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Hybrid steepest-descent viscosity methods for triple hierarchical variational inequalities with constraints of mixed equilibria and bilevel variational inequalities

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Abstract

In this paper, we introduce and analyze a hybrid steepest-descent viscosity algorithm for solving the triple hierarchical variational inequality problem with constraints of two problems: one generalized mixed equilibrium problem and another bilevel variational inequality problem in a real Hilbert space. Under mild conditions, the strong convergence of the iteration sequences generated by the algorithm is established. Our results improve and extend the corresponding results in the earlier and recent literature.

Keywords: Hybrid steepest-descent viscosity method, triple hierarchical variational inequality, generalized mixed equilibrium problem.
2010 MSC: 49J30, 47H09, 47J20, 49M05.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H, and P_C be the metric projection of H onto C. If $\{x_n\}$ is a sequence in H, then we denote by $x_n \to x$ (respectively, $x_n \rightharpoonup x$) the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geqslant 0$ such that

$$\|Sx - Sy\| \leqslant L\|x - y\|, \quad \forall x, y \in C.$$

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In particular, if $L = 1$, then S is called a nonexpansive mapping; if $L \in [0, 1)$, then S is called a contraction.

Let $\mathcal{A} : C \to H$ be a nonlinear mapping. The classical variational inequality problem (VIP) ([10, 15]) is to find $x \in C$ such that

$$\langle \mathcal{A}x, y - x\rangle \geqslant 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $VI(C, \mathcal{A})$.

It is well-known that, if A is a strongly monotone and Lipschitz-continuous mapping on C, then VIP (1.1) has a unique solution. In 1976, Korpelevich [14] proposed the following extragradient method for solving the VIP (1.1) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_k = P_C(x_k - \tau A x_k), \\ x_{k+1} = P_C(x_k - \tau A y_k), \quad \forall k \geqslant 0, \end{cases}$$

with $\tau > 0$ a given number. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see e.g., [4, 5, 18, 26, 27, 31] and references therein.

Let $A : C \to H$ and $B : H \to H$ be two mappings. Consider the following bilevel variational inequality problem (BVIP):

Problem 1.1. We find $x^* \in VI(C, B)$ such that

$$\langle Ax^*, x - x^*\rangle \geqslant 0, \quad \forall x \in VI(C, B),$$

where $VI(C, B)$ denotes the set of solutions of the VIP: find $y^* \in C$ such that

$$\langle By^*, y - y^*\rangle \geqslant 0, \quad \forall y \in C.$$

In particular, whenever $H = \mathbf{R}^n$, the BVIP was recently studied by Anh et al. [1]. Bilevel variational inequalities are special classes of quasivariational inequalities ([9, 24]) and of equilibrium with equilibrium constraints considered in [13]. However it covers some classes of mathematical programs with equilibrium constraints, bilevel minimization problems ([19]), variational inequalities ([28, 30]) and complementarity problems.

In what follows, suppose that A and B satisfy the following conditions:

(C1) B is pseudomonotone on H and A is β-strongly monotone on C;
(C2) A is L_1-Lipschitz continuous on C;
(C3) B is L_2-Lipschitz continuous on H;
(C4) $VI(C, B) \neq \emptyset$.

In 2012, Anh et al. [1] introduced the following extragradient iterative algorithm for solving the above bilevel variational inequality.

Algorithm 1.2 ([1, Algorithm 2.1])**.** Initialization: choose $u \in \mathbf{R}^n$ and $x_0 \in C$.

Step 1. Compute $y_k := P_C(x_k - \lambda_k B x_k)$ and $z_k := P_C(x_k - \lambda_k B y_k)$.
Step 2. Inner loop $j = 0, 1, \ldots$. Compute

$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_C(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} + \gamma_j P_C(x_{k,j} - \delta_j B y_{k,j}). \\ \text{If } \|x_{k,j+1} - P_{VI(C,B)} x_{k,0}\| \leqslant \bar{\epsilon}_k \text{ then set } h_k := x_{k,j+1} \text{ and go to Step 3,} \\ \text{Otherwise, increase } j \text{ by 1 and repeat the inner loop Step 2.} \end{cases}$$

Step 3. Set $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. Then increase k by 1 and go to Step 1.

Theorem 1.3. *Suppose that the assumptions* (C1)-(C4) *hold. Then the two sequences* $\{x_k\}$ *and* $\{z_k\}$ *in Algorithm* 1.2 *converge to the same point* x^* *which is a solution of the BVI.*

Furthermore, let $\varphi : C \to \mathbf{R}$ be a real-valued function, $\mathcal{A} : H \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [18] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle \mathcal{A}x, y - x\rangle \geqslant 0, \quad \forall y \in C. \tag{1.2}$$

We denote the set of solutions of GMEP (1.2) by GMEP$(\Theta, \varphi, \mathcal{A})$. The GMEP (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others. The GMEP is further considered and studied; see e.g., [7]. In particular, if $\varphi = 0$, then GMEP (1.2) reduces to the generalized equilibrium problem (GEP) ([22]) which is to find $x \in C$ such that

$$\Theta(x, y) + \langle \mathcal{A}x, y - x\rangle \geqslant 0, \quad \forall y \in C.$$

The set of solutions of GEP is denoted by GEP$(\Theta, \mathcal{A})$.

If $\mathcal{A} = 0$, then GMEP (1.2) reduces to the mixed equilibrium problem (MEP) ([8]), which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C.$$

The set of solutions of MEP is denoted by MEP(Θ, φ).

If $\varphi = 0$, $\mathcal{A} = 0$, then GMEP (1.2) reduces to the equilibrium problem (EP) ([2, 21]), which is to find $x \in C$ such that

$$\Theta(x, y) \geqslant 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of EP is denoted by EP(Θ). It is worth to mention that the EP is a unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, etc.

It is assumed as in [18] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)-(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restrictions (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leqslant 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1 - t)x, y) \leqslant \Theta(x, y);$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

Given a positive number $r > 0$. Let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$T_r^{(\Theta,\varphi)}(x) := \{y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geqslant 0, \forall z \in C\}.$$

Recall that, a mapping $T : C \to C$ is said to be semicompact if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^*$. A mapping $T : C \to C$ is called a ζ-strictly pseudocontractive mapping (or a ζ-strict pseudocontraction) if there exists a constant $\zeta \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \zeta\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

In 2009, Ceng et al. [2] proposed an iterative scheme for finding a common element of the set of solutions of the EP (1.3) and the set of fixed points of a strictly pseudocontractive mapping in a real Hilbert space H. They established some weak and strong convergence theorems by combining the ideas of Marino and Xu's result [16] and Takahashi and Takahashi's result [21].

Recall the variational inequality for a monotone operator $A_1 : H \to H$ over the fixed point set of a nonexpansive mapping $T : H \to H$:

$$\text{find } \bar{x} \in VI(Fix(T), A_1) := \{\bar{x} \in Fix(T) : \langle A_1\bar{x}, y - \bar{x}\rangle \geqslant 0, \forall y \in Fix(T)\},$$

where $Fix(T) := \{x \in H : Tx = x\} \neq \emptyset$. In [12], Iiduka introduced the following three-stage variational inequality problem, that is, the following monotone variational inequality with variational inequality constraint over the fixed point set of a nonexpansive mapping.

Problem 1.4. Assume that

(i) $T : H \to H$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$;
(ii) $A_1 : H \to H$ is α-inverse strongly monotone;
(iii) $A_2 : H \to H$ is β-strongly monotone and L-Lipschitz continuous;
(iv) $VI(Fix(T), A_1) \neq \emptyset$.

Then the objective is to

$$\text{find } x^* \in VI(VI(Fix(T), A_1), A_2) := \{x^* \in VI(Fix(T), A_1) : \langle A_2x^*, v - x^*\rangle \geqslant 0, \forall v \in VI(Fix(T), A_1)\}.$$

Since this problem has a triple structure in contrast with bilevel programming problems ([17]) or hierarchical constrained optimization problems or hierarchical fixed point problem, it is referred to as a triple-hierarchical constrained optimization problem (THCOP). More precisely, it is referred as a triple hierarchical variational inequality problem (THVIP); see Ceng et al. [6]. Very recently, some authors continued the study of Iiduka's THVIP (i.e., Problem 1.4) and its variant and extension; see e.g., [3, 6, 29].

Algorithm 1.5 ([12, Algorithm 4.1])**.** Let $T : H \to H$ and $A_i : H \to H$ $(i = 1, 2)$ satisfy the assumptions (i)-(iv) in Problem 1.4. The following steps are presented for solving Problem 1.4.

Step 0. Take $\{\alpha_k\}_{k=0}^{\infty}, \{\lambda_k\}_{k=0}^{\infty} \subset (0, \infty)$, and $\mu > 0$, choose $x_0 \in H$ arbitrarily, and let $k := 0$.

Step 1. Given $x_k \in H$, compute $x_{k+1} \in H$ as

$$\begin{cases} y_k := T(x_k - \lambda_k A_1 x_k), \\ x_{k+1} := y_k - \mu\alpha_k A_2 y_k. \end{cases}$$

Update $k := k + 1$ and go to Step 1.

Theorem 1.6 ([12, Theorem 4.1])**.** *Assume that $\{y_k\}_{k=0}^{\infty}$ in Algorithm 1.5 is bounded. If $\mu \in (0, \frac{2\beta}{L^2})$ is used and if $\{\alpha_k\}_{k=0}^{\infty} \subset (0, 1]$ and $\{\alpha_k\}_{k=0}^{\infty} \subset (0, 2\alpha]$ satisfying* (i) $\lim_{k\to\infty} \alpha_k = 0$, (ii) $\sum_{k=0}^{\infty} \alpha_k = \infty$, (iii) $\sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$, (iv) $\sum_{k=0}^{\infty} |\lambda_{k+1} - \lambda_k| < \infty$, *and* (v) $\lambda_k \leqslant \alpha_k$ $\forall k \geqslant 0$, *are used, then the sequence $\{x_k\}_{k=0}^{\infty}$ generated by Algorithm 1.5 satisfies the following properties:*

(a) *$\{x_k\}_{k=0}^{\infty}$ is bounded;*
(b) *$\lim_{k\to\infty} \|x_k - y_k\| = 0$ and $\lim_{k\to\infty} \|x_k - Tx_k\| = 0$ hold;*
(c) *If $\|x_k - y_k\| = o(\lambda_k)$, $\{x_k\}_{k=0}^{\infty}$ converges strongly to the unique solution of Problem 1.4.*

In this paper, we introduce and analyze a hybrid steepest-descent viscosity algorithm for solving the triple hierarchical variational inequality problem (THVIP) (for a strict pseudocontraction) with constraints of the GMEP (1.2) and the bilevel variational inequality problem (BVIP) in a real Hilbert space. The proposed algorithm is based on Korpelevich's extragradient method, Mann's iteration method, hybrid steepest-descent method (see [25]) and viscosity approximation method (see [21]) (including Halpern's iteration method). Under mild conditions, the strong convergence of the iteration sequences generated by the algorithm is derived. Our results improve and extend the corresponding results announced by some others, e.g., Iiduka [12, Theorem 4.1], Ceng et al. [2, Theorems 3.1-3.3], and Anh et al. [1, Theorem 3.1].

