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Viscosity approximation methods for the implicit midpoint rule of asymptotically nonexpansive mapping in complete $\mathrm{CAT}(0)$ spaces

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Abstract

In this paper, the implicit midpoint rule of asymptotically nonexpansive mapping in $\mathrm{CAT}(0)$ spaces is introduced. By the viscosity approximation method, we prove that the proposed implicit iteration converges strongly to a fixed point of asymptotically nonexpansive mapping under certain assumptions imposed on the sequence of parameters. The results presented in the paper improve and extend various results in the existing literature.

Keywords: Asymptotically nonexpansive, viscosity approximation methods, complete $\mathrm{CAT}(0)$ space, $\triangle$-convergence, implicit iteration.

2010 MSC: 47J05, 47H09, 49J25.

1. Introduction

The concept of variational inequalities plays an important role in various kinds of problems in pure and applied sciences. In particular, viscosity approximation methods have attracted the attention of many authors, and many important results about viscosity approximation methods of nonexpansive mappings are studied in $\mathrm{CAT}(0)$ spaces. In 1976, the concept of $\triangle$-convergence in general metric spaces was coined by Lim [18]. Then, Kirk et al. [16] specialized this concept to $\mathrm{CAT}(0)$ spaces, and proved that it is very similar to the weak convergence in the Banach space setting. Dhompongsa et al. [14] and Abbas et al. [1] obtained $\triangle$-convergence theorems for the Mann and Ishikawa iterations in $\mathrm{CAT}(0)$ space. Moreover, with the ideas of Attouch [4], viscosity approximation methods for nonexpansive mapping in Hilbert space was introduced by Moudafi [20]. In 2013, Wangkeeree et al. [23, 24] and Liu et al. [19] proved that viscosity approximation methods for nonexpansive mappings, hierarchical optimization problems and nonexpansive semigroups in $\mathrm{CAT}(0)$ spaces. Refinements in Hilbert spaces and extensions to Banach spaces of viscosity approximation methods were obtained by Xu [26].

The explicit viscosity method for nonexpansive mappings generates a sequence $\{x_n\}$ through the iteration process:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geqslant 1,$$

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where α_n is a sequence in $(0, 1)$. In 2004, Xu [26] proved that the sequence $\{x_n\}$ converges to a fixed point of T under certain conditions.

The implicit midpoint rule, which is one of the powerful methods for solving ordinary differential equations (see [5, 6, 21, 22] and the references therein), has been extended [3] to nonexpansive mappings, which generates a sequence $\{x_n\}$ by the implicit procedure:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad n \geqslant 1.$$

In 2015, Xu et al. [27] introduced the following process in a Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad n \geqslant 1,$$

where T is a nonexpansive mapping and f is a contraction, and proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Motivated and inspired by the known results [29], the purpose of this paper is to introduce the viscosity implicit midpoint rule for asymptotically nonexpansive mapping in complete CAT(0) spaces. More precisely, we consider the following implicit iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\frac{x_n \oplus x_{n+1}}{2}), \quad n \geqslant 1,$$

where T is a nonexpansive mapping and f is contractive. Under suitable conditions, some strong convergence theorems to a fixed point of the asymptotically nonexpansive mapping are proved. The results presented in the paper extend and improve some recent results announced in the current literatures.

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (more briefly, a geodesic from x to y) is a map c, from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x to y. When it is unique, this geodesic segment is denoted by $[x, y]$.

The space (X, d) is said to be a geodesic space, if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic, if there is exactly one geodesic joining x to y for each $x, y \in X$. A subset $Y \subset X$ is said to be convex, if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points $\triangle(x_1, x_2, x_3)$ in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space, if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality, if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leqslant d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1-t)d(x, y). \tag{2.1}$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is,

$$[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}.$$

A subset C of a CAT(0) space is convex if $[x, y] \subset C$ for all $x, y \in C$.

