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Solve the split equality problem by a projection algorithm with inertial effects

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Abstract

The split equality problem has wide applicability in many fields of applied mathematics. In this paper, by using the inertial extrapolation, we introduce an inertial projection algorithm for solving the split equality problem. The weak convergence of the proposed algorithm is shown. Finally, we present a numerical example to illustrate the efficiency of the inertial projection algorithm.

Keywords: Split equality problem, projection algorithm, inertial extrapolation.
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1. Introduction

In this article, we shall consider the split equality problem (SEP) which was firstly introduced by Moudafi and Oliny [18].

Problem 1.1. Find x, y with the property

$$x \in C, \ y \in Q, \ \text{such that} \ \ Ax = By,$$

where $C \subset H_1$, $Q \subset H_2$ are two nonempty closed convex sets, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, and H_1, H_2 and H_3 are real Hilbert spaces.

It is obvious that the SEP allows asymmetric and partial relations between the variables x and y. Many problems in mathematics and other sciences can be regarded as a split equality problem, such as the variational form of a PDE's in domain decomposition for PDE's [3], the agents who interplay only via some components of their decision variables in decision [2] and the weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity in the (IMRT) [7].

Many methods for computing the solution of Problem 1.1 are projection methods, which have been extensively studied in the literature [10, 15, 16, 19]. Byrne and Moudafi [6] introduced the classical

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projection gradient algorithm, which is also called as the simultaneous iterative methods [17]:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \tag{1.1}$$

where $\gamma_k \in (\varepsilon, 2/(\lambda_A + \lambda_B) - \varepsilon)$, λ_A and λ_B are the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B), respectively. To determine stepsize γ_k, one needs first calculate (or estimate) the operator norms $\|A\|$ and $\|B\|$. In general, it is difficult or even impossible.

In order to overcome this, the authors [11] proposed a choice of the stepsize γ_k for the projection algorithm (1.1) as follows:

Algorithm 1.2.

$$\gamma_k = \sigma_k \min\left\{ \frac{\|A\bar{x}_k - B\bar{y}_k\|^2}{\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2}, \frac{\|A\bar{x}_k - B\bar{y}_k\|^2}{\|B^*(A\bar{x}_k - B\bar{y}_k)\|^2} \right\}, \tag{1.2}$$

where $0 < \sigma_k < 1$. Note that the choice of the stepsize γ_k in (1.2) is independent of the norms $\|A\|$ and $\|B\|$.

As an acceleration process, the inertial extrapolation algorithms were widely studied. The researchers constructed many iterative algorithms by using inertial extrapolation, such as inertial forward-backward algorithm [14], inertial extragradient method [9, 12, 13] and fast iterative shrinkage thresholding algorithm (FISTA) ([5, 8]). The main feature of the inertial extrapolation algorithms is that the next iterate is defined by making use of the previous two iterates.

In this paper, by using the inertial extrapolation, we introduce an inertial projection algorithm (1.1) as follows:

Algorithm 1.3.

$$\begin{cases} (\bar{x}_k, \bar{y}_k) = (x_k, y_k) + \alpha_k(x_k - x_{k-1}, y_k - y_{k-1}), \\ x_{k+1} = P_C(\bar{x}_k - \gamma_k A^*(A\bar{x}_k - B\bar{y}_k)), \\ y_{k+1} = P_Q(\bar{y}_k + \gamma_k B^*(A\bar{x}_k - B\bar{y}_k)), \end{cases} \tag{1.3}$$

where $\alpha_k \in (0, 1)$ and the stepsize γ_k is chosen in the same way as (1.2).

The structure of the paper is as follows. In the next section, we present some lemmas which will be used in the main results. In Section 3, the weak convergence theorem of the inertial projection algorithm is given. In the final section, Section 4, some numerical results are provided, which show the advantages of the proposed algorithm.

2. Preliminaries

In this section, we present some lemmas which will be used in the proof of the main results.

Definition 2.1. Let K be a closed convex subset of a real Hilbert space H. P_K is called the projection from H on K, if for each $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$.

The following lemma is a useful characterization of projections.