2. Preliminaries

Throughout this paper, we assume that C is a nonempty closed convex subset of a real Hilbert space H. We use $\omega_w(x_k)$ to denote the weak ω-limit set of the sequence $\{x_k\}$, i.e.,

$$\omega_w(x_k) := \{x \in H : x_{k_i} \rightharpoonup x \text{ for some subsequence } \{x_{k_i}\} \text{ of } \{x_k\}\}.$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if
$$\langle Ax - Ay, x - y\rangle \geqslant 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that
$$\langle Ax - Ay, x - y\rangle \geqslant \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that
$$\langle Ax - Ay, x - y\rangle \geqslant \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *For given* $x \in H$ *and* $z \in C$*:*

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \leqslant 0, \ \forall y \in C$;
(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \leqslant \|x - y\|^2 - \|y - z\|^2, \ \forall y \in C$;
(iii) $\langle P_C x - P_C y, x - y\rangle \geqslant \|P_C x - P_C y\|^2, \ \forall y \in H$.

Consequently, P_C *is nonexpansive and monotone.*

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda A)u - (I - \lambda A)v\|^2 = \|(u - v) - \lambda(Au - Av)\|^2 \leqslant \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \tag{2.1}$$

So, if $\lambda \leqslant 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Definition 2.2. A mapping $T : H \to H$ is said to be:

(a) nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in H$;

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geqslant \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = \frac{1}{2}(I + S)$, where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Next we list some elementary conclusions for the MEP.

Proposition 2.3 ([8]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)-(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \forall y \in C\},$$

for all $x \in H$. Then the followings hold:

(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\|^2 \leqslant \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\rangle;$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(iv) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex;*

(v) $\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leqslant \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ *for all $s, t > 0$ and $x \in H$.*

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.4. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.5. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2.1 (i)) implies*

$$u \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad u = P_C(u - \lambda A u), \quad \forall \lambda > 0.$$

Lemma 2.6 ([11, Demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C with $\mathrm{Fix}(S) \neq \emptyset$. Then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Lemma 2.7 ([16, Proposition 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a mapping.*

(i) *If T is a ζ-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \leqslant \frac{1 + \zeta}{1 - \zeta}\|x - y\|, \quad \forall x, y \in C.$$

(ii) *If T is a ζ-strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - T)x_n \to 0$, then $(I - T)\tilde{x} = 0$.*

(iii) *If T is ζ-(quasi-)strict pseudocontraction, then the fixed-point set $\mathrm{Fix}(T)$ of T is closed and convex so that the projection $P_{\mathrm{Fix}(T)}$ is well-defined.*

Lemma 2.8 ([26]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a ζ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma+\delta)\zeta \leqslant \gamma$. Then*

$$\|\gamma(x-y)+\delta(Tx-Ty)\| \leqslant (\gamma+\delta)\|x-y\|, \quad \forall x,y \in C.$$

The following lemma can be easily proven, and therefore, we omit the proof.

Lemma 2.9. *Let $V : H \to H$ be an l-Lipschitzian mapping and $F : H \to H$ be an η-strongly monotone mapping. If $\mu\eta - \gamma l > 0$ for $\mu, \gamma \geqslant 0$, then $\mu F - \gamma V$ is $(\mu\eta - \gamma l)$-strongly monotone, that is,*

$$\langle(\mu F-\gamma V)x-(\mu F-\gamma V)y, x-y\rangle \geqslant (\mu\eta-\gamma l)\|x-y\|^2, \quad \forall x,y \in H.$$

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $S : C \to H$, we define the mapping $S^\lambda : C \to H$ by

$$S^\lambda x := Sx - \lambda\mu F(Sx), \quad \forall x \in C,$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\|Fx-Fy\| \leqslant \kappa\|x-y\| \quad \text{and} \quad \langle Fx-Fy, x-y\rangle \geqslant \eta\|x-y\|^2,$$

for all $x, y \in H$.

Lemma 2.10 ([23]). *S^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|S^\lambda x - S^\lambda y\| \leqslant (1-\lambda\tau)\|x-y\|, \quad \forall x,y \in C,$$

where $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$.

Lemma 2.11 ([23]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leqslant (1-\alpha_n)a_n + \alpha_n\beta_n, \quad \forall n \geqslant 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) *$\{\alpha_n\} \subset [0,1]$ and $\sum_{n=0}^\infty \alpha_n = \infty$, or equivalently, $\prod_{n=0}^\infty(1-\alpha_n) := \lim_{n\to\infty}\prod_{k=0}^n(1-\alpha_k) = 0$;*
(ii) *$\limsup_{n\to\infty}\beta_n \leqslant 0$, or $\sum_{n=0}^\infty|\alpha_n\beta_n| < \infty$.*

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 ([11]). *Let H be a real Hilbert space. Then the followings hold:*

(a) *$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ for all $x, y \in H$;*
(b) *$\|\lambda x+\mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ for all $x, y \in H$ and $\lambda, \mu \in [0,1]$ with $\lambda+\mu = 1$;*
(c) *If $\{x_k\}$ is a sequence in H such that $x_k \rightharpoonup x$, it follows that*

$$\limsup_{k\to\infty}\|x_k-y\|^2 = \limsup_{k\to\infty}\|x_k-x\|^2 + \|x-y\|^2, \quad \forall y \in H.$$

3. Iterative algorithm and convergence criteria

Let C be a nonempty closed convex subset of a real Hilbert space H. In this section, we always assume the followings:

- $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$;

- $T : H \to H$ is a ζ-strictly pseudocontractive mapping and $V : H \to H$ is an l-Lipschitzian mapping;
- $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)-(A4), $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function with restrictions (B1) or (B2), and $\mathcal{A} : H \to H$ is α-inverse strongly monotone;
- $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leqslant \gamma l < \tau$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $A : C \to H$ and $B : H \to H$ are two mappings satisfying the following hypotheses (H1)-(H4):
  (H1) B is monotone on H;
  (H2) A is β-inverse-strongly monotone on C;
  (H3) B is L_2-Lipschitz continuous on H;
  (H4) $\mathrm{VI}(\mathrm{VI}(\Omega, B), A) \neq \emptyset$ where $\Omega := \mathrm{GMEP}(\Theta, \varphi, \mathcal{A}) \cap \mathrm{Fix}(T)$.

Next, we introduce the following triple hierarchical variational inequality problem (THVIP) with constraints of the GMEP (1.2) and the bilevel variational inequality problem (BVIP).

Problem 3.1. The objective is to

$$\text{find } x^* \in \mathrm{VI}(\mathrm{VI}(\Omega, B), A) := \{x^* \in \mathrm{VI}(\Omega, B) : \langle Ax^*, x - x^* \rangle \geqslant 0, \forall x \in \mathrm{VI}(\Omega, B)\}.$$

That is, the objective is to find $x^* \in \mathrm{VI}(\Omega, B)$ such that

$$\langle Ax^*, x - x^* \rangle \geqslant 0, \quad \forall x \in \mathrm{VI}(\Omega, B),$$

where $\mathrm{VI}(\Omega, B)$ denotes the set of solutions of the VIP: find $y^* \in \Omega$ such that

$$\langle By^*, y - y^* \rangle \geqslant 0, \quad \forall y \in \Omega.$$

Algorithm 3.2. Initialization: choose $u \in H$, $x_0 \in H$, $k = 0$, $0 < \lambda \leqslant 2\beta$, positive sequences $\{\delta_k\}$, $\{\lambda_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, $\{\bar{\epsilon}_k\}$, and $\{r_k\}$ such that

$$\begin{cases} \lim\limits_{k\to\infty} \delta_k = 0, \ \sum\limits_{k=0}^{\infty} \bar{\epsilon}_k < \infty, \ \{r_k\} \subset [a, b] \subset (0, 2\alpha), \\ \alpha_k + \beta_k + \gamma_k = 1 \ \forall k \geqslant 0, \ \sum\limits_{k=0}^{\infty} \alpha_k = \infty, \ \lim\limits_{k\to\infty} |r_{k+1} - r_k| = 0, \\ \lim\limits_{k\to\infty} \alpha_k = 0, \ \bar{\epsilon}_k = o(\alpha_k), \ \lim\limits_{k\to\infty} \beta_k = \xi \in (\zeta, \frac{1}{2}], \ \lim\limits_{k\to\infty} \lambda_k = 0, \ \lambda_k \leqslant \frac{1}{L_2} \ \ \forall k \geqslant 0. \end{cases}$$

Step 1. Compute

$$\begin{cases} \Theta(u_k, y) + \varphi(y) - \varphi(u_k) + \frac{1}{r_k}\langle y - u_k, u_k - (x_k - r_k\mathcal{A}x_k)\rangle \geqslant 0, \quad \forall y \in C, \\ \tilde{u}_k = \beta_k u_k + (1 - \beta_k)Tu_k, \\ v_k = \alpha_k\gamma V x_k + \gamma_k x_k + ((1 - \gamma_k)I - \alpha_k\mu F)\tilde{u}_k, \\ y_k := P_\Omega(v_k - \lambda_k B v_k), \\ z_k := P_\Omega(v_k - \lambda_k B y_k). \end{cases}$$

Step 2. Inner loop $j = 0, 1, \ldots.$ Compute

$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_\Omega(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} + \gamma_j P_\Omega(x_{k,j} - \delta_j B y_{k,j}). \\ \text{If } \|x_{k,j+1} - P_{\mathrm{VI}(\Omega,B)} x_{k,0}\| \leqslant \bar{\epsilon}_k \text{ then set } h_k := x_{k,j+1} \text{ and go to Step 3.} \\ \text{Otherwise, increase } j \text{ by 1 and repeat the inner loop Step 2.} \end{cases}$$

Step 3. Set $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. Then increase k by 1 and go to Step 1.

Let C be a nonempty closed convex subset of H, $B : C \to H$ be monotone and L_2-Lipschitz continuous on C, and $S : C \to C$ be a nonexpansive mapping such that $VI(C, B) \cap Fix(S) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_k = P_C(x_k - \delta_k Bx_k), \\ x_{k+1} = \alpha_k x_0 + \beta_k x_k + \gamma_k SP_C(x_k - \delta_k By_k) \quad \forall k \geqslant 0, \end{cases}$$

where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$, and $\{\delta_k\}$ satisfy the following conditions:

$$\begin{cases} \delta_k > 0 \ \ \forall k \geqslant 0, \ \lim_{k\to\infty} \delta_k = 0, \\ \alpha_k + \beta_k + \gamma_k = 1, \ \ \forall k \geqslant 0, \\ \sum_{k=0}^{\infty} \alpha_k = \infty, \ \lim_{k\to\infty} \alpha_k = 0, \\ 0 < \liminf_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \beta_k < 1. \end{cases}$$

Under these conditions, Yao et al. [27] proved that the sequences $\{x_k\}$ and $\{y_k\}$ converge to the same point $P_{VI(C,B)\cap Fix(S)} x_0$.

Applying these iteration sequences with S being the identity mapping, we have the following lemma.

