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A new iterative scheme for finding attractive points of (α,β)-generalized hybrid set-valued mappings

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Abstract

In this paper, we first introduce the notions of (α,β)-generalized hybrid set-valued mappings, strongly attractive points, attractive points and condition I'. Then we construct an iterative method for finding attractive points of (α,β)-generalized hybrid set-valued mappings and obtain some convergence theorems of the proposed iterative scheme for (α,β)-generalized hybrid set-valued mappings defined on a uniformly convex Banach space by using of condition I' and demi-compact property, respectively. ©2017 All rights reserved.

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1. Introduction and preliminaries

Let X be a Banach space and C be a nonempty subset of X, and let N and R be the sets of positive integers and real numbers, respectively. We denote $CB(X)$ and $F(T)$ by the families of nonempty closed and bounded subsets and fixed points set of T, respectively. H is Hausdorff metric defined by

$$H(A,B)=\max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

where $d(x,B)=\inf\{\|x-z\| : z\in B\}$ and $d(y,A)=\inf\{\|y-z\| : z\in A\}$.

In 2010, Kocourek et al. [16] firstly introduced the notions of generalized hybrid mappings, which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings. A mapping $T : C \to C$ is called (α,β)-generalized hybrid if there exist $\alpha,\beta\in R$ such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \leqslant \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

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for all $x, y \in C$. T is called nonexpansive if T is $(1, 0)$-generalized hybrid; T is said to be hybrid if T is $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid [22], that is,

$$3\|Tx - Ty\|^2 \leqslant \|x - Ty\|^2 + \|Tx - y\|^2 + \|x - y\|^2, \text{ for each } x, y \in C.$$

T is called nonspreading if T is $(2, 1)$-generalized hybrid [17], that is,

$$2\|Tx - Ty\|^2 \leqslant \|x - Ty\|^2 + \|Tx - y\|^2, \text{ for each } x, y \in C.$$

In 2005, Sastry and Babu [19] introduced the Ishikawa iterative scheme for set-valued mappings in the following: let $T : C \to CB(C)$ be a set-valued mapping and fix $p \in F(T)$,

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \beta_n)x_n + \beta_n z_n \end{cases}$$

for all $n \in N$ and $\{\beta_n\} \subset (0, 1)$, $z_n \in Tx_n$ with $\|z_n - p\| = d(p, Tx_n)$ and

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n z'_n \end{cases}$$

for all $n \in N$ and $\{\alpha_n\} \subset (0, 1)$, $z'_n \in Ty_n$ with $\|z'_n - p\| = d(p, Ty_n)$.

In 2007, Agarwal et al. [2] introduced an iteration scheme for single-valued mappings. This iteration scheme is as the following:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n \end{cases}$$

for all $n \in N$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.

In 2011, Takahashi and Yao [24] got fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces. Kocourek et al. [16] also obtained fixed point theorems and weak convergence theorems of the Mann's iteration for generalized hybrid mappings in Hilbert spaces. This iteration scheme is as the following:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \end{cases} \tag{1.1}$$

for all $n \in N$ and $\{\alpha_n\} \subset (0, 1)$.

In 2012, Khan and Yildirim [15] introduced a multi-valued mapping version of the iteration scheme (1.1). This iteration scheme is as the following:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)v_n + \alpha_n u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n v_n \end{cases}$$

for all $n \in N$ and $v_n \in P_T(x_n)$, $u_n \in P_T(y_n)$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.

In 2015, Zheng [26] obtained convergence theorems of the Ishikawa iteration for (α, β)-generalized hybrid mappings. This iteration scheme is as the following:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n \end{cases}$$

for all $n \in N$ and $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.

A great deal of results involving (α, β)-generalized hybrid mappings, nonexpansive mappings and fixed points were obtained by several authors [1, 3, 4, 6–10, 12–14, 18, 21, 23].

$T : C \to CB(C)$ is said to be demi-compact [11] if for each sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} x_{n_k} = x \in C.$$

We now recall some basis definitions and useful lemmas.

