Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 1215–1220
Research Article

ISSN : 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Majorization by starlike functions

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Communicated by E. Savaş

Abstract

The main object of this paper is to investigate some majorization problems involving the subclass $S(\alpha, A, B)$ of starlike functions in the open unit disk U. Relevant connections of the results presented here with those given by earlier workers on the subject are also indicated.

Keywords: Analytic function, starlike function, convex function, subordination, majorization, Quasi-subordination.
2010 MSC: 30C45.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Definition 1.1. For two functions f and g, which are analytic in U, the function f is said to be subordinate to g, written as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z)$$

if there exists a Schwarz function w analytic in U, with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \qquad (z \in U)$$

and such that

$$f(z) = g(\omega(z)) \qquad (z \in U).$$

In particular, if the function g is univalent in U, the above subordination is equivalent to

$$f(0) = g(0) \qquad \text{and} \qquad f(U) \subset g(U). \tag{1.1}$$

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doi:10.22436/jnsa.010.03.29

Received 2016-06-22

Definition 1.2. For two functions f and g, which are analytic in U, the function f is said to be majorized to g, written as

$$f << g \quad \text{or} \quad f(z) << g(z)$$

if there exists a function φ analytic in U, with

$$|\varphi(z)| < 1 \quad (z \in U)$$

and such that

$$f(z) = \varphi(z)\, g(z) \quad (z \in U),$$

(see MacGregor [6]).

The majorization is closely related to the concept of quasi-subordination between analytic functions, which was considered recently by (for example) Altıntaş and Owa [3]. Some majorization problems were studied by Altıntaş et al. in [4, 5]. Therefore, various subclasses of univalent functions in U were studied by Akgul in [1, 2].

We purpose to investigate the majorization problems associated with the class $S(\alpha, A, B)$ of starlike functions.

Definition 1.3. We denote by $S(\alpha, A, B)$ the class of functions satisfying the condition

$$\frac{zf'(z)}{f(z)} + \alpha z \left(\frac{zf'(z)}{f(z)}\right)' \prec \frac{1+Az}{1+Bz}, \tag{1.2}$$

$(z \in U,\ f \in A,\ 0 \leqslant \alpha \leqslant 1,\ -1 \leqslant B < A \leqslant 1)$.

Clearly, we have the following relationships:

- $S(0, 1, -1) = S^*$ is the class of starlike functions;
- $S(0, 0, -1) = C$ is the class of convex functions;
- $S(0, 1-2\alpha, -1) = S^*(\alpha)$ is the class of starlike functions of order α, $(0 \leqslant \alpha < 1)$;
- $S(0, 1-\alpha, -1) = C(\alpha)$ is the class of convex functions of order α, $(0 \leqslant \alpha < 1)$.

2. Majorization problems for the class $S(\alpha, A, B)$

We first state and prove the following Lemma 2.1.

Lemma 2.1 ([9]). *If the function* $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *is analytic in* U *and satisfies the condition*

$$h(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U,\ -1 \leqslant B < A \leqslant 1), \tag{2.1}$$

then

$$\operatorname{Re} h(z) > \frac{1-A}{1-B} = \beta. \tag{2.2}$$

Proof. Using (1.1) and (2.1) we have

$$h(z) = \frac{1+A\omega(z)}{1+B\omega(z)} \quad (\omega(0) = 0,\ |\omega(z)| < 1)$$

and

$$|\omega(z)| = \left|\frac{h(z)-1}{A-Bh(z)}\right|,$$

for $h(z) = u + iv$.

Since $|h(z)|^2 \geqslant [\text{Re } h(z)]^2$, we have

$$\left(1-B^2\right)u^2 - 2(1-AB)u + 1 - A^2 < 0,$$

which implies that

$$\frac{1-A}{1-B} < u = \text{Re } h(z) < \frac{1+A}{1+B}.$$

□

Lemma 2.2 ([8]). *If the function* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *is analytic in* U *and satisfies the condition*

$$\text{Re}\left(p(z) + \alpha z p'(z)\right) > \beta, \tag{2.3}$$

then

$$\text{Re } p(z) > \frac{\alpha + 2\beta}{\alpha + 2} \quad (0 \leqslant \alpha \leqslant 1,\ 0 \leqslant \beta < 1). \tag{2.4}$$