Lemma 3.3. *Suppose that the hypotheses* (H1)-(H4) *hold. Then the sequence* $\{x_{k,j}\}$ *generated by Algorithm* 3.2 *converges strongly to the point* $P_{VI(\Omega,B)}(z_k - \lambda Az_k)$ *as* $j \to \infty$. *Consequently, we have*

$$\|h_k - P_{VI(\Omega,B)}(z_k - \lambda Az_k)\| \leqslant \bar{\epsilon}_k \quad \forall k \geqslant 0.$$

In the sequel we always suppose that the inner loop in Algorithm 3.2 terminates after a finite number of steps. This assumption, by Lemma 3.3, is satisfied when B is monotone on Ω.

Lemma 3.4. *Let sequences* $\{v_k\}, \{y_k\}$, *and* $\{z_k\}$ *be generated by Algorithm* 3.2, B *be* L_2*-Lipschitzian and monotone on* H, *and* $p \in VI(\Omega, B)$. *Then, we have*

$$\|z_k - p\|^2 \leqslant \|v_k - p\|^2 - (1 - \lambda_k L_2)\|v_k - y_k\|^2 - (1 - \lambda_k L_2)\|y_k - z_k\|^2. \tag{3.1}$$

Proof. Let $p \in VI(\Omega, B)$. This means $\langle Bp, x - p\rangle \geqslant 0, \forall x \in \Omega$. Then, for each $\lambda_k > 0$, p satisfies the fixed point equation $p = P_\Omega(p - \lambda_k Bp)$. Since B is monotone on H and $p \in VI(\Omega, B)$, we have

$$\langle By_k, y_k - p\rangle \geqslant \langle Bp, y_k - p\rangle \geqslant 0.$$

Then, applying Proposition 2.1 (ii) with $v_k - \lambda_k By_k$ and p, we obtain

$$\begin{aligned} \|z_k - p\|^2 &\leqslant \|v_k - \lambda_k By_k - p\|^2 - \|v_k - \lambda_k By_k - z_k\|^2 \\ &= \|v_k - p\|^2 - 2\lambda_k\langle By_k, v_k - p\rangle + \lambda_k^2\|By_k\|^2 - \|v_k - z_k\|^2 \\ &\quad - \lambda_k^2\|By_k\|^2 + 2\lambda_k\langle By_k, v_k - z_k\rangle \\ &= \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, p - z_k\rangle \\ &= \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, p - y_k\rangle + 2\lambda_k\langle By_k, y_k - z_k\rangle \\ &\leqslant \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, y_k - z_k\rangle. \end{aligned} \tag{3.2}$$

Applying Proposition 2.1 (i) with $v_k - \lambda_k Bv_k$ and z_k, we also have

$$\langle v_k - \lambda_k Bv_k - y_k, z_k - y_k\rangle \leqslant 0.$$

Combining this inequality with (3.2) and observing that B is L_2-Lipschitz continuous on H, we obtain

$$\begin{aligned}\|z_k-p\|^2 &\leqslant \|v_k-p\|^2-\|(v_k-y_k)+(y_k-z_k)\|^2+2\lambda_k\langle By_k,y_k-z_k\rangle\\ &=\|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2-2\langle v_k-y_k,y_k-z_k\rangle+2\lambda_k\langle By_k,y_k-z_k\rangle\\ &=\|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2-2\langle v_k-\lambda_k By_k-y_k,y_k-z_k\rangle\\ &=\|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2-2\langle v_k-\lambda_k Bv_k-y_k,y_k-z_k\rangle\\ &\quad+2\lambda_k\langle Bv_k-By_k,z_k-y_k\rangle\\ &\leqslant \|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2+2\lambda_k\langle Bv_k-By_k,z_k-y_k\rangle\\ &\leqslant \|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2+2\lambda_k\|Bv_k-By_k\|\|z_k-y_k\|\\ &\leqslant \|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2+2\lambda_k L_2\|v_k-y_k\|\|z_k-y_k\|\\ &\leqslant \|v_k-p\|^2-\|v_k-y_k\|^2-\|y_k-z_k\|^2+\lambda_k L_2(\|v_k-y_k\|^2+\|z_k-y_k\|^2)\\ &\leqslant \|v_k-p\|^2-(1-\lambda_k L_2)\|v_k-y_k\|^2-(1-\lambda_k L_2)\|y_k-z_k\|^2.\end{aligned}\tag{3.3}$$

Utilizing (2.1) and Proposition 2.3 (ii) we have $u_k=T_{r_k}^{(\Theta,\varphi)}(x_k-r_k\mathcal{A}x_k)$ for each $k\geqslant 0$ and hence

$$\begin{aligned}\|u_k-p\|^2 &= \|T_{r_k}^{(\Theta,\varphi)}(x_k-r_k\mathcal{A}x_k)-T_{r_k}^{(\Theta,\varphi)}(p-r_k\mathcal{A}p)\|^2\\ &\leqslant \|(x_k-r_k\mathcal{A}x_k)-(p-r_k\mathcal{A}p)\|^2\\ &\leqslant \|x_k-p\|^2+r_k(r_k-2\alpha)\|\mathcal{A}x_k-\mathcal{A}p\|^2\\ &\leqslant \|x_k-p\|^2.\end{aligned}\tag{3.4}$$

Since $\lim_{k\to\infty}\beta_k=\xi\in(\zeta,\frac{1}{2}]$, we may assume, without loss of generality, that $\{\beta_k\}\subset[c,d]\subset(\zeta,1)$ for all $k\geqslant 0$. Putting $\sigma_k=1-\beta_k$ and $\mathcal{B}=I-T$, we know that $\mathcal{B}$ is $\frac{1-\zeta}{2}$-inverse-strongly monotone since T is ζ-strictly pseudocontractive. Observe that $\tilde{u}_k=\beta_k u_k+(1-\beta_k)Tu_k=u_k-\sigma_k\mathcal{B}u_k$, which together with (2.1), yields

$$\begin{aligned}\|\tilde{u}_k-p\|^2 &= \|u_k-\sigma_k\mathcal{B}u_k-(p-\sigma_k\mathcal{B}p)\|^2\\ &= \|u_k-p-\sigma_k(\mathcal{B}u_k-\mathcal{B}p)\|^2\\ &\leqslant \|u_k-p\|^2-\sigma_k(1-\zeta-\sigma_k)\|\mathcal{B}u_k-\mathcal{B}p\|^2\\ &= \|u_k-p\|^2-(1-\beta_k)(\beta_k-\zeta)\|u_k-Tu_k\|^2.\end{aligned}\tag{3.5}$$

So, it follows from (3.4) and (3.5) that

$$\|\tilde{u}_k-p\|\leqslant\|u_k-p\|\leqslant\|x_k-p\|.\tag{3.6}$$

This completes the proof. □

Lemma 3.5. *Suppose that the conditions* (A1)-(A4) *and* (H1)-(H4) *hold and that the conditions* (B1) *or* (B2) *hold. Then the sequence* $\{x_k\}$ *generated by Algorithm 3.2 is bounded.*

Proof. Since $\lim_{k\to\infty}\alpha_k=0$, $\lim_{k\to\infty}\beta_k=\xi\in(\zeta,\frac{1}{2}]$ and $\alpha_k+\beta_k+\gamma_k=1$, we get $\lim_{k\to\infty}(1-\gamma_k)=\lim_{k\to\infty}(\alpha_k+\beta_k)=\xi$. Hence, we may assume, without loss of generality, that $0<\frac{\alpha_k}{1-\gamma_k}\leqslant 1$ for all $k\geqslant 0$.

Take an arbitrary $p\in VI(VI(B,\Omega),A)$. Then we have

$$\langle Ap,x-p\rangle\geqslant 0,\quad \forall x\in VI(\Omega,B),$$

which implies $p=P_{VI(\Omega,B)}(p-\lambda Ap)$. Then, it follows from (2.1), Proposition 2.1 (iii), β-inverse strong monotonicity of A, and $0<\lambda\leqslant 2\beta$ that

$$\begin{aligned}\|P_{VI(\Omega,B)}(z_k-\lambda Az_k)-p\|^2 &= \|P_{VI(\Omega,B)}(z_k-\lambda Az_k)-P_{VI(\Omega,B)}(p-\lambda Ap)\|^2\\ &\leqslant \|(I-\lambda A)z_k-(I-\lambda A)p\|^2\\ &\leqslant \|z_k-p\|^2+\lambda(\lambda-2\beta)\|Az_k-Ap\|^2\\ &\leqslant \|z_k-p\|^2.\end{aligned}\tag{3.7}$$

Furthermore, from Algorithm 3.2, Lemma 2.10, and (3.6), we have

$$\begin{aligned}
\|v_k - p\| &= \|\alpha_k\gamma V x_k + \gamma_k x_k + ((1-\gamma_k)I - \alpha_k\mu F)\tilde{u}_k - p\| \\
&= \|\alpha_k(\gamma V x_k - \mu F p) + \gamma_k(x_k - p) + ((1-\gamma_k)I - \alpha_k\mu F)\tilde{u}_k - ((1-\gamma_k)I - \alpha_k\mu F)p\| \\
&\leqslant \alpha_k\gamma l\|x_k - p\| + \alpha_k\|(\gamma V - \mu F)p\| + \gamma_k\|x_k - p\| \\
&\quad + \|((1-\gamma_k)I - \alpha_k\mu F)\tilde{u}_k - ((1-\gamma_k)I - \alpha_k\mu F)p\| \\
&\leqslant \alpha_k\gamma l\|x_k - p\| + \alpha_k\|(\gamma V - \mu F)p\| + \gamma_k\|x_k - p\| + (1-\gamma_k)(1 - \frac{\alpha_k}{1-\gamma_k}\tau)\|\tilde{u}_k - p\| \\
&\leqslant \alpha_k\gamma l\|x_k - p\| + \alpha_k\|(\gamma V - \mu F)p\| + \gamma_k\|x_k - p\| + (1 - \gamma_k - \alpha_k\tau)\|x_k - p\| \\
&= (1 - \alpha_k(\tau - \gamma l))\|x_k - p\| + \alpha_k(\tau - \gamma l)\frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l} \\
&\leqslant \max\{\|x_k - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\}.
\end{aligned} \tag{3.8}$$

Utilizing (3.3), (3.7), (3.8), and the assumptions $0 < \lambda \leqslant 2\beta$, $\sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty$ we obtain that

$$\begin{aligned}
\|x_{k+1} - p\| &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\| \\
&\leqslant \alpha_k\|u - p\| + \beta_k\|x_k - p\| + \gamma_k\|h_k - P_{VI(\Omega,B)}(z_k - \lambda A z_k)\| + \gamma_k\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - p\| \\
&\leqslant \alpha_k\|u - p\| + \beta_k\|x_k - p\| + \gamma_k\bar{\epsilon}_k + \gamma_k\|z_k - p\| \\
&\leqslant \alpha_k\|u - p\| + \beta_k\|x_k - p\| + \gamma_k\bar{\epsilon}_k + \gamma_k\|v_k - p\| \\
&\leqslant \alpha_k\|u - p\| + \beta_k\|x_k - p\| + \gamma_k\bar{\epsilon}_k + \gamma_k\max\{\|x_k - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} \\
&\leqslant \alpha_k\|u - p\| + (1 - \alpha_k)\max\{\|x_k - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} + \bar{\epsilon}_k \\
&\leqslant \max\{\|x_k - p\|, \|u - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} + \bar{\epsilon}_k \\
&\leqslant \max\{\|x_0 - p\|, \|u - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\} + \sum_{k=0}^{\infty}\bar{\epsilon}_k < \infty,
\end{aligned}$$

which shows that the sequence $\{x_k\}$ is bounded, and so are the sequences $\{u_k\}, \{\tilde{u}_k\}, \{v_k\}, \{y_k\}$, and $\{z_k\}$. □

Lemma 3.6 ([20]). *Let $\{x_k\}$ and $\{y_k\}$ be two bounded sequences in a real Banach space X. Let $\{\beta_k\}$ be a sequence in $[0, 1]$. Suppose that $\liminf_{k\to\infty}\beta_k \leqslant \limsup_{k\to\infty}\beta_k < 1$, $x_{k+1} = (1-\beta_k)y_k + \beta_k x_k$ and*

$$\limsup_{k\to\infty}(\|y_{k+1} - y_k\| - \|x_{k+1} - x_k\|) \leqslant 0.$$

Then, $\lim_{k\to\infty}\|y_k - x_k\| = 0$.