Lemma 2.1. *Let X be a $\mathrm{CAT}(0)$ space. Then for any $x, y, z, w \in X$ and $t, s \in [0, 1]$,*

(i) *(see [14])* $d((1-t)x \oplus ty, z) \leqslant (1-t)d(x, z) + td(y, z)$;

(ii) *(see [12])* $d((1-t)x \oplus ty, (1-s)x \oplus sy) \leqslant |t-s|d(x, y)$;

(iii) *(see [9])* $d((1-t)x \oplus ty, (1-t)z \oplus tw) \leqslant (1-t)d(x, z) + td(y, w)$;

(iv) *(see [16])* $d((1-t)z \oplus tx, (1-t)z \oplus ty) \leqslant td(x, y)$;

(v) *(see [14])* $d^2((1-t)x \oplus ty, z) \leqslant (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y)$.

If x, y_1, y_2 are points in a $\mathrm{CAT}(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$, then the $\mathrm{CAT}(0)$ inequality implies

$$d^2(y_0, x) \leqslant \frac{1}{2}d^2(y_1, x) + \frac{1}{2}d^2(y_2, x) - \frac{1}{4}d^2(y_1, y_2).$$

This is the (CN)-inequality of Bruhat and Tits [11]. In fact, a geodesic space is a $\mathrm{CAT}(0)$ space, if and only if it satisfies the (CN)-inequality ([9, p.163]).

It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $\mathrm{CAT}(0)$ space. Other examples include pre-Hilbert spaces, R-trees (see [9]), Euclidean buildings (see [10]), the complex Hilbert ball with a hyperbolic metric (see [15]), and many others. Complete $\mathrm{CAT}(0)$ spaces are often called Hadamard spaces.

In order to study our results in the general setup of $\mathrm{CAT}(0)$ spaces, we first collect some basic concepts. Let $\{x_n\}$ be a bounded sequence in $\mathrm{CAT}(0)$ space X. For $p \in X$, define a continuous functional $r(., \{x_n\}) : X \to [0, +\infty)$ by

$$r(p, \{x_n\}) = \limsup_{n\to\infty} d(p, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(p, \{x_n\}) : p \in X\}.$$

The asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf\{r(p, \{x_n\}) : p \in C\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{p \in E : r(p, \{x_n\}) = r(\{x_n\})\}.$$

The asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is the set

$$A_C(\{x_n\}) = \{p \in C : r(p, \{x_n\}) = r_C(\{x_n\})\}.$$

A sequence $\{x_n\}$ in $\mathrm{CAT}(0)$ space X is said to $\triangle$-converge to $p \in X$, if p is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we call p the $\triangle$-limit of $\{x_n\}$.

Remark 2.2. The uniqueness of an asymptotic center implies that the $\mathrm{CAT}(0)$ space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ $\triangle$-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Lemma 2.3 ([14]). *If C is a closed convex subset of a complete $\mathrm{CAT}(0)$ space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.4 ([14, 17]). *Every bounded sequence in a complete $\mathrm{CAT}(0)$ space has a $\triangle$-convergent subsequence.*

In 2008, Berg and Nikolaev [7] introduced the concept of quasilinearization as follows.

Let us formally denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is defined as a map $\langle .,. \rangle : (X \times X, X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \quad a,b,c,d \in X. \tag{2.2}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a,b,c,d,x \in X$.

We say that X satisfies the Cauchy-Schwarz inequality, if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leqslant d(a,b)d(c,d), \quad a,b,c,d \in X. \tag{2.3}$$

It is known [7] that a geodesically connected metric space is a CAT(0) space, if and only if it satisfies the Cauchy-Schwarz inequality.

In 2012, Dehghan and Rooin [13] introduced the duality mapping in CAT(0) spaces and studied its relation with subdifferential, by using the concept of quasilinearization. Then they presented a characterization of metric projection in CAT(0) spaces as follows.

Lemma 2.5 ([13, Theorem 2.4])**.** *Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C x$, if and only if $\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geqslant 0$ for all $y \in C$.*

Lemma 2.6 ([2, Theorem 2.6])**.** *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X, and $x \in X$. Then $\{x_n\}$ $\triangle$-converges to x, if and only if $\limsup_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{xy} \rangle \leqslant 0$ for all $y \in X$.*

Lemma 2.7 ([24])**.** *Let X be a complete CAT(0) space. Then for all $u,x,y \in X$, the following inequality holds:*

$$d^2(x,u) \leqslant d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Now, we present a lemma, which is very important to the proof of theorem later.