Lemma 2.2. *Let K be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in K$. Then $z = P_K x$, if and only if there holds the relation:*

$$\langle x - z, y - z \rangle \leqslant 0, \quad \forall y \in K.$$

Lemma 2.3. *For any $x, y \in H$ and $z \in \Omega$, it holds*

$$\|P_\Omega(x) - z\|^2 \leqslant \|x - z\|^2 - \|P_\Omega(x) - x\|^2.$$

Lemma 2.4 ([1, Lemma 3]). *Let (ψ_n), (δ_n) and (α_n) be sequences in $[0, +\infty)$ such that*

$$\psi_{n+1} \leqslant \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n,$$

for all $n \geqslant 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number α with $0 \leqslant \alpha_n \leqslant \alpha < 1$, for all $n \in \mathbb{N}$. Then the following hold:

(i) $\sum_{n \geqslant 1}[\psi_n - \psi_{n-1}]_+ < +\infty$, *where* $[t]_+ = \max\{t, 0\}$;

(ii) *there exists* $\psi^* \in [0, +\infty)$ *such that* $\lim_{n \to +\infty} \psi_n = \psi^*$.

Finally, we recall a well-known result on weak convergence in Hilbert spaces.

Lemma 2.5 ([4]). *Let H be a Hilbert space and $\|\cdot\|$ be a norm on H, then,*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \quad \forall t \in R,\ \forall x, y \in H.$$

3. The main results

In this section, we present the weak convergence theorem and its proof for the inertial projection algorithm (1.3).

We make the following assumptions to $\{\alpha_k\}$:

(C1) $0 \leqslant \alpha_k \leqslant \alpha$, where $\alpha \in [0, 1)$;

(C2) $\sum_{k=1}^{+\infty} \alpha_k(\|x_k - x_{k-1}\|^2 + \|y_k - y_{k-1}\|^2) < +\infty$;

(C3) $\lim_{k \to \infty} \alpha_k(\|x_k - x_{k-1}\| + \|y_k - y_{k-1}\|) = 0$.

Remark 3.1.

(1) Comparing other works [8, 16], we need the additional assumptions (C3) to present the convergence theorem.

(2) According to Moudafi's comments in [16], assumptions (C2) and (C3) involve the iterates that are a priori unknown. In practice, it is easy to enforce it by applying an appropriate on-line rule. For example, choosing

$$\alpha_k = \min\left(\frac{1}{(k+1)^4 \eta_k}, \frac{1}{(k+1)\zeta_k}, 0.94\right),$$

where

$$\eta_k = \|x_k - x_{k-1}\|^2 + \|y_k - y_{k-1}\|^2, \zeta_k = \|x_k - x_{k-1}\| + \|y_k - y_{k-1}\|.$$

Theorem 3.2. *Let the sequence (x_k, y_k) be generated by Algorithm 1.3. Suppose the assumptions (C1) and (C2) hold. Then $(\|x_k - x^*\|^2 + \|y_k - y^*\|^2)$ is convergent with $(x^*, y^*) \in \Gamma$.*

Proof. Take $(x^*, y^*) \in \Gamma$, i.e., $x^* \in C$, $y^* \in Q$, $Ax^* = By^*$. From Lemma 2.3, the second equality of (1.3) successively gives

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &= \|P_C(\bar{x}_k - \gamma_k A^*(A\bar{x}_k - B\bar{y}_k) - x^*)\|^2 \\
&\leqslant \|\bar{x}_k - \gamma_k A^*(A\bar{x}_k - B\bar{y}_k) - x^*\|^2 \\
&\leqslant \|\bar{x}_k - x^*\|^2 + \gamma_k^2\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2 - 2\gamma_k\langle A\bar{x}_k - Ax^*, A\bar{x}_k - B\bar{y}_k\rangle.
\end{aligned}$$

Using the equality

$$-2\langle A\bar{x}_k - Ax^*, A\bar{x}_k - B\bar{y}_k\rangle = -\|A\bar{x}_k - B\bar{y}_k\|^2 - \|A\bar{x}_k - Ax^*\|^2 + \|B\bar{y}_k - Ax^*\|^2,$$

we obtain

$$\begin{aligned}\|x_{k+1}-x^*\|^2 \leqslant \|\bar{x}_k - x^*\|^2 + \gamma_k^2\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2 - \gamma_k\|A\bar{x}_k - B\bar{y}_k\|^2 \\ - \gamma_k\|A\bar{x}_k - Ax^*\|^2 + \gamma_k\|B\bar{y}_k - Ax^*\|^2.\end{aligned}$$