Lemma 3.7. *Suppose that the conditions (A1)-(A4) and (H1)-(H4) hold. Assume that the conditions (B1) or (B2) hold and that the sequences $\{v_k\}$ and $\{z_k\}$ are generated by Algorithm 3.2. Then, we have*

$$\|z_{k+1} - z_k\| \leqslant (1 + \lambda_{k+1}L_2)\|v_{k+1} - v_k\| + \lambda_k\|By_k\| + \lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|). \tag{3.9}$$

Moreover, $\{z_k\}$ is bounded and $\lim_{k\to\infty}\|z_{k+1} - z_k\| = \lim_{k\to\infty}\|v_{k+1} - v_k\| = 0$.

Proof. Since B is L_2-Lipschitzian on H, for each $x, y \in H$, we have

$$\|(I - \lambda_k B)x - (I - \lambda_k B)y\| \leqslant \|x - y\| + \lambda_k\|Bx - By\| \leqslant (1 + \lambda_k L_2)\|x - y\|.$$

Combining this inequality with Proposition 2.1 (iii), we have

$$\begin{aligned}\|z_{k+1}-z_k\| &= \|P_C(v_{k+1}-\lambda_{k+1}By_{k+1})-P_C(v_k-\lambda_k By_k)\|\\ &\leqslant \|(v_{k+1}-\lambda_{k+1}By_{k+1})-v_k+\lambda_k By_k\|\\ &= \|(v_{k+1}-\lambda_{k+1}Bv_{k+1})-(v_k-\lambda_{k+1}Bv_k)+\lambda_{k+1}(Bv_{k+1}-By_{k+1}-Bv_k)+\lambda_k By_k\|\\ &\leqslant (1+\lambda_{k+1}L_2)\|v_{k+1}-v_k\|+\lambda_k\|By_k\|+\lambda_{k+1}(\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|).\end{aligned}$$

This is the desired result (3.9).

Now we denote $x_{k+1}=(1-\beta_k)w_k+\beta_k x_k$. Then, we have

$$\begin{aligned}w_{k+1}-w_k &= \frac{\alpha_{k+1}u+\gamma_{k+1}h_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k u+\gamma_k h_k}{1-\beta_k}\\ &= \Big(\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\Big)u+\Big(\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big)h_k+\frac{\gamma_{k+1}}{1-\beta_{k+1}}(h_{k+1}-h_k).\end{aligned} \tag{3.10}$$

Note that, for $0<\lambda\leqslant 2\beta$, we have from (2.1) that

$$\begin{aligned}\|P_{VI(\Omega,B)}(z_{k+1}-\lambda Az_{k+1})-P_{VI(\Omega,B)}(z_k-\lambda Az_k)\|^2 &\leqslant \|(I-\lambda A)z_{k+1}-(I-\lambda A)z_k\|^2\\ &\leqslant \|z_{k+1}-z_k\|^2+\lambda(\lambda-2\beta)\|Az_{k+1}-Az_k\|^2\\ &\leqslant \|z_{k+1}-z_k\|^2.\end{aligned}$$

Then, combining (3.9) with $\lambda\leqslant 2\beta$ and (3.10) we get

$$\begin{aligned}&\|w_{k+1}-w_k\|-\|x_{k+1}-x_k\|\\ &\leqslant \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\Big|\|u\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|\|h_k\|\\ &\quad+\frac{\gamma_{k+1}}{1-\beta_{k+1}}\|h_{k+1}-h_k\|-\|x_{k+1}-x_k\|\\ &\leqslant \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\Big|\|u\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|(\|P_{VI(\Omega,B)}(z_k-\lambda Az_k)\|+\bar{\epsilon}_k)\\ &\quad+\frac{\gamma_{k+1}}{1-\beta_{k+1}}\|P_{VI(\Omega,B)}(z_{k+1}-\lambda Az_{k+1})-P_{VI(\Omega,B)}(z_k-\lambda Az_k)\|\\ &\quad+\frac{\gamma_{k+1}}{1-\beta_{k+1}}(\|P_{VI(\Omega,B)}(z_{k+1}-\lambda Az_{k+1})-h_{k+1}\|\\ &\quad+\|P_{VI(\Omega,B)}(z_k-\lambda Az_k)-h_k\|)-\|x_{k+1}-x_k\|\\ &\leqslant \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\Big|\|u\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|(\|P_{VI(\Omega,B)}(z_k-\lambda Az_k)\|+\bar{\epsilon}_k)\\ &\quad+\frac{\gamma_{k+1}}{1-\beta_{k+1}}\|z_{k+1}-z_k\|+\frac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1}+\bar{\epsilon}_k)-\|x_{k+1}-x_k\|.\end{aligned}$$

Hence,

$$\begin{aligned}&\|w_{k+1}-w_k\|-\|x_{k+1}-x_k\|\\ &\leqslant \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\Big|\|u\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|(\|P_{VI(\Omega,B)}(z_k-\lambda Az_k)\|+\bar{\epsilon}_k)\\ &\quad+\frac{\gamma_{k+1}(1+\lambda_{k+1}L_2)}{1-\beta_{k+1}}\|v_{k+1}-v_k\|+\frac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1}+\bar{\epsilon}_k)\\ &\quad+\frac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\|)-\|x_{k+1}-x_k\|\\ &\leqslant \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k}{1-\beta_k}\Big|\|u\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|(\|P_{VI(\Omega,B)}(z_k-\lambda Az_k)\|+\bar{\epsilon}_k)\\ &\quad+\frac{\gamma_{k+1}(1+\lambda_{k+1}L_2)}{1-\beta_{k+1}}\|v_{k+1}-v_k\|-\|x_{k+1}-x_k\|+\frac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1}+\bar{\epsilon}_k)\\ &\quad+\frac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\|).\end{aligned} \tag{3.11}$$

On the other hand, utilizing Proposition 2.3 (ii), (v), we obtain

$$\begin{aligned}
\|u_{k+1}-u_k\| &= \|T^{(\Theta,\varphi)}_{r_{k+1}}(I-r_{k+1}\mathcal{A})x_{k+1}-T^{(\Theta,\varphi)}_{r_k}(I-r_k\mathcal{A})x_k\| \\
&\leqslant \|T^{(\Theta,\varphi)}_{r_{k+1}}(I-r_{k+1}\mathcal{A})x_{k+1}-T^{(\Theta,\varphi)}_{r_k}(I-r_k\mathcal{A})x_{k+1}\| \\
&\quad + \|T^{(\Theta,\varphi)}_{r_k}(I-r_k\mathcal{A})x_{k+1}-T^{(\Theta,\varphi)}_{r_k}(I-r_k\mathcal{A})x_k\| \\
&\leqslant \|T^{(\Theta,\varphi)}_{r_{k+1}}(I-r_{k+1}\mathcal{A})x_{k+1}-T^{(\Theta,\varphi)}_{r_k}(I-r_{k+1}\mathcal{A})x_{k+1}\| \\
&\quad + \|T^{(\Theta,\varphi)}_{r_k}(I-r_{k+1}\mathcal{A})x_{k+1}-T^{(\Theta,\varphi)}_{r_k}(I-r_k\mathcal{A})x_{k+1}\| \\
&\quad + \|(I-r_k\mathcal{A})x_{k+1}-(I-r_k\mathcal{A})x_k\| \\
&\leqslant \frac{|r_{k+1}-r_k|}{r_{k+1}}\|T^{(\Theta,\varphi)}_{r_{k+1}}(I-r_{k+1}\mathcal{A})x_{k+1}-(I-r_{k+1}\mathcal{A})x_{k+1}\| \\
&\quad + |r_{k+1}-r_k|\|\mathcal{A}x_{k+1}\|+\|x_{k+1}-x_k\| \\
&= |r_{k+1}-r_k|\left[\|\mathcal{A}x_{k+1}\|+\frac{1}{r_{k+1}}\|T^{(\Theta,\varphi)}_{r_{k+1}}(I-r_{k+1}\mathcal{A})x_{k+1}-(I-r_{k+1}\mathcal{A})x_{k+1}\|\right] \\
&\quad + \|x_{k+1}-x_k\| \\
&\leqslant M_0|r_{k+1}-r_k|+\|x_{k+1}-x_k\|,
\end{aligned} \tag{3.12}$$

where $\sup_{k\geqslant 0}\{\|\mathcal{A}x_{k+1}\|+\frac{1}{r_{k+1}}\|T^{(\Theta,\varphi)}_{r_{k+1}}(I-r_{k+1}\mathcal{A})x_{k+1}-(I-r_{k+1}\mathcal{A})x_{k+1}\|\}\leqslant M_0$ for some $M_0>0$. Also, utilizing Algorithm 3.2 and Lemma 2.8, we have

$$\begin{aligned}
\|\tilde{u}_{k+1}-\tilde{u}_k\| &= \|\beta_{k+1}u_{k+1}+(1-\beta_{k+1})Tu_{k+1}-(\beta_k u_k+(1-\beta_k)Tu_k)\| \\
&= \|(1-\beta_{k+1})(Tu_{k+1}-Tu_k)-(\beta_{k+1}-\beta_k)Tu_k \\
&\quad + \beta_{k+1}(u_{k+1}-u_k)+(\beta_{k+1}-\beta_k)u_k\| \\
&\leqslant \|u_{k+1}-u_k\|+|\beta_{k+1}-\beta_k|\|u_k-Tu_k\|.
\end{aligned} \tag{3.13}$$