Lemma 2.8. *Let X be a complete CAT(0) space. For all $u,x,y \in X$, let $z_1 = \alpha x \oplus (1-\alpha)u$, $z_2 = \alpha y \oplus (1-\alpha)u$, the following inequality holds:*

$$\langle \overrightarrow{z_1 z_2}, \overrightarrow{x z_2} \rangle \leqslant \alpha \langle \overrightarrow{xy}, \overrightarrow{x z_2} \rangle.$$

Proof. By Lemma 2.1, (2.1) and (2.2), we have that

$$\begin{aligned}
\langle \overrightarrow{z_1 z_2}, \overrightarrow{x z_2} \rangle - \alpha \langle \overrightarrow{xy}, \overrightarrow{x z_2} \rangle &= d^2(z_1,z_2) + d^2(x,z_2) - d^2(z_1,x) - \alpha(d^2(x,z_2) + d^2(x,y) - d^2(z_2,y)) \\
&\leqslant \alpha d^2(x,z_2) + (1-\alpha)d^2(u,z_2) - \alpha(1-\alpha)d^2(x,u) + \alpha d^2(x,y) + (1-\alpha)d^2(u,x) \\
&\quad - \alpha(1-\alpha)d^2(y,u) - d^2(x,z_1) - \alpha(d^2(x,z_2) + d^2(x,y) - d^2(z_2,y)) \\
&= (1-\alpha)d^2(u,z_2) - \alpha(1-\alpha)d^2(x,u) + (1-\alpha)d^2(u,x) \\
&\quad - \alpha(1-\alpha)d^2(y,u) - d^2(x,z_1) + \alpha d^2(z_2,y) \\
&= (1-\alpha)\alpha^2 d^2(u,y) - \alpha(1-\alpha)d^2(x,u) + (1-\alpha)d^2(u,x) \\
&\quad - \alpha(1-\alpha)d^2(y,u) - (1-\alpha)^2 d^2(x,u) + \alpha(1-\alpha)^2 d^2(u,y) \\
&= 0.
\end{aligned}$$

This completes the proof. ☐

Definition 2.9. Let C be a nonempty subset of a complete CAT(0) space X. A mapping $T : C \to C$ is called nonexpansive, if $d(Tx,Ty) \leqslant d(x,y)$ for all $x,y \in C$.

Definition 2.10. Let C be a nonempty subset of a complete CAT(0) space X. A mapping $T : C \to C$ is called asymptotically nonexpansive, iff $d(T^n x, T^n y) \leqslant k_n d(x,y)$ for all $x,y \in C$, where $k_n \in [1,+\infty)$ and $\lim_{n\to\infty} k_n = 1$.

A point $x \in C$ is called a fixed point of T, if $x \in Tx$. We denote by $F(T)$ the set of all fixed points of T.

Remark 2.11. The existence of fixed points for asymptotically nonexpansive mappings in a CAT(0) space was proved by Kirk et al. [16].

Definition 2.12. A mapping f of C into itself is called contraction with coefficient $\alpha \in (0,1)$ if

$$d(f(x), f(y)) \leqslant \alpha d(x,y),$$

for all $x, y \in C$.

Remark 2.13. Banach's contraction principle guarantees that f has a unique fixed point when C is a nonempty closed convex subset of a complete metric space.

Lemma 2.14 ([28]). *If C is a closed convex subset of X and $T : C \to C$ is a asymptotically nonexpansive mapping, then the conditions $\{x_n\}$ $\triangle$-converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in C$ and $Tx = x$.*

Lemma 2.15 ([25, Lemma 2.1]). *Let $\{a_n\}$ be a sequence of nonnegative numbers such that*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \delta_n, \quad \forall n \geqslant 1,$$

where sequences $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the following property

(1) $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\} \subset \mathbb{R}$;
(2) $\sum_{n=1}^{+\infty} \gamma_n = +\infty$;
(3) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0$, *or* $\sum_{n=1}^{+\infty} |\delta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let C be a closed convex subset of a complete CAT(0) space X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For an arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\frac{x_n \oplus x_{n+1}}{2}), \quad n \geqslant 0,$$

where $\{\alpha_n\}$ satisfies the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;
(iii) $\lim_{n\to\infty} \frac{k_n^2 - 1}{\alpha_n} = 0$.