Similarly, the third equality of (1.3) leads to

$$\begin{aligned}\|y_{k+1}-y^*\|^2 \leqslant \|\bar{y}_k - y^*\|^2 + \gamma_k^2\|B^*(A\bar{x}_k - B\bar{y}_k)\|^2 - \gamma_k\|A\bar{x}_k - B\bar{y}_k\|^2 \\ - \gamma_k\|B\bar{y}_k - By^*\|^2 + \gamma_k\|A\bar{x}_k - By^*\|^2.\end{aligned}$$

By adding the two last inequalities and taking into account the fact that $Ax^* = By^*$, we finally get

$$\begin{aligned}\|x_{k+1}-x^*\|^2 + \|y_{k+1}-y^*\|^2 \leqslant \|\bar{x}_k - x^*\|^2 + \|\bar{y}_k - y^*\|^2 - \gamma_k(\|A\bar{x}_k - B\bar{y}_k\|^2 - \gamma_k\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2) \\ - \gamma_k(\|A\bar{x}_k - B\bar{y}_k\|^2 - \gamma_k\|B^*(A\bar{x}_k - B\bar{y}_k)\|^2).\end{aligned} \tag{3.1}$$

The first equality of (1.3) and Lemma 2.5 imply

$$\begin{aligned}\|\bar{x}_k - x^*\|^2 &= \|(1+\alpha_k)(x_k - x^*) - \alpha_k(x_{k-1} - x^*)\|^2 \\ &= (1+\alpha_k)\|x_k - x^*\|^2 - \alpha_k \| x_{k-1} - x^*\|^2 + \alpha_k(1+\alpha_k)\|x_k - x_{k-1}\|^2.\end{aligned} \tag{3.2}$$

Similarly, we obtain

$$\|\bar{y}_k - y^*\|^2 = (1+\alpha_k)\|y_k - y^*\|^2 - \alpha_k\|y_{k-1} - y^*\|^2 + \alpha_k(1+\alpha_k)\|y_k - y_{k-1}\|^2. \tag{3.3}$$

Set $\varphi_k := \|x_k - x^*\|^2 + \|y_k - y^*\|^2$. Combining (3.1), (3.2), (3.3), we obtain

$$\begin{aligned}\varphi_{k+1} - \varphi_k &\leqslant \alpha_k(\varphi_k - \varphi_{k-1}) - \gamma_k(\|A\bar{x}_k - B\bar{y}_k\|^2 - \gamma_k\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2) \\ &\quad - \gamma_k(\|A\bar{x}_k - B\bar{y}_k\|^2 - \gamma_k\|B^*(A\bar{x}_k - B\bar{y}_k)\|^2) \\ &\quad + \alpha_k(1+\alpha_k)[\|x_k - x_{k-1}\|^2 + \|y_k - y_{k-1}\|^2] \\ &\leqslant \alpha_k(\varphi_k - \varphi_{k-1}) + \delta_k,\end{aligned} \tag{3.4}$$

where $\delta_k := \alpha_k(1+\alpha_k)[\|x_k - x_{k-1}\|^2 + \|y_k - y_{k-1}\|^2]$. By using (C2) and Lemma 2.4, we obtain that φ_k consequently converges to some finite limit, say $\varphi(x^*, y^*)$. □

Theorem 3.3. *Let the sequence* (x_k, y_k) *be generated by Algorithm 1.3. Assume* $\sigma_k \in [\varepsilon, 1-\varepsilon]$, $\varepsilon \in (0, 1/2]$ *and suppose the assumptions* (C1)-(C3) *hold. Then*

$$\lim_{k\to\infty} \|Ax_k - By_k\| = \lim_{k\to\infty} \|A\bar{x}_k - B\bar{y}_k\| = 0.$$

Proof. Now, we divide the proof of the first conclusion into two parts.