Theorem 2.3. *Let the function* $f(z)$ *be in the class* A *and suppose that* $g \in S(\alpha, A, B)$. *If* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$\left|f'(z)\right| \leqslant \left|g'(z)\right| \quad (|z| \leqslant r_1),$$

where

$$r_1 = r_1(\alpha, A, B) = \frac{3 + |1-2\gamma| - \sqrt{|1-2\gamma|^2 + 2|1-2\gamma| + 9}}{2|1-2\gamma|} \tag{2.5}$$

and

$$\gamma = \frac{\alpha(1-B) + 2(1-A)}{(\alpha+2)(1-B)} \quad (0 \leqslant \alpha \leqslant 1,\ -1 \leqslant B < A < 1). \tag{2.6}$$

Proof. Since $g \in S(\alpha, A, B)$, if we let

$$\frac{zg'(z)}{g(z)} = p(z) \quad \text{and} \quad \left(p(z) + \alpha z p'(z)\right) = h(z)$$

and $\beta = \frac{1-A}{1-B}$, then using (1.2), (2.2), (2.3), and (2.4) we find

$$\text{Re}\frac{zg'(z)}{g(z)} > \frac{\alpha + 2\beta}{\alpha + 2}.$$

Letting $\gamma = \frac{\alpha+2\beta}{\alpha+2}$, we obtain

$$\frac{zg'(z)}{g(z)} = \frac{1 - (1-2\gamma)\omega(z)}{1 + \omega(z)},$$

where $\omega(0) = 0$ and $|\omega(z)| < 1$.

Hence we find the inequality

$$|g(z)| \leqslant \left(\frac{(1+|z|)|z|}{1 - |1-2\gamma||z|}\right)\left|g'(z)\right| \quad (z \in U). \tag{2.7}$$

Since $f(z)$ is majorized by $g(z)$ in U, from (1.1) we have

$$f'(z) = \varphi(z)g'(z) + \varphi'(z)g(z). \tag{2.8}$$

We know that $\varphi(z)$ satisfies the inequality (Nehari, [7, p.168])

$$\left|\varphi'(z)\right| \leqslant \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in U), \tag{2.9}$$

and using (2.7) and (2.9) in (2.8), we get

$$\left|f'(z)\right| \leqslant \left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}\frac{(1+|z|)\,|z|}{1-|1-2\gamma|\,|z|}\right)\left|g'(z)\right|,$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \mu \quad (0 \leqslant \mu \leqslant 1)$$

we have the inequality

$$\left|f'(z)\right| \leqslant \frac{\Theta(\mu)}{(1-r)\,(1-|1-2\gamma|\,r)}\left|g'(z)\right| \quad (z \in \mathbb{U}), \tag{2.10}$$

where the function $\Theta(\mu)$ defined by

$$\Theta(\mu) = -r\mu^2 + (1-r)\,(1-|1-2\gamma|\,r)\,\mu + r \quad (0 \leqslant \mu \leqslant 1)$$

takes the maximum value at $\mu = 1$ with $r = r_1(\gamma)$ given by (2.5).

Furthermore, if $0 \leqslant q \leqslant r_1(\gamma)$ is given by (2.5), then we have

$$\Lambda(\mu) \leqslant \Lambda(1) = (1-r)\,(1-|1-2\gamma|\,r) \quad (0 \leqslant \mu \leqslant 1,\ 0 \leqslant q \leqslant r_1(\gamma)).$$

Hence, upon setting $\mu = 1$ in (2.10), we conclude that the inequality in (2.5) holds true for $|z| \leqslant r_1(\gamma)$ and is given by (2.6). The proof of Theorem 2.4 is based on Lemma 1 in [4],

$$f \in C(\gamma) \quad \Longrightarrow f \in S\left(\frac{1}{2}\gamma\right).$$

□

Theorem 2.4. *Let the function $f(z)$ be analytic in $\mathbb{U}$ and suppose that $g \in C(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$\left|f'(z)\right| \leqslant \left|g'(z)\right| \quad (|z| \leqslant r_2),$$

where

$$r_2 = r_2(\alpha, A, B) = \frac{3 + |1-\gamma| - \sqrt{|1-\gamma|^2 + 2\,|1-\gamma| + 9}}{2\,|1-\gamma|}$$

and

$$\gamma = \frac{\alpha(1-B) + 2(1-A)}{(\alpha+2)(1-B)} \quad (0 \leqslant \alpha \leqslant 1,\ -1 \leqslant B < A \leqslant 1).$$

Proof. Upon replacing γ in Theorem 2.3 by $\frac{1}{2}\gamma$, the conclusion follows. □

Letting special values for α, A, B we have the following corollaries.