If $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$ and $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly as $n \to \infty$ to $q = P_{F(T)} f(q)$, which solves the following variational inequality:

$$\langle \overrightarrow{f(q)q}, \overrightarrow{qx} \rangle \geqslant 0, \quad \forall x \in F(T).$$

Proof. (I) We prove that $\{x_n\}$ is bounded.

In fact, by Lemma 2.1, for any $p \in F(T)$, we have that

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\frac{x_n \oplus x_{n+1}}{2}), p) \\ &\leqslant \alpha_n d(f(x_n), p) + (1-\alpha_n) d(T^n(\frac{x_n \oplus x_{n+1}}{2}), p) \end{aligned}$$

$$\leqslant \alpha_n(d(f(x_n), f(p)) + d(f(p), p)) + (1-\alpha_n)k_n d(\frac{x_n \oplus x_{n+1}}{2}, p)$$
$$\leqslant \alpha_n(\alpha d(x_n, p) + d(f(p), p)) + \frac{(1-\alpha_n)k_n}{2}(d(x_{n+1}, p) + d(x_n, p)),$$

that is

$$(1 - \frac{1-\alpha_n}{2}k_n)d(x_{n+1}, p) \leqslant (\frac{1-\alpha_n}{2}k_n + \alpha\alpha_n)d(x_n, p) + \alpha_n d(f(p), p). \tag{3.1}$$

By condition (iii), for any given positive number $\varepsilon(0 < \varepsilon < 1-\alpha)$, there exists a sufficient large positive integer N, such that for any $n > N$,

$$k_n - 1 \leqslant \frac{1}{2}(k_n^2 - 1) \leqslant \varepsilon\alpha_n, \tag{3.2}$$

and

$$1 - \frac{1-\alpha_n}{2}k_n = \frac{1}{2}(1-(k_n-1)+k_n\alpha_n) \geqslant \frac{1}{2}(1-\varepsilon\alpha_n+k_n\alpha_n) \geqslant \frac{1}{2}(1+(1-\varepsilon)\alpha_n. \tag{3.3}$$

Substituting (3.3) into (3.1), after simplifying, for any $n > N$, we have

$$\begin{aligned}(1 - \frac{1-\alpha_n}{2}k_n)d(x_{n+1}, p) &= \frac{1-(k_n-1)+k_n\alpha_n}{2}d(x_{n+1}, p)\\ &\leqslant \frac{1+(k_n-1)-k_n\alpha_n+2\alpha\alpha_n}{2}d(x_n, p) + \alpha_n d(f(p), p)\\ &\leqslant \frac{1+\varepsilon\alpha_n-k_n\alpha_n+2\alpha\alpha_n}{2}d(x_n, p) + \alpha_n d(f(p), p)\\ &= \frac{1-(k_n-2\alpha-\varepsilon)\alpha_n}{2}d(x_n, p) + \alpha_n d(f(p), p)\\ &\leqslant \frac{1-(1-2\alpha-\varepsilon)\alpha_n}{2}d(x_n, p) + \alpha_n d(f(p), p).\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned}d(x_{n+1}, p) &\leqslant \frac{1-(1-2\alpha-\varepsilon)\alpha_n}{1+(1-\varepsilon)\alpha_n}d(x_n, p) + \frac{2\alpha_n}{1+(1-\varepsilon)\alpha_n}d(f(p), p)\\ &\leqslant (1 - \frac{2(1-\alpha-\varepsilon)\alpha_n}{1+(1-\varepsilon)\alpha_n}d(x_n, p) + \frac{2(1-\alpha-\varepsilon)\alpha_n}{1+(1-\varepsilon)\alpha_n}.\frac{d(f(p), p)}{1-\alpha-\varepsilon}\\ &\leqslant \max\{d(x_n, p), \frac{d(f(p), p)}{1-\alpha-\varepsilon}\}\\ &\leqslant \max\{d(x_{N+1}, p), \frac{d(f(p), p)}{1-\alpha-\varepsilon}\}.\end{aligned}$$

Hence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$, $\{T^n x_n)\}$ and $\{T^n(\frac{x_n \oplus x_{n+1}}{2})\}$.