Case 1. Suppose that there exists k_0 such that $\|A^*(A\bar{x}_k - B\bar{y}_k)\| \geqslant \|B^*(A\bar{x}_k - B\bar{y}_k)\|$, for all $k \geqslant k_0$. In this situation, $\gamma_k = \sigma_k \frac{\|A\bar{x}_k - B\bar{y}_k\|^2}{\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2}$. Using (3.4) and Theorem 3.2, we obtain

$$\lim_{k\to\infty} \gamma_k(\|A\bar{x}_k - B\bar{y}_k\|^2 - \gamma_k\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2) = \lim_{k\to\infty} \sigma_k(1-\sigma_k)\frac{\|A\bar{x}_k - B\bar{y}_k\|^4}{\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2} = 0,$$

which together with $\sigma_k \in [\varepsilon, 1-\varepsilon]$ implies

$$\lim_{k\to\infty} \frac{\|A\bar{x}_k - B\bar{y}_k\|^2}{\|A^*(A\bar{x}_k - B\bar{y}_k)\|} = 0.$$

Using $\gamma_k\|A^*(A\bar{x}_k - B\bar{y}_k)\| = \sigma_k \frac{\|A\bar{x}_k - B\bar{y}_k\|^2}{\|A^*(A\bar{x}_k - B\bar{y}_k)\|}$, we get

$$\lim_{k\to\infty} \gamma_k\|A^*(A\bar{x}_k - B\bar{y}_k)\| = 0.$$

From the assumption $\|A^*(A\bar{x}_k - B\bar{y}_k)\| \geqslant \|B^*(A\bar{x}_k - B\bar{y}_k)\|$, it follows

$$\lim_{k\to\infty} \gamma_k \|B^*(A\bar{x}_k - B\bar{y}_k)\| = 0.$$

It is easy to show that

$$\|A\bar{x}_k - B\bar{y}_k\| \leqslant \|A\| \frac{\|A\bar{x}_k - B\bar{y}_k\|^2}{\|A^*(A\bar{x}_k - B\bar{y}_k)\|}.$$

Consequently, we get

$$\lim_{k\to\infty} \|A\bar{x}_k - B\bar{y}_k\| = 0.$$

Conversely, suppose that there exists k_1 such that $\|A^*(A\bar{x}_k - B\bar{y}_k)\| \leqslant \|B^*(A\bar{x}_k - B\bar{y}_k)\|$, for all $k \geqslant k_1$, following the above process, we obtain the results.

Case 2. Suppose that there does not exist k_0 such that $\|A^*(A\bar{x}_k - B\bar{y}_k)\| \geqslant \|B^*(A\bar{x}_k - B\bar{y}_k)\|$ or $\|A^*(A\bar{x}_k - B\bar{y}_k)\| \leqslant \|B^*(A\bar{x}_k - B\bar{y}_k)\|$, for all $k \geqslant k_0$. We divide the sequence $\|A^*(A\bar{x}_k - B\bar{y}_k)\|$ into two subsequences: one subsequence satisfies $\|A^*(A\bar{x}_k - B\bar{y}_k)\| \geqslant \|B^*(Ax_k - By_k)\|$ denoted by $A^*(Ax_{k_m} - By_{k_m})$ and the other subsequence satisfies $\|A^*(A\bar{x}_k - B\bar{y}_k)\| < \|B^*(A\bar{x}_k - B\bar{y}_k)\|$, denoted by $A^*(A\bar{x}_{k_n} - B\bar{y}_{k_n})$. Following the process of Case 1, we show that the results hold for the subsequences with k_m and k_n. Hence we obtain $\lim_{k\to\infty} \|A\bar{x}_k - B\bar{y}_k\| = 0$.

Next we show $\lim_{k\to\infty} \|Ax_k - By_k\| = 0$. From the linearity of the operators A and B, and the first equality of (1.3), we obtain

$$\begin{aligned}
\|Ax_k - By_k\| &= \|[A\bar{x}_k - \alpha_k A(x_k - x_{k-1})] - [B\bar{y}_k - \alpha_k B(y_k - y_{k-1})]\| \\
&\leqslant \|A\bar{x}_k - B\bar{y}_k\| + \alpha_k(\|A(x_k - x_{k-1})\| + \alpha_k\|B(y_k - y_{k-1})\|) \\
&\leqslant \|A\bar{x}_k - B\bar{y}_k\| + (\|A\| + \|B\|)\alpha_k(\|x_k - x_{k-1}\| + \|y_k - y_{k-1}\|),
\end{aligned}$$

which with $\lim_{k\to\infty} \|A\bar{x}_k - B\bar{y}_k\| = 0$ and (C3) implies

$$\lim_{k\to\infty} \|Ax_k - By_k\| = 0.$$

□

Theorem 3.4. *Assume* $\sigma_k \in [\varepsilon, 1-\varepsilon]$, $\varepsilon \in (0, 1/2]$ *and suppose the assumptions* (C1)-(C3) *hold. Then the sequence* (x_k, y_k) *generated by Algorithm* 1.3 *weakly converges to a solution of the SEP. Furthermore, both* (x_k) *and* (y_k) *are asymptotically regular.*