Definition 1.1 ([20])**.** Let X be a Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to satisfy condition I if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(a) > 0$ for each $a \in (0, +\infty)$ such that

$$\|x - Tx\| \geqslant f(d(x, A(T))) \text{ for each } x \in C,$$

where $d(x, A(T)) = \inf\{\|x - p\| : p \in A(T)\}$.

Definition 1.2 ([5])**.** A Banach space X is said to be uniformly convex if for each $\varepsilon \in [0, 2]$, there exists $\delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \Rightarrow \|\frac{x + y}{2}\| < 1 - \delta_\varepsilon,$$

whenever $\|x - y\| \geqslant \varepsilon$.

Lemma 1.3 ([25])**.** *Let $q > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^q \leqslant \lambda\|x\|^q + (1 - \lambda)\|y\|^q - \omega_q(\lambda)g(\|x - y\|)$$

for each $x, y \in B_r(0) = \{x \in X : \|x\| \leqslant r\}$ and $\lambda \in [0, 1]$, where $\omega_q(\lambda) = \lambda^q(1 - \lambda) + \lambda(1 - \lambda)^q$.

In this paper, we first introduce the notions of (α, β)-generalized hybrid set-valued mappings, strongly attractive points, attractive points and condition I'. Moreover, we propose a new iteration for finding attractive points of an (α, β)-generalized hybrid set-valued mapping and obtain convergence theorems of an (α, β)-generalized hybrid set-valued mapping. This iterative scheme is denoted by the following:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \\ z_n = (1 - \gamma_n)y_n + \gamma_n w_n \end{cases} \tag{1.2}$$

for all $n \in N$ and $u_n \in Tz_n$, $y_n \in T((1 - \beta_n)x_n + \beta_n w_n)$, $w_n \in Tx_n$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$.

2. Main results

We begin with this section by introducing the notions of (α, β)-generalized hybrid set-valued mappings, strongly attractive points, and attractive points.

Definition 2.1. A mapping $T : C \to C$ is called (α, β)-generalized hybrid set-valued if there exist $\alpha, \beta \in R$ such that

$$\alpha H^2(Tx, Ty) + (1 - \alpha)d^2(x, Ty) \leqslant \beta d^2(y, Tx) + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$.

Definition 2.2. Let X be a Banach space and C be a nonempty subset of X, and let $T : C \to 2^X \backslash \{\emptyset\}$. A point $p \in X$ is called a strongly attractive point of T if for all $x \in C$, we have

$$H(p, Tx) \leqslant \|p - x\|.$$

We denote by $SA(T)$ the set of all strongly attractive points of T, that is,

$$SA(T) = \{p \in X : \ H(p, Tx) \leqslant \|p - x\| \text{ for all } x \in C\}.$$

Definition 2.3. Let X be a Banach space and C be a nonempty subset of X, and let $T : C \to 2^X \backslash \{\emptyset\}$. A point $p \in X$ is called an attractive point of T if for all $x \in C$, we have

$$d(p, Tx) \leqslant \|p - x\|.$$

We denote by $A(T)$ the set of all attractive points of T, that is,

$$A(T) = \{p \in X : \ d(p, Tx) \leqslant \|p - x\| \text{ for all } x \in C\}.$$

It is obvious that $SA(T) \subseteq A(T)$. Now, using condition I and the set $SA(T)$ we can introduce the notion of condition I'.

Definition 2.4. Let X be a Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to satisfy condition I', if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(a) > 0$ for each $a \in (0, +\infty)$ such that

$$\|x - Tx\| \geqslant f(d(x, SA(T))) \text{ for each } x \in C,$$

where $d(x, SA(T)) = \inf\{\|x - p\| : p \in SA(T)\}$.

It is not difficult to see that if a mapping T satisfies condition I', then T satisfies condition I. Next, we discuss convergence theorems of an (α, β)-generalized hybrid set-valued mapping in a uniformly convex Banach space.