Moreover, we define $\tilde{w}_k=\frac{v_k-\gamma_k x_k}{1-\gamma_k}$, which implies that $v_k=(1-\gamma_k)\tilde{w}_k+\gamma_k x_k$. Simple calculations show that

$$\begin{aligned}
\tilde{w}_{k+1}-\tilde{w}_k &= \frac{v_{k+1}-\gamma_{k+1}x_{k+1}}{1-\gamma_{k+1}}-\frac{v_k-\gamma_k x_k}{1-\gamma_k} \\
&= \frac{\alpha_{k+1}\gamma Vx_{k+1}+((1-\gamma_{k+1})I-\alpha_{k+1}\mu F)\tilde{u}_{k+1}}{1-\gamma_{k+1}}-\frac{\alpha_k\gamma Vx_k+((1-\gamma_k)I-\alpha_k\mu F)\tilde{u}_k}{1-\gamma_k} \\
&= \frac{\alpha_{k+1}}{1-\gamma_{k+1}}\gamma Vx_{k+1}-\frac{\alpha_k}{1-\gamma_k}\gamma Vx_k+\tilde{u}_{k+1}-\tilde{u}_k+\frac{\alpha_k}{1-\gamma_k}\mu F\tilde{u}_k-\frac{\alpha_{k+1}}{1-\gamma_{k+1}}\mu F\tilde{u}_{k+1} \\
&= \frac{\alpha_{k+1}}{1-\gamma_{k+1}}(\gamma Vx_{k+1}-\mu F\tilde{u}_{k+1})+\frac{\alpha_k}{1-\gamma_k}(\mu F\tilde{u}_k-\gamma Vx_k)+\tilde{u}_{k+1}-\tilde{u}_k.
\end{aligned}$$

So, it follows from (3.12) and (3.13) that

$$\begin{aligned}
\|\tilde{w}_{k+1}-\tilde{w}_k\| &\leqslant \frac{\alpha_{k+1}}{1-\gamma_{k+1}}(\|\gamma Vx_{k+1}\|+\|\mu F\tilde{u}_{k+1}\|)+\frac{\alpha_k}{1-\gamma_k}(\|\mu F\tilde{u}_k\|+\|\gamma Vx_k\|)+\|\tilde{u}_{k+1}-\tilde{u}_k\| \\
&\leqslant \frac{\alpha_{k+1}}{1-\gamma_{k+1}}(\|\gamma Vx_{k+1}\|+\|\mu F\tilde{u}_{k+1}\|)+\frac{\alpha_k}{1-\gamma_k}(\|\mu F\tilde{u}_k\|+\|\gamma Vx_k\|)+\|u_{k+1}-u_k\| \\
&\quad +|\beta_{k+1}-\beta_k|\|u_k-Tu_k\| \\
&\leqslant \frac{\alpha_{k+1}}{1-\gamma_{k+1}}(\|\gamma Vx_{k+1}\|+\|\mu F\tilde{u}_{k+1}\|)+\frac{\alpha_k}{1-\gamma_k}(\|\mu F\tilde{u}_k\|+\|\gamma Vx_k\|)+M_0|r_{k+1}-r_k| \\
&\quad +\|x_{k+1}-x_k\|+|\beta_{k+1}-\beta_k|\|u_k-Tu_k\| \\
&\leqslant \|x_{k+1}-x_k\|+(\frac{\alpha_{k+1}}{1-\gamma_{k+1}}+\frac{\alpha_k}{1-\gamma_k}+|r_{k+1}-r_k|+|\beta_{k+1}-\beta_k|)M_1,
\end{aligned} \tag{3.14}$$

where $\sup_{k\geqslant 0}\{\|\mu F\tilde{u}_k\| + \|\gamma V x_k\| + \|u_k - Tu_k\| + M_0\} \leqslant M_1$ for some $M_1 > 0$. In the meantime, from $v_k = (1-\gamma_k)\tilde{w}_k + \gamma_k x_k$, together with (3.14), we get

$$\begin{aligned}
\|v_{k+1} - v_k\| &= \|\gamma_{k+1}x_{k+1} + (1-\gamma_{k+1})\tilde{w}_{k+1} - (\gamma_k x_k + (1-\gamma_k)\tilde{w}_k)\| \\
&= \|(1-\gamma_{k+1})(\tilde{w}_{k+1} - \tilde{w}_k) - (\gamma_{k+1} - \gamma_k)\tilde{w}_k + \gamma_{k+1}(x_{k+1} - x_k) + (\gamma_{k+1} - \gamma_k)x_k\| \\
&\leqslant (1-\gamma_{k+1})\|\tilde{w}_{k+1} - \tilde{w}_k\| + \gamma_{k+1}\|x_{k+1} - x_k\| + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\| \\
&\leqslant (1-\gamma_{k+1})\left[\|x_{k+1} - x_k\| + (\frac{\alpha_{k+1}}{1-\gamma_{k+1}} + \frac{\alpha_k}{1-\gamma_k} + |r_{k+1} - r_k| + |\beta_{k+1} - \beta_k|)M_1\right] \\
&\quad + \gamma_{k+1}\|x_{k+1} - x_k\| + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\| \\
&\leqslant \|x_{k+1} - x_k\| + (\frac{\alpha_{k+1}}{1-\gamma_{k+1}} + \frac{\alpha_k}{1-\gamma_k} + |r_{k+1} - r_k| + |\beta_{k+1} - \beta_k|)M_1 \\
&\quad + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\|.
\end{aligned} \tag{3.15}$$

Combining (3.11) and (3.15) we have

$$\begin{aligned}
&\|w_{k+1} - w_k\| - \|x_{k+1} - x_k\| \\
&\leqslant |\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}|\|u\| + |\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k}|(\|P_{VI(\Omega,B)}(z_k - \lambda A z_k)\| + \bar{\epsilon}_k) \\
&\quad + \frac{\gamma_{k+1}(1+\lambda_{k+1}L_2)}{1-\beta_{k+1}}[\|x_{k+1} - x_k\| + (\frac{\alpha_{k+1}}{1-\gamma_{k+1}} + \frac{\alpha_k}{1-\gamma_k} + |r_{k+1} - r_k| + |\beta_{k+1} - \beta_k|)M_1 \\
&\quad + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\|] - \|x_{k+1} - x_k\| + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) \\
&\quad + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|) \\
&= |\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}|\|u\| + |\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k}|(\|P_{VI(\Omega,B)}(z_k - \lambda A z_k)\| + \bar{\epsilon}_k) \\
&\quad + \frac{\gamma_{k+1}(1+\lambda_{k+1}L_2)}{1-\beta_{k+1}}[(\frac{\alpha_{k+1}}{1-\gamma_{k+1}} + \frac{\alpha_k}{1-\gamma_k} + |r_{k+1} - r_k| + |\beta_{k+1} - \beta_k|)M_1 \\
&\quad + |\gamma_{k+1} - \gamma_k|\|x_k - \tilde{w}_k\|] + (\frac{\gamma_{k+1}(1+\lambda_{k+1}L_2)}{1-\beta_{k+1}} - 1)\|x_{k+1} - x_k\| + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1} + \bar{\epsilon}_k) \\
&\quad + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\| + \|By_{k+1}\| + \|Bv_k\|) + \lambda_k\|By_k\|).
\end{aligned} \tag{3.16}$$

From the assumptions $\alpha_k + \beta_k + \gamma_k = 1$, $\lim_{k\to\infty}\beta_k = \xi \in (\zeta, \frac{1}{2}]$, $\lim_{k\to\infty}\alpha_k = 0$, and $\lim_{k\to\infty}\lambda_k = 0$, it follows that $\lim_{k\to\infty}|\beta_{k+1} - \beta_k| = \lim_{k\to\infty}|\gamma_{k+1} - \gamma_k| = 0$,

$$\lim_{k\to\infty}|\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}| = \lim_{k\to\infty}|\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k}| = 0 \quad \text{and} \quad \lim_{k\to\infty}\frac{\gamma_{k+1}(1+\lambda_{k+1}L_2)}{1-\beta_{k+1}} = 1.$$

Combining these equalities with (3.16), we obtain from Lemma 3.5, $\lim_{k\to\infty}\bar{\epsilon}_k = 0$, and $\lim_{k\to\infty}|r_{k+1} - r_k| = 0$ that

$$\limsup_{k\to\infty}(\|w_{k+1} - w_k\| - \|x_{k+1} - x_k\|) \leqslant 0.$$

Now applying Lemma 3.6, we have

$$\lim_{k\to\infty}\|w_k - x_k\| = 0.$$

Hence by $x_{k+1} = (1-\beta_k)w_k + \beta_k x_k$, we deduce that

$$\lim_{k\to\infty}\|x_{k+1} - x_k\| = \lim_{k\to\infty}(1-\beta_k)\|w_k - x_k\| = 0, \tag{3.17}$$

which together with $\lim_{k\to\infty}\lambda_k = 0$, (3.9), and (3.15), implies that

$$\lim_{k\to\infty}\|v_{k+1} - v_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|z_{k+1} - z_k\| = 0.$$

□

Lemma 3.8. *Suppose that the conditions* (A1)-(A4) *and* (H1)-(H4) *hold and that the conditions* (B1) *or* (B2) *hold. Then for any* $p \in VI(VI(\Omega, B), A)$ *we have*

$$\begin{aligned} \|x_{k+1} - p\|^2 \leqslant \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 + \gamma_k \|v_k - p\|^2 + 2\gamma_k \bar{\epsilon}_k \|z_k - p\| \\ + \gamma_k \bar{\epsilon}_k^2 - \gamma_k (1 - \lambda_k L_2)(\|v_k - y_k\|^2 + \|y_k - z_k\|^2). \end{aligned} \tag{3.18}$$

Moreover

$$\lim_{k\to\infty} \|P_{VI(\Omega,B)}(z_k - \lambda_k A z_k) - z_k\| = \lim_{k\to\infty} \|P_{VI(\Omega,B)}(y_k - \lambda_k A y_k) - y_k\| = 0.$$

Proof. By Lemma 3.3, we know that $\lim_{j\to\infty} x_{k,j} = P_{VI(\Omega,B)}(z_k - \lambda A z_k)$ which together with $0 < \lambda \leqslant 2\beta$, inequality (3.1), $\lim_{k\to\infty} \beta_k = \xi \in (\zeta, \frac{1}{2}]$, and $p \in VI(VI(\Omega, B), A)$, implies that

$$\begin{aligned} \|x_{k+1} - p\|^2 &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\|^2 \\ &\leqslant \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 + \gamma_k \|h_k - p\|^2 \\ &\leqslant \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 + \gamma_k (\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - p\| + \bar{\epsilon}_k)^2 \\ &= \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 \\ &\quad + \gamma_k (\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - P_{VI(\Omega,B)}(p - \lambda A p)\| + \bar{\epsilon}_k)^2 \\ &\leqslant \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 + \gamma_k (\|(I - \lambda A) z_k - (I - \lambda A) p\| + \bar{\epsilon}_k)^2 \\ &\leqslant \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 + \gamma_k (\|z_k - p\| + \bar{\epsilon}_k)^2 \\ &= \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 + \gamma_k \|z_k - p\|^2 + 2\gamma_k \bar{\epsilon}_k \|z_k - p\| + \gamma_k \bar{\epsilon}_k^2 \\ &\leqslant \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 + 2\gamma_k \bar{\epsilon}_k \|z_k - p\| + \gamma_k \bar{\epsilon}_k^2 \\ &\quad + \gamma_k (\|v_k - p\|^2 - (1 - \lambda_k L_2)\|v_k - y_k\|^2 - (1 - \lambda_k L_2)\|y_k - z_k\|^2) \\ &= \alpha_k \|u - p\|^2 + \beta_k \|x_k - p\|^2 + \gamma_k \|v_k - p\|^2 + 2\gamma_k \bar{\epsilon}_k \|z_k - p\| + \gamma_k \bar{\epsilon}_k^2 \\ &\quad - \gamma_k (1 - \lambda_k L_2)(\|v_k - y_k\|^2 + \|y_k - z_k\|^2). \end{aligned} \tag{3.19}$$