Proof. We first prove that (x_k) and (y_k) are asymptotically regular. Indeed, the first and the second equalities of (1.3) give

$$\begin{aligned}
\|x_{k+1} - x_k\| &= \|P_C(\bar{x}_k - \gamma_k A^*(A\bar{x}_k - B\bar{y}_k)) - x_k\| \\
&\leqslant \|\bar{x}_k - \gamma_k A^*(A\bar{x}_k - B\bar{y}_k) - x_k\| \\
&\leqslant \|\alpha_k(x_k - x_{k-1}) - \gamma_k A^*(A\bar{x}_k - B\bar{y}_k)\| \\
&\leqslant \alpha_k\|x_k - x_{k-1}\| + \gamma_k\|A^*(A\bar{x}_k - B\bar{y}_k)\| \\
&\leqslant \alpha_k\|x_k - x_{k-1}\| + \gamma_k\|A\|\|A\bar{x}_k - B\bar{y}_k\|.
\end{aligned}$$

Using (C3) and Theorem 3.3, we have

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.5}$$

Similarly, we get

$$\lim_{k\to\infty} \|y_{k+1} - y_k\| = 0.$$

So, (x_k) and (y_k) are asymptotically regular.

The first equality of (1.3) gives

$$\begin{aligned}\|\bar{x}_k - x_{k+1}\| &= \|x_k - x_{k+1} + \alpha_k(x_k - x_{k-1})\| \\ &\leqslant \|x_k - x_{k+1}\| + \alpha_k\|x_k - x_{k-1}\|,\end{aligned}$$

which with (C3) and (3.5) yields

$$\lim_{k\to\infty} \|\bar{x}_k - x_{k+1}\| = 0.$$

Similarly, we have

$$\lim_{k\to\infty} \|\bar{y}_k - y_{k+1}\| = 0.$$

From (1.2), we get

$$\gamma_k \geqslant \varepsilon \min\left\{\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}\right\}.$$

So, it follows

$$\lim_{k\to\infty} \frac{\bar{x}_k - x_{k+1}}{\gamma_k} = 0, \quad \text{and} \quad \lim_{k\to\infty} \frac{\bar{y}_k - y_{k+1}}{\gamma_k} = 0.$$

Let $(\hat{x}, \hat{y}) \in \omega_w(x_k, y_k)$, then there exists a subsequence of (x_k) (resp. (y_k)) (again labeled (x_k) (resp. (y_k))) which converges weakly to $\hat{x}$ (resp. $\hat{y}$). The two equalities in (1.3) can be rewritten as:

$$\begin{cases} \dfrac{\bar{x}_k - x_{k+1}}{\gamma_k} - A^*(A\bar{x}_k - B\bar{y}_k) \in N_C(x_{k+1}), \\ \dfrac{\bar{y}_k - y_{k+1}}{\gamma_k} - B^*(A\bar{x}_k - B\bar{y}_k) \in N_Q(y_{k+1}). \end{cases}$$

Since the graphs of the maximal monotone operators N_C, N_Q are weakly-strongly closed, by passing to the limit in the last inclusions and using Theorem 3.3, we obtain $0 \in N_C(\hat{x})$ and $0 \in N_Q(\hat{y})$ which are equivalent to

$$\hat{x} \in C \quad \text{and} \quad \hat{y} \in Q.$$

Furthermore, the weak convergence of $(Ax_k - By_k)$ to $A\hat{x} - B\hat{y}$ and lower semicontinuity of the norm imply

$$\|A\hat{x} - B\hat{y}\| \leqslant \liminf_{k\to\infty} \|Ax_k - By_k\| = 0,$$

where the equality comes from Theorem 3.3. Hence, $(\hat{x}, \hat{y}) \in \Gamma$.