Theorem 2.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T : C \to CB(C)$ be an (α, β)-generalized hybrid set-valued mapping with $SA(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by the iterative scheme (1.2), where $u_n \in Tz_n$, $y_n \in T((1 - \beta_n)x_n + \beta_n w_n)$, $w_n \in Tx_n$, $\{\alpha_n\}$, and $\{\beta_n\}$ and $\{\gamma_n\}$ belong to $(0, 1)$ such that*

$$\liminf_{n\to\infty} \alpha_n \beta_n (1 - \beta_n)(1 - \gamma_n) > 0. \tag{2.1}$$

Then the following conclusions hold:

(1) *the sequence $\{x_n\}$ is bounded;*
(2) $\lim_{n\to\infty} \|x_n - p\|$ *exists for each* $p \in SA(T)$;
(3) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Let $p \in SA(T)$, we have

$$\begin{aligned}
\|y_n - p\| &\leqslant H(T((1 - \beta_n)x_n + \beta_n w_n), p) \\
&\leqslant \|(1 - \beta_n)x_n + \beta_n w_n - p\| \\
&\leqslant (1 - \beta_n)\|x_n - p\| + \beta_n\|w_n - p\| \\
&\leqslant (1 - \beta_n)\|x_n - p\| + \beta_n H(Tx_n, p) \\
&\leqslant (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
&= \|x_n - p\|,
\end{aligned}$$

$$\begin{aligned}\|z_n - p\| &= \|(1-\gamma_n)y_n + \gamma_n w_n - p\|\\ &\leqslant (1-\gamma_n)\|y_n - p\| + \gamma_n\|w_n - p\|\\ &\leqslant (1-\gamma_n)\|y_n - p\| + \gamma_n H(Tx_n, p)\\ &\leqslant (1-\gamma_n)\|y_n - p\| + \gamma_n\|x_n - p\|\\ &\leqslant (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\|\\ &= \|x_n - p\|,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n u_n - p\|\\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|u_n - p\|\\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n H(Tz_n, p)\\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\|\\ &\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\|\\ &= \|x_n - p\|,\end{aligned}$$

which implies the sequence $\{\|x_n - p\|\}$ is nonincreasing. Therefore, the limit $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in SA(T)$. Hence the sequence $\{x_n\}$ is bounded.

Now we show the last conclusion holds. Let $r \geqslant \|x_1 - p\|$, then we get

$$\begin{aligned}&\|u_n - p\| \leqslant H(Tz_n, p) \leqslant \|z_n - p\| \leqslant \|x_n - p\| \leqslant r,\\ &\|y_n - p\| \leqslant \|x_n - p\| \leqslant r,\end{aligned}$$

and

$$\|w_n - p\| \leqslant H(Tx_n, p) \leqslant \|x_n - p\| \leqslant r.$$

It follows from Lemma 1.3 that

$$\begin{aligned}\|x_{n+1} - p\|^2 =&\|(1-\alpha_n)x_n + \alpha_n u_n - p\|^2\\ \leqslant&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|u_n - p\|^2\\ \leqslant&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n H^2(Tz_n, p)\\ \leqslant&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|z_n - p\|^2\\ =&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|(1-\gamma_n)y_n + \gamma_n w_n - p\|^2\\ \leqslant&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1-\gamma_n)\|y_n - p\|^2 + \alpha_n\gamma_n\|w_n - p\|^2\\ \leqslant&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1-\gamma_n)\|y_n - p\|^2 + \alpha_n\gamma_n H^2(Tx_n, p)\\ \leqslant&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1-\gamma_n)\|y_n - p\|^2 + \alpha_n\gamma_n\|x_n - p\|^2\\ \leqslant&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1-\gamma_n)H^2(T((1-\beta_n)x_n + \beta_n w_n), p) + \alpha_n\gamma_n\|x_n - p\|^2\\ \leqslant&(1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1-\gamma_n)\|(1-\beta_n)(x_n - p) + \beta_n(w_n - p)\|^2 + \alpha_n\gamma_n\|x_n - p\|^2\\ \leqslant&(1-\alpha_n + \alpha_n\gamma_n)\|x_n - p\|^2 + \alpha_n(1-\gamma_n)[(1-\beta_n)\|x_n - p\|^2\\ &+\beta_n\|w_n - p\|^2 - \beta_n(1-\beta_n)g(\|w_n - x_n\|)]\\ \leqslant&(1-\alpha_n + \alpha_n\gamma_n)\|x_n - p\|^2 + \alpha_n(1-\gamma_n)[(1-\beta_n)\|x_n - p\|^2\\ &\quad+\beta_n H^2(Tx_n, p) - \beta_n(1-\beta_n)g(\|w_n - x_n\|)]\\ \leqslant&(1-\alpha_n + \alpha_n\gamma_n)\|x_n - p\|^2 + \alpha_n(1-\gamma_n)[(1-\beta_n)\|x_n - p\|^2\\ &\quad+\beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|w_n - x_n\|)]\\ =&\|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)(1-\gamma_n)g(\|w_n - x_n\|)\end{aligned}$$