Corollary 2.5. *If $g \in S(\alpha, 1, -1)$ and $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$\left|f'(z)\right| \leqslant \left|g'(z)\right| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = \frac{8 + 2\alpha - \sqrt{8\alpha^2 + 32\alpha + 48}}{2(2-\alpha)} \quad (0 \leqslant \alpha \leqslant 1).$$

Proof. We let $A = 1, B = -1$ in (2.6) and $\gamma = \frac{\alpha}{\alpha+2}$ in Theorem 2.3. □

Corollary 2.6. *If $g \in S(0, 1, -1)$ and $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$\left|f'(z)\right| \leqslant \left|g'(z)\right| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = 2 - \sqrt{3}.$$

Proof. We let $\alpha = 0, A = 1, B = -1$ in (2.6) and $\gamma = 0$ in Theorem 2.3. □

Corollary 2.7. *If* $g \in S(\alpha, 0, -1)$ *and* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \leqslant |g'(z)| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = \frac{2\alpha + 3 - \sqrt{3\alpha^2 + 10\alpha + 9}}{\alpha} \quad (0 \leqslant \alpha \leqslant 1).$$

Proof. We let $A = 0, \ B = -1$ in (2.6) and $\gamma = \frac{\alpha+1}{\alpha+2}$ in Theorem 2.3. □

Corollary 2.8. *If* $g \in S(1, 1, -1)$ *and* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \leqslant |g'(z)| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = 5 - \sqrt{22}.$$

Proof. We let $\alpha = 1, A = 1, B = -1$ in (2.6) and $\gamma = \frac{1}{3}$ in Theorem 2.3. □

Corollary 2.9. *If* $g \in C(\alpha, 1, -1)$ *and* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \leqslant |g'(z)| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = \frac{8 + 3\alpha - \sqrt{9\alpha^2 + 40\alpha + 48}}{4} \quad (0 \leqslant \alpha \leqslant 1).$$

Proof. We let $A = 1, B = -1$ in (2.6) and $\gamma = \frac{\alpha}{\alpha+2}$ in Theorem 2.4. □

Corollary 2.10. *If* $g \in C(0, 1, -1)$ *and* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \leqslant |g'(z)| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = 2 - \sqrt{3}.$$

Proof. We let $\alpha = 0, A = 1, B = -1$ in (2.6) and $\gamma = 0$ in Theorem 2.4. □

Corollary 2.11. *If* $g \in C(\alpha, 0, -1)$ *and* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \leqslant |g'(z)| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = \frac{7 + 3\alpha - \sqrt{9\alpha^2 + 38\alpha + 41}}{2} \quad (0 \leqslant \alpha \leqslant 1).$$

Proof. We let $A = 0, B = -1$ in (2.6) and $\gamma = \frac{\alpha+1}{\alpha+2}$ in Theorem 2.4. □

Corollary 2.12. *If* $g \in C(1, 1, -1)$ *and* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \leqslant |g'(z)| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = \frac{11 - \sqrt{97}}{4}.$$

Proof. We let $\alpha = 1, A = 1, B = -1$ in (2.6) and $\gamma = \frac{1}{3}$ in Theorem 2.4. □

Corollary 2.13. *If* $g \in S(0,0,-1)$ *and* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$\left|f'(z)\right| \leqslant \left|g'(z)\right| \quad (|z| \leqslant r),$$

where

$$|z| \leqslant r = \frac{1}{3}.$$

Remark 2.14. $S(0,0,-1) = S^*\left(\frac{1}{2}\right)$ and $C \subset S^*\left(\frac{1}{2}\right)$.

Acknowledgment

The authors would like to express their thanks to the referee and the editors for their helpful comments and advices.

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