On the other hand, from Algorithm 3.2 we have

$$\begin{aligned} \|v_k - p\|^2 &= \|\alpha_k \gamma V x_k + \gamma_k x_k + ((1 - \gamma_k) I - \alpha_k \mu F) \tilde{u}_k - p\|^2 \\ &= \|\alpha_k (\gamma V x_k - \mu F p) + \gamma_k (x_k - p) + ((1 - \gamma_k) I - \alpha_k \mu F) \tilde{u}_k - ((1 - \gamma_k) I - \alpha_k \mu F) p\|^2 \\ &= \|\alpha_k \gamma (V x_k - V p) + \alpha_k (\gamma V p - \mu F p) + \gamma_k (x_k - p) \\ &\quad + ((1 - \gamma_k) I - \alpha_k \mu F) \tilde{u}_k - ((1 - \gamma_k) I - \alpha_k \mu F) p\|^2 \\ &\leqslant \|\alpha_k \gamma (V x_k - V p) + \gamma_k (x_k - p) + ((1 - \gamma_k) I - \alpha_k \mu F) \tilde{u}_k \\ &\quad - ((1 - \gamma_k) I - \alpha_k \mu F) p\|^2 + 2\alpha_k \langle (\gamma V - \mu F) p, v_k - p \rangle \\ &\leqslant [\alpha_k \gamma \|V x_k - V p\| + \gamma_k \|x_k - p\| + \|((1 - \gamma_k) I - \alpha_k \mu F) \tilde{u}_k \\ &\quad - ((1 - \gamma_k) I - \alpha_k \mu F) p\|]^2 + 2\alpha_k \langle (\gamma V - \mu F) p, v_k - p \rangle \\ &\leqslant [\alpha_k \gamma l \|x_k - p\| + \gamma_k \|x_k - p\| + (1 - \gamma_k - \alpha_k \tau) \|\tilde{u}_k - p\|]^2 + 2\alpha_k \langle (\gamma V - \mu F) p, v_k - p \rangle \\ &= [\alpha_k \tau \frac{\gamma l}{\tau} \|x_k - p\| + \gamma_k \|x_k - p\| + (1 - \gamma_k - \alpha_k \tau) \|\tilde{u}_k - p\|]^2 + 2\alpha_k \langle (\gamma V - \mu F) p, v_k - p \rangle \\ &\leqslant \alpha_k \tau \frac{(\gamma l)^2}{\tau^2} \|x_k - p\|^2 + \gamma_k \|x_k - p\|^2 + (1 - \gamma_k - \alpha_k \tau) \|\tilde{u}_k - p\|^2 + 2\alpha_k \langle (\gamma V - \mu F) p, v_k - p \rangle \\ &\leqslant \alpha_k \frac{(\gamma l)^2}{\tau} \|x_k - p\|^2 + \gamma_k \|x_k - p\|^2 + (1 - \gamma_k - \alpha_k \tau) [\|u_k - p\|^2 \\ &\quad - (1 - \beta_k)(\beta_k - \zeta) \|u_k - T u_k\|^2] + 2\alpha_k \langle (\gamma V - \mu F) p, v_k - p \rangle \\ &\leqslant \alpha_k \frac{(\gamma l)^2}{\tau} \|x_k - p\|^2 + \gamma_k \|x_k - p\|^2 + (1 - \gamma_k - \alpha_k \tau) [\|x_k - p\|^2 + r_k (r_k - 2\alpha) \|\mathcal{A} x_k - \mathcal{A} p\|^2 \end{aligned}$$

$$- (1-\beta_k)(\beta_k-\zeta)\|u_k - Tu_k\|^2] + 2\alpha_k\langle(\gamma V - \mu F)p, v_k - p\rangle.$$

Thus,

$$\begin{aligned}\|v_k - p\|^2 &\leqslant (1-\alpha_k\frac{\tau^2-(\gamma l)^2}{\tau})\|x_k-p\|^2 - (1-\gamma_k-\alpha_k\tau)[r_k(2\alpha-r_k)\|\mathcal{A}x_k-\mathcal{A}p\|^2 \\ &\quad + (1-\beta_k)(\beta_k-\zeta)\|u_k-Tu_k\|^2] + 2\alpha_k\langle(\gamma V-\mu F)p, v_k-p\rangle \\ &\leqslant \|x_k-p\|^2 - (1-\gamma_k-\alpha_k\tau)[r_k(2\alpha-r_k)\|\mathcal{A}x_k-\mathcal{A}p\|^2 \\ &\quad + (1-\beta_k)(\beta_k-\zeta)\|u_k-Tu_k\|^2] + 2\alpha_k\langle(\gamma V-\mu F)p, v_k-p\rangle.\end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20), we get

$$\begin{aligned}\|x_{k+1}-p\|^2 &\leqslant \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|v_k-p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| \\ &\quad + \gamma_k\bar{\epsilon}_k^2 - \gamma_k(1-\lambda_k L_2)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2) \\ &\leqslant \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\{\|x_k-p\|^2 \\ &\quad - (1-\gamma_k-\alpha_k\tau)[r_k(2\alpha-r_k)\|\mathcal{A}x_k-\mathcal{A}p\|^2 + (1-\beta_k)(\beta_k-\zeta)\|u_k-Tu_k\|^2] \\ &\quad + 2\alpha_k\langle(\gamma V-\mu F)p, v_k-p\rangle\} + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| \\ &\quad + \gamma_k\bar{\epsilon}_k^2 - \gamma_k(1-\lambda_k L_2)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2) \\ &\leqslant \alpha_k\|u-p\|^2 + \|x_k-p\|^2 - \gamma_k(1-\gamma_k-\alpha_k\tau)[r_k(2\alpha-r_k)\|\mathcal{A}x_k-\mathcal{A}p\|^2 \\ &\quad + (1-\beta_k)(\beta_k-\zeta)\|u_k-Tu_k\|^2] + 2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\| \\ &\quad + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| + \gamma_k\bar{\epsilon}_k^2 - \gamma_k(1-\lambda_k L_2)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2),\end{aligned}$$

which immediately yields

$$\begin{aligned}&\gamma_k(1-\gamma_k-\alpha_k\tau)[r_k(2\alpha-r_k)\|\mathcal{A}x_k-\mathcal{A}p\|^2 + (1-\beta_k)(\beta_k-\zeta)\|u_k-Tu_k\|^2] \\ &\quad + \gamma_k(1-\lambda_k L_2)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2) \\ &\leqslant \alpha_k\|u-p\|^2 + \|x_k-p\|^2 - \|x_{k+1}-p\|^2 + 2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\| + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| + \gamma_k\bar{\epsilon}_k^2 \\ &\leqslant \alpha_k\|u-p\|^2 + \|x_k-x_{k+1}\|(\|x_k-p\| + \|x_{k+1}-p\|) + 2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\| \\ &\quad + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| + \gamma_k\bar{\epsilon}_k^2.\end{aligned}$$

Since $\alpha_k+\beta_k+\gamma_k = 1$, $\alpha_k \to 0$, $\beta_k \to \xi \in (\zeta, \frac{1}{2}]$, $\bar{\epsilon}_k \to 0$, $\lambda_k \to 0$, $\|x_k - x_{k+1}\| \to 0$, and $\{r_k\} \subset [a, b] \subset (0, 2\alpha)$, from the boundedness of $\{x_k\}, \{v_k\}$, and $\{z_k\}$ we obtain

$$\lim_{k\to\infty}\|\mathcal{A}x_k-\mathcal{A}p\| = \lim_{k\to\infty}\|u_k-Tu_k\| = \lim_{k\to\infty}\|v_k-y_k\| = \lim_{k\to\infty}\|y_k-z_k\| = 0. \tag{3.21}$$

Also, utilizing Proposition 2.3 (ii) and Lemma 2.12 (a), we obtain from (2.1) and $\{r_k\} \subset [a, b] \subset (0, 2\alpha)$ that

$$\begin{aligned}\|u_k-p\|^2 &= \|T_{r_k}^{(\Theta,\varphi)}(I-r_k\mathcal{A})x_k - T_{r_k}^{(\Theta,\varphi)}(I-r_k\mathcal{A})p\|^2 \\ &\leqslant \langle(I-r_k\mathcal{A})x_k - (I-r_k\mathcal{A})p, u_k-p\rangle \\ &= \frac{1}{2}(\|(I-r_k\mathcal{A})x_k-(I-r_k\mathcal{A})p\|^2 + \|u_k-p\|^2 - \|(I-r_k\mathcal{A})x_k-(I-r_k\mathcal{A})p-(u_k-p)\|^2) \\ &\leqslant \frac{1}{2}(\|x_k-p\|^2 + \|u_k-p\|^2 - \|(I-r_k\mathcal{A})x_k-(I-r_k\mathcal{A})p-(u_k-p)\|^2) \\ &= \frac{1}{2}(\|x_k-p\|^2 + \|u_k-p\|^2 - \|x_k-u_k-r_k(\mathcal{A}x_k-\mathcal{A}p)\|^2),\end{aligned}$$

which immediately leads to

$$\begin{aligned}\|u_k-p\|^2 &\leqslant \|x_k-p\|^2 - \|x_k-u_k-r_k(\mathcal{A}x_k-\mathcal{A}p)\|^2 \\ &= \|x_k-p\|^2 - \|x_k-u_k\|^2 - r_k^2\|\mathcal{A}x_k-\mathcal{A}p\|^2 + 2r_k\langle\mathcal{A}x_k-\mathcal{A}p, x_k-u_k\rangle \\ &\leqslant \|x_k-p\|^2 - \|x_k-u_k\|^2 + 2r_k\|\mathcal{A}x_k-\mathcal{A}p\|\|x_k-u_k\|.\end{aligned} \tag{3.22}$$