To show the uniqueness of the weak cluster points, we will use the same trick as in the celebrated Opial Lemma. Indeed, let$(\bar{x}, \bar{y})$ be other weak cluster point of (x_k, y_k). By passing to the limit in the relation

$$\varphi_k(\hat{x}, \hat{y}) = \varphi_k(\bar{x}, \bar{y}) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2 + 2\langle x_k - \bar{x}, \bar{x} - \hat{x}\rangle + 2\langle y_k - \bar{y}, \bar{y} - \hat{y}\rangle,$$

we obtain

$$\varphi(\hat{x}, \hat{y}) = \varphi(\bar{x}, \bar{y}) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2.$$

Reversing the role of $(\hat{x}, \hat{y})$ and $(\bar{x}, \bar{y})$, we also have

$$\varphi(\bar{x}, \bar{y}) = \varphi(\hat{x}, \hat{y}) + \|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2.$$

By adding the two last equalities, we obtain

$$\|\hat{x} - \bar{x}\|^2 + \|\hat{y} - \bar{y}\|^2 = 0.$$

Hence, $(\hat{x}, \hat{y}) = (\bar{x}, \bar{y})$. This implies that the whole sequence (x_k, y_k) weakly converges to a solution of the SEP, which completes the proof. $\square$

4. Preliminary numerical results

In this section, we consider a numerical example in [11] to demonstrate the effectiveness of Algorithm 1.3. We apply Algorithm 1.3 to solve a numerical example, and compare the numerical results with those of Algorithm 1.2.

We denote the vector with all elements 0 by e_0, and the vector with all elements 1 by e_1 in what follows. In the numerical results listed in the following table, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

Example 4.1. The SEP with $A = (a_{ij})_{J\times N}$, $B = (b_{ij})_{J\times M}$, $C = \{x \in \mathbb{R}^N | \|x\| \leqslant 0.25\}$, $Q = \{y \in \mathbb{R}^M | e_0 \leqslant y \leqslant u\}$, where $a_{ij} \in [0,1]$, $b_{ij} \in [0,1]$ and $u \in [e_1, 2e_1]$ are all generated randomly. In the implementations, we take $\|Ax - By\| < \varepsilon = 10^{-4}$ as the stopping criterion. Take the initial value $x_0 = 10e_1$, $y_0 = -10e_1$ for two algorithms.

We make comparison of Algorithm 1.3 with Algorithm 1.2 with different J, N, M, and report the results in Table 1. We take $\sigma_k = 0.69$, $\gamma_k = 0.69 \times \min\{\frac{\|A\bar{x}_k - B\bar{y}_k\|^2}{\|A^*(A\bar{x}_k - B\bar{y}_k)\|^2}, \frac{\|A\bar{x}_k - B\bar{y}_k\|^2}{\|B^*(A\bar{x}_k - B\bar{y}_k)\|^2}\}$, and $\alpha_k = \min\left(\frac{1}{(k+1)^{3/2}\eta_k}, \frac{1}{(k+1)\zeta_k}, 0.94\right)$, where $\eta_k = \|x_k - x_{k-1}\|^2 + \| y_k - y_{k-1}\|^2$, $\zeta_k = \|x_k - x_{k-1}\| + \| y_k - y_{k-1}\|$. For comparison, the same random values are taken in each test for two algorithms.

Table 1: Computational results for Example 4.1 with different dimensions.

	J		20	40	60	80	100
$(N, M) = (50, 50)$	Algorithm 1.2	Iter.	720	4870	24162	14806	131045
		Sec.	0.172	2.293	6.318	5.990	36.504
$(N, M) = (50, 50)$	Algorithm 1.3	Iter.	247	2509	7539	8055	78855
		Sec.	0.078	1.264	2.231	3.526	24.633
$(N, M) = (80, 100)$	Algorithm 1.2	Iter.	182	1668	3482	11198	22431
		Sec.	0.062	0.484	1.295	4.602	11.528
$(N, M) = (80, 100)$	Algorithm 1.3	Iter.	168	337	719	2529	4575
		Sec.	0.078	0.125	0.296	1.154	2.730
$(N, M) = (200, 150)$	Algorithm 1.2	Iter.	150	1011	2360	3669	5288
		Sec.	0.047	0.437	1.357	2.605	5.288
$(N, M) = (200, 150)$	Algorithm 1.3	Iter.	139	244	329	707	876
		Sec.	0.047	0.125	0.218	0.562	0.920

From Table 1, we could observe that the inertial projection Algorithm 1.3 behaves far better than Algorithm 1.2 from the number of iterations and the cpu time. It really speeds up Algorithm 1.2.

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