$$\leqslant \|x_n - p\|^2 - \alpha_n \beta_n (1-\beta_n)(1-\gamma_n) g(d(x_n, Tx_n)).$$

Then

$$\alpha_n \beta_n (1-\beta_n)(1-\gamma_n) g(d(x_n, Tx_n)) \leqslant \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Hence

$$\Sigma_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n)(1-\gamma_n) g(d(x_n, Tx_n)) \leqslant \|x_1 - p\|^2 < +\infty.$$

In view of

$$\liminf_{n\to\infty} \alpha_n \beta_n (1-\beta_n)(1-\gamma_n) > 0,$$

which implies

$$\lim_{n\to\infty} g(d(x_n, Tx_n)) = 0.$$

Since g is continuous, strictly increasing, convex, and $g(0) = 0$, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

which completes the proof. □

By Theorem 2.5, we show a strong convergence theorem of an (α, β)-generalized hybrid set-valued mapping in a uniformly convex Banach space.

Theorem 2.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T : C \to CB(C)$ be an (α, β)-generalized hybrid set-valued mapping with $SA(T) \neq \emptyset$ and satisfy condition I'. Suppose that the sequence $\{x_n\}$ is generated by the iterative scheme (1.2), where $u_n \in Tz_n$, $y_n \in T((1-\beta_n)x_n + \beta_n w_n)$, $w_n \in Tx_n$, and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ belonging to $(0, 1)$ satisfy (2.1). Then the sequence $\{x_n\}$ converges strongly to an attractive point of T.*

Proof. It follows from Theorem 2.5 that the sequence $\{x_n\}$ is bounded, the sequence $\{\|x_n - p\|\}$ is nonincreasing, and

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists for each } p \in SA(T).$$

We also have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

In view of Definition 2.4, we obtain

$$\lim_{n\to\infty} f(d(x_n, SA(T))) = 0,$$

which implies

$$\lim_{n\to\infty} d(x_n, SA(T)) = 0. \tag{2.2}$$

Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Indeed, for any $n, m \in N$, without loss of generality, we suppose $m > n$, then

$$\|x_m - p\| \leqslant \|x_n - p\|, \text{ for each } p \in SA(T),$$

and

$$\|x_n - x_m\| \leqslant \|x_n - p\| + \|p - x_m\| \leqslant 2\|x_n - p\|.$$

Thus, we obtain

$$\|x_n - x_m\| \leqslant 2 \inf\{\|x_n - p\| : p \in SA(T)\} = 2d(x_n, SA(T)).$$

Combining with (2.2), we get

$$\lim_{m,n\to\infty} \|x_n - x_m\| = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is uniformly convex, then there exists $u \in X$ such that

$$\lim_{n\to\infty} \|x_n - u\| = 0.$$

Then

$$\lim_{n\to\infty} d(u, Tx_n) \leqslant \lim_{n\to\infty} \|x_n - u\| + \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

It follows that

$$\lim_{n\to\infty} d(u, Tx_n) = 0.$$

Now, we prove $u \in A(T)$. Since T is an (α, β)-generalized hybrid set-valued mapping, for each $x \in C$, we have

$$\alpha H^2(Tx_n, Tx) + (1-\alpha)d^2(x_n, Tx) \leqslant \beta d^2(x, Tx_n) + (1-\beta)\|x_n - x\|^2.$$