(II) We show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

In fact,

$$\begin{aligned}d(x_{n+1}, x_n) &\leqslant d(x_{n+1}, T^n x_n) + d(T^n x_n, x_n)\\ &= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\frac{x_n \oplus x_{n+1}}{2}), T^n x_n) + d(T^n x_n, x_n)\\ &\leqslant \alpha_n d(f(x_n), T^n x_n) + (1-\alpha_n)d(T^n(\frac{x_n \oplus x_{n+1}}{2}), T^n x_n) + d(T^n x_n, x_n)\\ &\leqslant \alpha_n d(f(x_n), T^n x_n) + d(T^n x_n, x_n) + (1-\alpha_n)k_n d(\frac{x_n \oplus x_{n+1}}{2}, x_n)\end{aligned}$$

$$\leqslant \alpha_n d(f(x_n), T^n x_n) + d(T^n x_n, x_n) + \frac{1}{2}(1-\alpha_n)k_n d(x_{n+1}, x_n).$$

Since $\{f(x_n)\}$ and $\{T^n x_n)\}$ are bounded, hence there exists $M > 0$, such that $M \geqslant \sup_{n\geqslant 1} d(f(x_n), T^n x_n)$. We get

$$d(x_{n+1}, x_n) \leqslant \alpha_n M + d(T^n x_n, x_n) + \frac{1}{2}(1-\alpha_n)k_n d(x_{n+1}, x_n). \tag{3.5}$$

From (3.3) and (3.5), we have

$$d(x_{n+1}, x_n) \leqslant \frac{2\alpha_n}{1+(1-\varepsilon)\alpha_n} M + \frac{1}{1+(1-\varepsilon)\alpha_n} d(T^n x_n, x_n), \quad n > N.$$

By virtue of the conditions (i) and $\lim_{n\to\infty} d(T^n x_n, x_n) = 0$, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.6}$$

(III) We show that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Indeed,

$$\begin{aligned} d(x_{n+1}, T^n x_{n+1}) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n) T^n(\frac{x_n \oplus x_{n+1}}{2}), T^n x_{n+1}) \\ &\leqslant \alpha_n d(f(x_n), T^n x_{n+1}) + (1-\alpha_n)k_n d(\frac{x_n \oplus x_{n+1}}{2}, x_{n+1}) \\ &\leqslant \alpha_n d(f(x_n), T^n x_{n+1}) + \frac{1}{2}(1-\alpha_n)k_n d(x_n, x_{n+1}) \\ &\leqslant \alpha_n M + \frac{1}{2}(1-\alpha_n)k_n d(x_n, x_{n+1}). \end{aligned}$$

It follows from condition (i) and (3.6) that

$$\lim_{n\to\infty} d(x_{n+1}, T^n x_{n+1}) = 0. \tag{3.7}$$

Hence

$$\begin{aligned} d(Tx_{n+1}, x_{n+1}) &\leqslant d(Tx_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, x_{n+1}) \\ &\leqslant k_1 d(x_{n+1}, T^n x_{n+1}) + d(T^{n+1}x_{n+1}, x_{n+1}). \end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that $\lim_{n\to\infty} d(Tx_{n+1}, x_{n+1}) = 0$, which implies that

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{3.9}$$

(IV) We prove that

$$w_\triangle\{x_n\} := \bigcup_{\{u_n\}\subset\{x_n\}} \{A(\{u_n\})\} \subset F(T), \tag{3.10}$$

where $A(\{u_n\})$ is the asymptotic center of $\{u_n\}$.

Let $u \in w_\triangle\{x_n\}$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. It follows from Lemma 2.4 that there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} v_n = u$. In view of (3.9),

$$\lim_{n\to\infty} d(Tv_n, v_n) = 0,$$

and T is demi-closed at 0. By Lemma 2.14, $Tu = u$, that is $u \in F(T)$. Hence $w_\triangle\{x_n\} \subset F(T)$.

(V) We prove that $x_n \to q$ as $n \to \infty$, where $q \in F(T)$ is the unique fixed point of contraction $P_{F(T)}f$, that is, $q = P_{F(T)}f(q)$.

