} - p) \leqslant \Delta(M_U(Tp_0 - p_{0,1}), \cdots, M_U(Tp_0 - p_{0,N})) \frac{h^n}{1-h}.$$

From the definition of Minkowski functional $M_U(x)$, above inequality can be rewritten as

$$\inf\{t > 0 : Tp_{n-1} - p \in tU\} \leqslant \Delta(\inf\{t > 0 : Tp_0 - p_{0,1} \in tU\}, \cdots, \inf\{t > 0 : Tp_0 - p_{0,N} \in tU\}) \frac{h^n}{1-h}.$$

We can take positive real numbers $t_1, t_2, t_3, \cdots, t_N$ such that $Tp_0 - p_{0,i} \in t_i U,\ i = 1, 2, 3, \cdots, N$. It follows from the above inequality that

$$\inf\{t > 0 : Tp_{n-1} - p \in tU\} \leqslant \frac{h^n}{1-h} \Delta(t_1, t_2, t_3, \cdots, t_N).$$

Hence

$$Tp_{n-1} - p \in \frac{h^{n-1}}{1-h} \Delta(t_1, t_2, t_3, \cdots, t_N) U.$$

From above inequality, we can get the following error estimate formulas.

Error estimate formula 4.1. For any $U \in \Omega$, we take positive real numbers $t_1, t_2, t_3, \cdots, t_N$ such that $Tp_0 - p_{0,i} \in t_i U,\ i = 1, 2, 3, \cdots, N$. If

$$\frac{h^{n-1}}{1-h} \Delta(t_1, t_2, t_3, \cdots, t_N) \leqslant 1,$$

then

$$Tp_{n-1} - p \in U,$$

where p is the multivariate fixed point of T. Let $t_i = \dfrac{1-h}{h^{n-1}},\ i = 1, 2, 3, \cdots, N$, the error estimate formula

4.1 reduce to the following result.

Error estimate formula 4.2. For any $U \in \Omega$, we take n such that

$$Tp_0 - p_{0,i} \in \frac{1-h}{h^{n-1}}U, \quad i = 1,2,3,\cdots,N,$$

then

$$Tp_{n-1} - p \in U,$$

where p is the multivariate fixed point of T.

5. Application for equation of functions

Let $\Omega \subset R^m$ be a nonempty open set, let $K_n \subset \Omega$, $n = 1,2,3,\cdots$ be compact sets such that $K_n \subset K^0_{n+1}$, $\bigcup_{n=1}^{\infty} K_n = \Omega$, for all $n \geqslant 1$. Let $C(\Omega)$ denote the linear space of all real continuous functions defined on Ω. Let

$$p_n(f) = \sup_{x\in K_n} |f(x)|, \quad \forall\, f \in C(\Omega), \quad n = 1,2,3,\cdots,$$

it is easy to show that $\{p_n\}_{n=1}^{\infty}$ is a family of semi-norms such that $p_n(f) = 0$, for all $n = 1,2,3,\cdots$ implies $f(x) \equiv 0$. Hence $\{p_n\}_{n=1}^{\infty}$ generates a locally convex topology on the linear vector space $C(\Omega)$. Therefore $C(\Omega)$ is a complete metrizable topological vector space with the following basis of convex neighborhood of zero:

$$V_n = \{f \in C(\Omega) : p_n(x) < \frac{1}{n}\}, \quad n = 1,2,3,\cdots.$$

In addition, the convergence in topology is equivalent to the inner-closed uniformly convergence (see [12]).

We have the following conclusion.

Theorem 5.1. *Let $T : C(\Omega)^N \longrightarrow C(\Omega)$ be an N-variables mapping satisfying the following condition:*

$$\sup_{x\in K_n} |f_i(x) - g_i(x)| < \frac{t_i}{n},$$

implies there exists a real number t_0 such that

$$0 < t_0 < h\triangle(t_1,t_2,t_3,\cdots,t_N),$$

and

$$\sup_{x\in K_n} |Tf(x) - Tg(x)| < \frac{t_0 h}{n},$$

for all $f = (f_1,f_2,f_3,\cdots,f_N), g = (g_1,g_2,g_3,\cdots,g_N) \in C(\Omega)^N, n = 1,2,3,\cdots$ and all $t_i > 0$, $i = 1,2,3,\cdots,N$, where $0 < h < 1$ is a constant and Δ is a multiply metric function. Then T has a unique multivariate fixed point $f^*(x)$, that is, the equation of function $T(f(x),f(x),\cdots,f(x)) = f(x)$ has a unique solution $f^*(x)$. In addition, for any $f_0 \in X^N$, the iterative sequence $\{f_n\} \subset X^N$ defined by

$$\begin{aligned} f_1 &= (Tf_0,Tf_0,\cdots,Tf_0),\\ f_2 &= (Tf_1,Tf_1,\cdots,Tf_1),\\ f_3 &= (Tf_2,Tf_2,\cdots,Tf_2),\\ &\vdots\\ f_{n+1} &= (Tf_n,Tf_n,\cdots,Tf_n),\\ &\vdots \end{aligned}$$

converges, in the topological space $(C(\Omega)^N,\tau^N)$ to $(f,f,\cdots,f) \in X^N$ and the iterative sequence $\{Tf_n\} \subset C(\Omega)$ inner-closed uniformly converges to $f^* \in C(\Omega)$.

Proof. From the conditions of Theorem 5.1, we have that,

$$f_i - g_i \in t_i V_n,$$

implies there exists a real number t_0 such that

$$0 < t_0 \leqslant h\Delta(t_1, t_2, t_3, \cdots, t_N),$$

and

$$Tf - Tg \in t_0 h V_n,$$

for all $f = (f_1, f_2, f_3, \cdots, f_N)$, $g = (g_1, g_2, g_3, \cdots, g_N) \in C(\Omega)^N, n = 1, 2, 3, \cdots$ and all $t_i > 0$, $i = 1, 2, 3, \cdots, N$. By using Theorem 3.4, we get the conclusion of Theorem 5.1. This completes the proof. □

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