Combining (3.20) and (3.22) we have

$$\begin{aligned}\|v_k-p\|^2 &\leqslant \alpha_k\frac{(\gamma l)^2}{\tau}\|x_k-p\|^2+\gamma_k\|x_k-p\|^2+(1-\gamma_k-\alpha_k\tau)[\|u_k-p\|^2\\&\quad-(1-\beta_k)(\beta_k-\zeta)\|u_k-Tu_k\|^2]+2\alpha_k\langle(\gamma V-\mu F)p,v_k-p\rangle\\&\leqslant \alpha_k\frac{(\gamma l)^2}{\tau}\|x_k-p\|^2+\gamma_k\|x_k-p\|^2+(1-\gamma_k-\alpha_k\tau)\|u_k-p\|^2\\&\quad+2\alpha_k\langle(\gamma V-\mu F)p,v_k-p\rangle\\&\leqslant \alpha_k\frac{(\gamma l)^2}{\tau}\|x_k-p\|^2+\gamma_k\|x_k-p\|^2+(1-\gamma_k-\alpha_k\tau)[\|x_k-p\|^2\\&\quad-\|x_k-u_k\|^2+2r_k\|\mathcal{A}x_k-\mathcal{A}p\|\|x_k-u_k\|]+2\alpha_k\langle(\gamma V-\mu F)p,v_k-p\rangle\\&\leqslant (1-\alpha_k\frac{\tau^2-(\gamma l)^2}{\tau})\|x_k-p\|^2-(1-\gamma_k-\alpha_k\tau)\|x_k-u_k\|^2\\&\quad+2r_k\|\mathcal{A}x_k-\mathcal{A}p\|\|x_k-u_k\|+2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\|\\&\leqslant \|x_k-p\|^2-(1-\gamma_k-\alpha_k\tau)\|x_k-u_k\|^2+2r_k\|\mathcal{A}x_k-\mathcal{A}p\|\|x_k-u_k\|\\&\quad+2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\|,\end{aligned}$$

which together with (3.18), implies that

$$\begin{aligned}\|x_{k+1}-p\|^2 &\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|v_k-p\|^2+2\gamma_k\bar{\epsilon}_k\|z_k-p\|\\&\quad+\gamma_k\bar{\epsilon}_k^2-\gamma_k(1-\lambda_kL_2)(\|v_k-y_k\|^2+\|y_k-z_k\|^2)\\&\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|v_k-p\|^2+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2\\&\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k[\|x_k-p\|^2-(1-\gamma_k-\alpha_k\tau)\|x_k-u_k\|^2\\&\quad+2r_k\|\mathcal{A}x_k-\mathcal{A}p\|\|x_k-u_k\|+2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\|]+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2\\&\leqslant \alpha_k\|u-p\|^2+\|x_k-p\|^2-\gamma_k(1-\gamma_k-\alpha_k\tau)\|x_k-u_k\|^2\\&\quad+2r_k\|\mathcal{A}x_k-\mathcal{A}p\|\|x_k-u_k\|+2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\|+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2.\end{aligned}$$

So, it follows that

$$\begin{aligned}\gamma_k(1-\gamma_k-\alpha_k\tau)\|x_k-u_k\|^2 &\leqslant \alpha_k\|u-p\|^2+\|x_k-p\|^2-\|x_{k+1}-p\|^2+2r_k\|\mathcal{A}x_k-\mathcal{A}p\|\|x_k-u_k\|\\&\quad+2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\|+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2\\&\leqslant \alpha_k\|u-p\|^2+\|x_k-x_{k+1}\|(\|x_k-p\|+\|x_{k+1}-p\|)+2r_k\|\mathcal{A}x_k-\mathcal{A}p\|\\&\quad\times\|x_k-u_k\|+2\alpha_k\|(\gamma V-\mu F)p\|\|v_k-p\|+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2.\end{aligned}$$

Since $\alpha_k+\beta_k+\gamma_k=1$, $\alpha_k\to 0$, $\beta_k\to\xi\in(\zeta,\frac{1}{2}]$, $\bar{\epsilon}_k\to 0$, $\|x_k-x_{k+1}\|\to 0$, $\|\mathcal{A}x_k-\mathcal{A}p\|\to 0$, and $\{r_k\}\subset[a,b]\subset(0,2\alpha)$, from the boundedness of $\{x_k\},\{u_k\},\{v_k\}$, and $\{z_k\}$ we obtain

$$\lim_{k\to\infty}\|x_k-u_k\|=0. \tag{3.23}$$

Thus, from $\alpha_k\to 0$, Algorithm 3.2, (3.21), and (3.23) it follows that as $k\to\infty$,

$$\|\tilde{u}_k-u_k\|=(1-\beta_k)\|Tu_k-u_k\|\leqslant\|Tu_k-u_k\|\to 0,$$

and hence

$$\begin{aligned}\|v_k-x_k\| &= \|\alpha_k\gamma Vx_k+\gamma_kx_k+((1-\gamma_k)I-\alpha_k\mu F)\tilde{u}_k-x_k\|\\&=\|\alpha_k(\gamma Vx_k-\mu F\tilde{u}_k)+(1-\gamma_k)(\tilde{u}_k-x_k)\|\\&\leqslant \alpha_k\|\gamma Vx_k-\mu F\tilde{u}_k\|+(1-\gamma_k)\|\tilde{u}_k-x_k\|\\&\leqslant \alpha_k\|\gamma Vx_k-\mu F\tilde{u}_k\|+(1-\gamma_k)(\|\tilde{u}_k-u_k\|+\|u_k-x_k\|)\\&\leqslant \alpha_k\|\gamma Vx_k-\mu F\tilde{u}_k\|+\|Tu_k-u_k\|+\|u_k-x_k\|\to 0.\end{aligned}$$

That is,

$$\lim_{k\to\infty}\|\tilde{u}_k - u_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|v_k - x_k\| = 0. \tag{3.24}$$

Taking into consideration that $\|v_k - z_k\| \leqslant \|v_k - y_k\| + \|y_k - z_k\|$ and $\|z_k - x_k\| \leqslant \|z_k - v_k\| + \|v_k - x_k\|$, we deduce from (3.21) and (3.24) that

$$\lim_{k\to\infty}\|v_k - z_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|z_k - x_k\| = 0. \tag{3.25}$$

It is clear from (3.21) and (3.24) that

$$\lim_{k\to\infty}\|x_k - y_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|y_k - z_k\| = 0. \tag{3.26}$$

Again by Proposition 2.1 (iii) and Lemma 3.3 we have

$$\begin{aligned}
&\|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - x_{k+1}\| \\
&\leqslant \|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - P_{VI(\Omega,B)}(z_k - \lambda A z_k)\| + \|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - x_{k+1}\| \\
&\leqslant (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - u\| \\
&\quad + \beta_k\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - x_k\| + \bar{\epsilon}_k \\
&\leqslant (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - u\| + \bar{\epsilon}_k \\
&\quad + \beta_k\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - P_{VI(\Omega,B)}(y_k - \lambda A y_k)\| \\
&\quad + \beta_k\|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| + \beta_k\|y_k - x_k\| \\
&\leqslant (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - u\| + \bar{\epsilon}_k \\
&\quad + \beta_k(1+\lambda L_1)\|z_k - y_k\| + \beta_k\|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| + \beta_k\|y_k - x_k\|.
\end{aligned} \tag{3.27}$$

Consequently, from (3.27), we have

$$\begin{aligned}
\|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| &\leqslant \|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - x_{k+1}\| + \|x_{k+1} - x_k\| + \|x_k - y_k\| \\
&\leqslant (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - u\| + \bar{\epsilon}_k \\
&\quad + \beta_k(1+\lambda L_1)\|z_k - y_k\| + \beta_k\|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| + \beta_k\|y_k - x_k\| \\
&\quad + \|x_{k+1} - x_k\| + \|x_k - y_k\| \\
&= (1+\beta_k)(1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - u\| + \bar{\epsilon}_k \\
&\quad + \beta_k\|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| + (1+\beta_k)\|y_k - x_k\| + \|x_{k+1} - x_k\|,
\end{aligned}$$

which immediately yields

$$\begin{aligned}
\|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| \leqslant \frac{1+\beta_k}{1-\beta_k}(1+\lambda L_1)\|y_k - z_k\| + \frac{\alpha_k}{1-\beta_k}\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - u\| + \frac{\bar{\epsilon}_k}{1-\beta_k} \\
+ \frac{1+\beta_k}{1-\beta_k}\|y_k - x_k\| + \frac{1}{1-\beta_k}\|x_{k+1} - x_k\|.
\end{aligned}$$

Since $\alpha_k + \beta_k + \gamma_k = 1$, $\alpha_k \to 0$, $\beta_k \to \xi \in (\zeta, \frac{1}{2}]$, $\bar{\epsilon}_k \to 0$, $\|y_k - z_k\| \to 0$, $\|y_k - x_k\| \to 0$, and $\|x_{k+1} - x_k\| \to 0$ (due to (3.17) and (3.26)), we conclude that

$$\lim_{k\to\infty}\|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| = 0. \tag{3.28}$$

From Proposition 2.1 (iii), it follows that

$$\begin{aligned}
\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - z_k\| &\leqslant \|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - P_{VI(\Omega,B)}(y_k - \lambda A y_k)\| \\
&\quad + \|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| + \|y_k - z_k\| \\
&\leqslant (1+\lambda L_1)\|z_k - y_k\| + \|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| + \|y_k - z_k\| \\
&\leqslant \|P_{VI(\Omega,B)}(y_k - \lambda A y_k) - y_k\| + (2+\lambda L_1)\|y_k - z_k\|.
\end{aligned}$$

Utilizing the last inequality we obtain from (3.26) and (3.28) that

$$\lim_{k\to\infty} \|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - z_k\| = 0.$$

□

Theorem 3.9. *Suppose that the conditions* (A1)-(A4) *and* (H1)-(H4) *hold and that the conditions* (B1) *or* (B2) *hold. Then the two sequences* $\{x_k\}$ *and* $\{z_k\}$ *in Algorithm* 3.2 *converge strongly to the same point* $x^* \in VI(VI(\Omega, B), A)$ *provided* $\|x_{k+1} - x_k\| = o(\alpha_k)$*, which is a unique solution to the VIP*

$$\langle (I + \bar{\xi}\mu F - \bar{\xi}\gamma V)x^* - u, p - x^*\rangle \geqslant 0, \quad \forall p \in VI(VI(\Omega, B), A),$$

where $\bar{\xi} = 1 - \xi \in [\frac{1}{2}, 1)$.

Proof. Note that Lemma 3.5 shows the boundedness of $\{x_k\}$. Since H is reflexive, there is at least a weak convergence subsequence of $\{x_k\}$. First, let us assert that $\omega_w(x_k) \subset VI(VI(\Omega, B), A)$. As a matter of fact, take an arbitrary $w \in VI(VI(\Omega, B), A)$. Then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup w$. From (3.26), we know that $y_{k_i} \rightharpoonup w$. It is easy to see that the mapping $P_{VI(\Omega,B)}(I - \lambda A) : C \to VI(\Omega, B) \subset C$ is nonexpansive because $P_{VI(\Omega,B)}$ is nonexpansive and $I - \lambda A$ is nonexpansive for β-inverse-strongly monotone mapping A with $0 < \lambda \leqslant 2\beta$. So, utilizing Lemma 2.6 and (3.28), we obtain $w = P_{VI(\Omega,B)}(w - \lambda A w)$, which leads to $w \in VI(VI(\Omega, B), A)$. Thus, the assertion is valid.