First, we show that

$$\limsup_{n\to\infty}\langle\overrightarrow{qf(q)},\overrightarrow{qx_n}\rangle \leqslant 0.$$

As a matter of fact, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$, which $\triangle$-converges to a point p. By Lemma 2.5, Lemma 2.6 and (3.10), we get

$$\limsup_{n\to\infty}\langle\overrightarrow{qf(q)},\overrightarrow{qx_n}\rangle = \lim_{k\to\infty}\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n_k}}\rangle = \langle\overrightarrow{qf(q)},\overrightarrow{qp}\rangle \leqslant 0. \tag{3.11}$$

Next, for $n \geqslant 0$, let $z_n := \alpha_n q \oplus (1-\alpha_n)T^n(\frac{x_n \oplus x_{n+1}}{2})$. It follows from Lemma 2.8 that

$$\langle\overrightarrow{z_n x_{n+1}},\overrightarrow{qx_{n+1}}\rangle \leqslant \alpha_n\langle\overrightarrow{qf(x_n)},\overrightarrow{qx_{n+1}}\rangle.$$

Hence, it follows from Lemma 2.1 (v), Lemma 2.7, (2.2) and (2.3) that

$$\begin{aligned}
d^2(x_{n+1},q) &\leqslant d^2(z_n,q) + 2\langle\overrightarrow{z_n x_{n+1}},\overrightarrow{qx_{n+1}}\rangle\\
&= d^2(\alpha_n q \oplus (1-\alpha_n)T^n(\frac{x_n \oplus x_{n+1}}{2}),q) + 2\langle\overrightarrow{z_n x_{n+1}},\overrightarrow{qx_{n+1}}\rangle\\
&\leqslant (1-\alpha_n)^2 d^2(T^n(\frac{x_n \oplus x_{n+1}}{2}),q) + 2\langle\overrightarrow{z_n x_{n+1}},\overrightarrow{qx_{n+1}}\rangle\\
&\leqslant (1-\alpha_n)^2 k_n^2 d^2(\frac{x_n \oplus x_{n+1}}{2},q) + 2\alpha_n\langle\overrightarrow{qf(x_n)},\overrightarrow{qx_{n+1}}\rangle\\
&= (1-\alpha_n)^2 k_n^2 d^2(\frac{x_n \oplus x_{n+1}}{2},q) + 2\alpha_n\langle\overrightarrow{f(q)f(x_n)},\overrightarrow{qx_{n+1}}\rangle + 2\alpha_n\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n+1}}\rangle\\
&\leqslant (1-\alpha_n)^2 k_n^2 d^2(\frac{x_n \oplus x_{n+1}}{2},q) + 2\alpha_n d(f(x_n),f(q))d(x_{n+1},q) + 2\alpha_n\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n+1}}\rangle\\
&\leqslant (1-\alpha_n)^2 k_n^2 d^2(\frac{x_n \oplus x_{n+1}}{2},q) + 2\alpha\alpha_n d(x_n,q)d(x_{n+1},q) + 2\alpha_n\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n+1}}\rangle\\
&\leqslant (1-\alpha_n)^2 k_n^2(\frac{1}{2}d^2(x_n,q) + \frac{1}{2}d^2(x_{n+1},q) - \frac{1}{4}d^2(x_{n+1},x_n))\\
&\quad + \alpha\alpha_n(d^2(x_n,q) + d^2(x_{n+1},q)) + 2\alpha_n\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n+1}}\rangle\\
&= \frac{(1-2\alpha_n)k_n^2 + 2\alpha\alpha_n}{2}(d^2(x_n,q) + d^2(x_{n+1},q))\\
&\quad + \frac{1}{2}\alpha_n^2 k_n^2(d^2(x_n,q) + d^2(x_{n+1},q)) + 2\alpha_n\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n+1}}\rangle.
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\geqslant 1}\{k_n^2 d^2(x_n,q)\} \leqslant M$. Hence we have

$$d^2(x_{n+1},q) \leqslant \frac{(1-2\alpha_n)k_n^2 + 2\alpha\alpha_n}{2}(d^2(x_n,q) + d^2(x_{n+1},q)) + \alpha_n^2 M + 2\alpha_n\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n+1}}\rangle. \tag{3.12}$$