Then

$$\alpha H^2(Tx_n, Tx) + (1-\alpha)d^2(x_n, Tx) \leqslant \beta[d(x, u) + d(u, Tx_n)]^2 + (1-\beta)\|x - u\|^2. \tag{2.3}$$

Since $d(x_n, Tx) \geqslant d(u, Tx) - d(u, x_n)$ and $d(u, x_n) < d(u, Tx)$ for n large enough, then

$$d^2(x_n, Tx) \geqslant [d(u, Tx) - d(u, x_n)]^2,$$

which implies

$$\alpha H^2(Tx_n, Tx) + (1-\alpha)[d(u, Tx) - d(u, x_n)]^2 \leqslant \alpha H^2(Tx_n, Tx) + (1-\alpha)d^2(x_n, Tx) \tag{2.4}$$

for n large enough. Since $d(u, Tx_n) \to 0$, then there exists $y_n \in Tx_n$ such that $\|u - y_n\| \to 0$ $(n \to \infty)$. From the definition of Hausdorff metric, it follows that

$$H(Tx_n, Tx) = \max\{\sup_{y\in Tx_n} d(y, Tx), \sup_{z\in Tx} d(z, Tx_n)\} \geqslant \sup_{y\in Tx_n} d(y, Tx) \geqslant d(y_n, Tx).$$

Since

$$\begin{aligned} d(u, Tx) = \inf_{y\in Tx} \|u - y\| &\leqslant \inf_{y\in Tx} \{\|u - y_n\| + \|y_n - y\|\} \\ &= \|u - y_n\| + \inf_{y\in Tx} \|y_n - y\| \\ &= \|u - y_n\| + d(y_n, Tx), \end{aligned}$$

we deduce that

$$d(y_n, Tx) \geqslant d(u, Tx) - \|u - y_n\|.$$

Therefore

$$H(Tx_n, Tx) \geqslant d(u, Tx) - \|u - y_n\|.$$

We notice that $\|u - y_n\| < d(u, Tx)$ for n large enough, thus

$$H^2(Tx_n, Tx) \geqslant [d(u, Tx) - \|u - y_n\|]^2. \tag{2.5}$$

Combining with (2.3), (2.4), and (2.5), we have

$$\begin{aligned} \alpha[d(u, Tx) - \|u - y_n\|]^2 + (1-\alpha)[d(u, Tx) - d(u, x_n)]^2 &\leqslant \alpha H^2(Tx_n, Tx) + (1-\alpha)d^2(x_n, Tx) \\ &\leqslant \beta[d(x, u) + d(u, Tx_n)]^2 + (1-\beta)\|x - u\|^2. \end{aligned}$$

Let $n \to \infty$, we obtain

$$d^2(u, Tx) \leqslant \|x - u\|^2,$$

which implies

$$d(u, Tx) \leqslant \|x - u\|, \text{for any } x \in C.$$

Hence $u \in A(T)$. This completes the proof. □

Using Theorem 2.5 and demi-compact property, we get the following theorem.

Theorem 2.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T: C \to CB(C)$ be an (α, β)-generalized hybrid and demi-compact set-valued mapping with $SA(T) = A(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is generated by the iterative scheme (1.2), where $u_n \in Tz_n$, $y_n \in T((1-\beta_n)x_n + \beta_n w_n)$, $w_n \in Tx_n$, and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ belonging to $(0,1)$ satisfy (2.1). Then the sequence $\{x_n\}$ converges strongly to an attractive point of T.*