It is clear that

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \geqslant (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x, y \in H.$$

Hence, it follows from $0 \leqslant \gamma l < \tau \leqslant \mu\eta$ that $\mu F - \gamma V$ is $(\mu\eta - \gamma l)$-strongly monotone. In the meantime, it is clear that $\mu F - \gamma V$ is Lipschitzian with constant $\mu\kappa + \gamma l > 0$. We define the mapping $\Gamma : H \to H$ as below

$$\Gamma x = (\mu F - \gamma V)x + \frac{1}{\xi}(x - u), \quad \forall x \in H,$$

where $u \in H$ and $\xi \in (\zeta, \frac{1}{2}]$. Then it is easy to see that Γ is $(\mu\eta - \gamma l + \frac{1}{\xi})$-strongly monotone and Lipschitzian with constant $\mu\kappa + \gamma l + \frac{1}{\xi} > 0$. Thus, there exists a unique solution $x^* \in VI(VI(\Omega, B), A)$ to the VIP

$$\langle (\mu F - \gamma V)x^* + \frac{1}{\xi}(x^* - u), p - x^*\rangle \geqslant 0, \quad \forall p \in VI(VI(\Omega, B), A). \tag{3.29}$$

Next, let us show that $x_k \rightharpoonup x^*$. Indeed, take an arbitrary $p \in VI(VI(\Omega, B), A)$. In terms of Algorithm 3.2 and Lemma 2.4, we conclude from (3.1), (3.3), and the β-inverse-strong monotonicity of A with $\lambda \leqslant 2\beta$, that

$$\begin{aligned}
\|x_{k+1} - p\|^2 &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\|^2 \\
&\leqslant \|\beta_k(x_k - p) + \gamma_k(h_k - p)\|^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle \\
&\leqslant \beta_k\|x_k - p\|^2 + \gamma_k\|h_k - p\|^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle \\
&\leqslant \beta_k\|x_k - p\|^2 + \gamma_k(\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - p\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle \\
&= \beta_k\|x_k - p\|^2 + \gamma_k(\|P_{VI(\Omega,B)}(z_k - \lambda A z_k) - P_{VI(\Omega,B)}(p - \lambda A p)\| + \bar{\epsilon}_k)^2 \\
&\quad + 2\alpha_k\langle u - p, x_{k+1} - p\rangle \\
&\leqslant \beta_k\|x_k - p\|^2 + \gamma_k(\|(I - \lambda A)z_k - (I - \lambda A)p\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle \\
&\leqslant \beta_k\|x_k - p\|^2 + \gamma_k(\|z_k - p\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - p, x_{k+1} - p\rangle \\
&= \beta_k\|x_k - p\|^2 + \gamma_k\|z_k - p\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - p, x_{k+1} - p\rangle \\
&\leqslant \beta_k\|x_k - p\|^2 + \gamma_k\|v_k - p\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - p\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - p, x_{k+1} - p\rangle.
\end{aligned} \tag{3.30}$$

Combining (3.20) and (3.30), we get

$$\begin{aligned}\|x_{k+1}-p\|^2 &\leqslant \beta_k\|x_k-p\|^2+\gamma_k\|v_k-p\|^2+\gamma_k\bar{\epsilon}_k(2\|z_k-p\|+\bar{\epsilon}_k)+2\alpha_k\langle u-p,x_{k+1}-p\rangle\\
&\leqslant \beta_k\|x_k-p\|^2+\gamma_k\{(1-\alpha_k\frac{\tau^2-(\gamma l)^2}{\tau})\|x_k-p\|^2\\
&\quad -(1-\gamma_k-\alpha_k\tau)[r_k(2\alpha-r_k)\|\mathcal{A}x_k-\mathcal{A}p\|^2+(1-\beta_k)(\beta_k-\zeta)\|u_k-Tu_k\|^2]\\
&\quad +2\alpha_k\langle(\gamma V-\mu F)p,v_k-p\rangle\}+\gamma_k\bar{\epsilon}_k(2\|z_k-p\|+\bar{\epsilon}_k)+2\alpha_k\langle u-p,x_{k+1}-p\rangle\\
&\leqslant \beta_k\|x_k-p\|^2+\gamma_k\{(1-\alpha_k\frac{\tau^2-(\gamma l)^2}{\tau})\|x_k-p\|^2+2\alpha_k\langle(\gamma V-\mu F)p,v_k-p\rangle\}\\
&\quad +\gamma_k\bar{\epsilon}_k(2\|z_k-p\|+\bar{\epsilon}_k)+2\alpha_k\langle u-p,x_{k+1}-p\rangle\\
&=(\beta_k+\gamma_k-\alpha_k\gamma_k\frac{\tau^2-(\gamma l)^2}{\tau})\|x_k-p\|^2+2\alpha_k\gamma_k\langle(\gamma V-\mu F)p,v_k-x_{k+1}\rangle\\
&\quad +\gamma_k\bar{\epsilon}_k(2\|z_k-p\|+\bar{\epsilon}_k)+2\alpha_k\gamma_k\langle(\gamma V-\mu F)p+\frac{1}{\gamma_k}(u-p),x_{k+1}-p\rangle\\
&\leqslant (1-\alpha_k\gamma_k\frac{\tau^2-(\gamma l)^2}{\tau})\|x_k-p\|^2+2\alpha_k\|(\gamma V-\mu F)p\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\
&\quad +\gamma_k\bar{\epsilon}_k(2\|z_k-p\|+\bar{\epsilon}_k)+2\alpha_k\gamma_k\langle(\gamma V-\mu F)p+\frac{1}{\gamma_k}(u-p),x_{k+1}-p\rangle\\
&\leqslant \|x_k-p\|^2+2\alpha_k\|(\gamma V-\mu F)p\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\
&\quad +\gamma_k\bar{\epsilon}_k(2\|z_k-p\|+\bar{\epsilon}_k)+2\alpha_k\gamma_k\langle(\gamma V-\mu F)p+\frac{1}{\gamma_k}(u-p),x_{k+1}-p\rangle,
\end{aligned}\tag{3.31}$$

which immediately yields

$$\begin{aligned}&\langle(\mu F-\gamma V)p+\frac{1}{\gamma_k}(p-u),x_{k+1}-p\rangle\\
&\leqslant \frac{1}{2\alpha_k\gamma_k}(\|x_k-p\|^2-\|x_{k+1}-p\|^2)+\frac{1}{\gamma_k}\|(\gamma V-\mu F)p\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\
&\quad +\frac{\bar{\epsilon}_k}{2\alpha_k}(2\|z_k-p\|+\bar{\epsilon}_k)\\
&\leqslant \frac{\|x_k-x_{k+1}\|}{2\alpha_k\gamma_k}(\|x_k-p\|+\|x_{k+1}-p\|)+\frac{1}{\gamma_k}\|(\gamma V-\mu F)p\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\
&\quad +\frac{\bar{\epsilon}_k}{2\alpha_k}(2\|z_k-p\|+\bar{\epsilon}_k).
\end{aligned}\tag{3.32}$$

Since for any $w\in\omega_w(x_k)$ there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i}\rightharpoonup w$, we deduce from (3.24), (3.32), $\frac{1}{\gamma_k}\to\frac{1}{\xi}$, and $\|x_{k+1}-x_k\|=o(\alpha_k)$ that

$$\begin{aligned}\langle(\mu F-\gamma V)p+\frac{1}{\xi}(p-u),w-p\rangle &= \lim_{i\to\infty}\langle(\mu F-\gamma V)p+\frac{1}{\gamma_{k_i}}(p-u),x_{k_i}-p\rangle\\
&\leqslant \limsup_{k\to\infty}\langle(\mu F-\gamma V)p+\frac{1}{\gamma_k}(p-u),x_k-p\rangle\\
&= \limsup_{k\to\infty}\langle(\mu F-\gamma V)p+\frac{1}{\gamma_k}(p-u),x_{k+1}-p\rangle\\
&\leqslant \limsup_{k\to\infty}\frac{\|x_k-x_{k+1}\|}{2\alpha_k\gamma_k}(\|x_k-p\|+\|x_{k+1}-p\|)\\
&\quad +\limsup_{k\to\infty}\frac{1}{\gamma_k}\|(\gamma V-\mu F)p\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)
\end{aligned}$$

$$+\limsup_{k\to\infty}\frac{\bar{\epsilon}_k}{2\alpha_k}(2\|z_k-p\|+\bar{\epsilon}_k)$$
$$=0.$$

So, it follows that

$$\langle(\mu F-\gamma V)p+\frac{1}{\bar{\xi}}(p-u),p-w\rangle\geqslant 0,\quad \forall p\in VI(VI(\Omega,B),A).$$

Since $w\in\omega_w(x_k)\subset VI(VI(\Omega,B),A)$, by Minty's lemma, we have

$$\langle(\mu F-\gamma V)w+\frac{1}{\bar{\xi}}(w-u),p-w\rangle\geqslant 0,\quad \forall p\in VI(VI(\Omega,B),A);$$

that is, w is a solution of VIP (3.29). Utilizing the uniqueness of solutions of VIP (3.29), we get $w=x^*$, which hence implies that $\omega_w(x_k)=\{x^*\}$. Therefore, it is known that $\{x_k\}$ converges weakly to the unique solution $x^*\in VI(VI(\Omega,B),A)$ of VIP (3.29).

Finally, let us show that $\|x_k-x^*\|\to 0$ as $k\to\infty$. Indeed, utilizing (3.31) with $p=x^*$, we have

$$\begin{aligned}
\|x_{k+1}-x^*\|^2 &\leqslant (1-\alpha_k\gamma_k\frac{\tau^2-(\gamma l)^2}{\tau})\|x_k-x^*\|^2+2\alpha_k\|(\gamma V-\mu F)x^*\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\
&\quad+\gamma_k\bar{\epsilon}_k(2\|z_k-x^*\|+\bar{\epsilon}_k)+2\alpha_k\gamma_k\langle(\gamma V-\mu F)x^*+\frac{1}{\gamma_k}(u-x^*),x_{k+1}-x^*\rangle\\
&=(1-\alpha_k\gamma_k\frac{\tau^2-(\gamma l)^2}{\tau})\|x_k-x^*\|^2\\
&\quad+\alpha_k\gamma_k\frac{\tau^2-(\gamma l)^2}{\tau}\times\frac{\tau}{\tau^2-(\gamma l)^2}[\frac{2}{\gamma_k}\|(\gamma V-\mu F)x^*\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\
&\quad+\frac{\bar{\epsilon}_k}{\alpha_k}(2\|z_k-x^*\|+\bar{\epsilon}_k)+2\langle(\gamma V-\mu F)x^*+\frac{1}{\gamma_k}(u-x^*),x_{k+1}-x^*\rangle].
\end{aligned}\tag{3.33}$$

Since $\alpha_k\to 0,\ \alpha_k+\beta_k+\gamma_k=1,\ \sum_{k=0}^{\infty}\alpha_k=\infty,\ \beta_k\to\xi\in(\zeta,\frac{1}{2}],\ \bar{\epsilon}_k=o(\alpha_k)$, and $x_k\rightharpoonup x^*$, from (3.17) and (3.21) we conclude that $\sum_{k=0}^{\infty}\alpha_k\gamma_k\frac{\tau^2-(\gamma l)^2}{\tau}=\infty$ and

$$\begin{aligned}
&\limsup_{k\to\infty}\frac{\tau}{\tau^2-(\gamma l)^2}[\frac{2}{\gamma_k}\|(\gamma V-\mu F)x^*\|(\|v_k-x_k\|+\|x_k-x_{k+1}\|)\\
&+\frac{\bar{\epsilon}_k}{\alpha_k}(2\|z_k-x^*\|+\bar{\epsilon}_k)+2\langle(\gamma V-\mu F)x^*+\frac{1}{\gamma_k}(u-x^*),x_{k+1}-x^*\rangle]\leqslant 0.
\end{aligned}$$

Therefore, applying Lemma 2.11 to (3.33), we obtain that $\|x_k-x^*\|\to 0$ as $k\to\infty$. Utilizing (3.25) we also obtain that $\|z_k-x^*\|\to 0$ as $k\to\infty$. This completes the proof. □

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