Proof. It follows from Theorem 2.5 that the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists for each } p \in SA(T),$$

and

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Noticing that T is demi-compact, there is a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and a point $q \in X$ such that

$$\lim_{i\to\infty} \|x_{n_i} - q\| = 0.$$

Thus

$$\lim_{i\to\infty} d(q, Tx_{n_i}) \leqslant \lim_{i\to\infty} [d(x_{n_i}, Tx_{n_i}) + \|x_{n_i} - q\|],$$

which implies

$$\lim_{i\to\infty} d(q, Tx_{n_i}) = 0.$$

From the definition of (α, β)-generalized hybrid set-valued mapping, it follows that

$$\alpha H^2(Tx_{n_i}, Tx) + (1-\alpha) d^2(x_{n_i}, Tx) \leqslant \beta d^2(x, Tx_{n_i}) + (1-\beta)\|x_{n_i} - x\|^2$$

for each $x \in C$. In a similar way to Theorem 2.6, we deduce $q \in A(T)$. Since $\lim_{i\to\infty} \|x_{n_i} - q\| = 0$ and $\lim_{n\to\infty} \|x_n - q\|$ exists, we get

$$\lim_{n\to\infty} \|x_n - q\| = 0.$$

This completes the proof. □

To end this section, we give an example to show that an (α, β)-generalized hybrid set-valued mapping which fails to be nonexpansive has an attractive point.

Example 2.8. Let $C = [0, 3]$ and $T: C \to CB(C)$ is defined by

$$Tx = \begin{cases} \{0\}, & \text{if } x \neq 3, \\ [0.5, 1], & \text{if } x = 3. \end{cases}$$

We pick $x = \frac{8}{3}$, $y = 3$, then

$$H(Tx, Ty) = H(\{0\}, [0.5, 1]) = 1 > \frac{1}{3} = \|x - y\|.$$

Therefore, T is not a nonexpansive mapping. Let $\alpha = 2, \beta = \frac{1}{2}$, we verify that T is a $(2, \frac{1}{2})$-generalized hybrid set-valued mapping, that is,

$$2H^2(Tx, Ty) \leqslant d^2(x, Ty) + \frac{1}{2} d^2(Tx, y) + \frac{1}{2}\|x - y\|^2.$$

Next, we consider the following four cases:
Case I. Let $x, y \in [0, 3)$, then

$$2H^2(Tx, Ty) = 2H^2(\{0\}, \{0\}) = 0 \leqslant d^2(x, Ty) + \frac{1}{2} d^2(Tx, y) + \frac{1}{2}\|x - y\|^2.$$

Case II. Let $x = 3, y \in [0, 3)$, then

$$2H^2(Tx, Ty) = 2H^2([0.5, 1], \{0\}) = 2,$$

and

$$d^2(x, Ty) + \frac{1}{2}d^2(Tx, y) + \frac{1}{2}\|x - y\|^2 = d^2(3, \{0\}) + \frac{1}{2}d^2(y, [0.5, 1]) + \frac{1}{2}\|3 - y\|^2 \geqslant 9.$$

Hence

$$2H^2(Tx, Ty) < d^2(x, Ty) + \frac{1}{2}d^2(Tx, y) + \frac{1}{2}\|x - y\|^2.$$

Case III. Let $x \in [0, 3), y = 3$, then

$$2H^2(Tx, Ty) = 2H^2(\{0\}, [0.5, 1]) = 2,$$

and

$$d^2(x, Ty) + \frac{1}{2}d^2(Tx, y) + \frac{1}{2}\|x - y\|^2 = d^2(x, [0.5, 1]) + \frac{1}{2}d^2(\{0\}, 3) + \frac{1}{2}\|x - 3\|^2 \geqslant \frac{9}{2}.$$

Thus

$$2H^2(Tx, Ty) < d^2(x, Ty) + \frac{1}{2}d^2(Tx, y) + \frac{1}{2}\|x - y\|^2.$$

Case IV. Let $x = y = 3$, then

$$2H^2(Tx, Ty) = 2H^2([0.5, 1], [0.5, 1]) = 0 \leqslant d^2(x, Ty) + \frac{1}{2}d^2(Tx, y) + \frac{1}{2}\|x - y\|^2.$$

Therefore, T is a $(2, \frac{1}{2})$-generalized hybrid set-valued mapping. For each $x \in [0, 3]$, we have

$$H(Tx, 0) \leqslant \|x - 0\|,$$

which implies 0 is an attractive point of T.

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