It follows from (3.2), (3.3) and (3.12) that for all $n > N$

$$\begin{aligned}
&(1 - \frac{(1-2\alpha_n)k_n^2 + 2\alpha\alpha_n}{2})d^2(x_{n+1},q)\\
&\qquad\leqslant \frac{(1-2\alpha_n)k_n^2 + 2\alpha\alpha_n}{2}d^2(x_n,q) + \alpha_n^2 M + 2\alpha_n\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n+1}}\rangle\\
&\qquad\leqslant \frac{1 + 2\varepsilon\alpha_n - 2(1-\alpha)\alpha_n}{2}d^2(x_n,q) + \alpha_n^2 M + 2\alpha_n\langle\overrightarrow{qf(q)},\overrightarrow{qx_{n+1}}\rangle
\end{aligned}$$

$$= \frac{1-2(1-\varepsilon-\alpha)\alpha_n}{2} d^2(x_n, q) + \alpha_n^2 M + 2\alpha_n \langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}} \rangle.$$

From (3.2) and (3.3), we have

$$1 - \frac{(1-2\alpha_n)k_n^2 + 2\alpha\alpha_n}{2} \geqslant \frac{1+2(1-\varepsilon-\alpha)\alpha_n}{2},$$

and arrive at

$$\begin{aligned} d^2(x_{n+1}, q) &\leqslant \frac{1-2(1-\varepsilon-\alpha)\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n} d^2(x_n, q) + \frac{\alpha_n^2 M}{1+2(1-\varepsilon-\alpha)\alpha_n} + \frac{4\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n} \langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}} \rangle \\ &= (1-\gamma_n) d^2(x_n, q) + \delta_n, \end{aligned}$$

where

$$\begin{aligned} \gamma_n &= \frac{4(1-\varepsilon-\alpha)\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n}, \\ \delta_n &= \frac{\alpha_n^2 M}{1+2(1-\varepsilon-\alpha)\alpha_n} + \frac{4\alpha_n}{1+2(1-\varepsilon-\alpha)\alpha_n} \langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}} \rangle. \end{aligned}$$

It follows from conditions (i), (ii) and (3.11) that

$$\gamma_n \subset (0,1), \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n = \infty,$$

$$\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty} \frac{\alpha_n M + 2\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n+1}} \rangle}{2(1-\varepsilon-\alpha)} \leqslant 0.$$

By Lemma 2.5 and Lemma 2.15, we get that $x_n \to q = P_{F(T)} f(q)$, which solves the following variational inequality:

$$\langle \overrightarrow{f(q)q}, \overrightarrow{qx} \rangle \geqslant 0, \quad \forall x \in F(T).$$

This completes the proof of Theorem 3.1. □

Remark 3.2. Since a real Hilbert space is a complete CAT(0), and every nonexpansive mapping is an asymptotically nonexpansive mapping, Theorem 3.1 is an improvement and generalization of the main results in Alghamdi et al. [3], Xu et al. [26, 27] and Zhao et al. [29].

The following result can be obtained from Theorem 3.1 immediately.

Theorem 3.3. *Let C be a closed convex subset of a complete $CAT(0)$ space X. Let $T : C \to C$ be a nonexpansive mapping. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For an arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \quad n \geqslant 0,$$

where $\{\alpha_n\}$ satisfies the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $q = P_{F(T)} f(q)$, which solves the following variational inequality:

$$\langle \overrightarrow{f(q)q}, \overrightarrow{qx} \rangle \geqslant 0, \quad \forall x \in F(T).$$

Remark 3.4. Note that the condition $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ is not needed, since it suffices to prove that the condition is satisfied.

In fact, similar to the result given in [29], we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Therefore,

$$\begin{aligned}
d(x_{n+1}, Tx_{n+1}) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T(\frac{x_n \oplus x_{n+1}}{2}), Tx_{n+1}) \\
&\leqslant \alpha_n d(f(x_n), Tx_{n+1}) + (1-\alpha_n)d(T(\frac{x_n \oplus x_{n+1}}{2}), Tx_{n+1}) \\
&\leqslant \alpha_n d(f(x_n), Tx_{n+1}) + (1-\alpha_n)d(\frac{x_n \oplus x_{n+1}}{2}, x_{n+1}) \\
&\leqslant \alpha_n d(f(x_n), Tx_{n+1}) + \frac{1}{2}(1-\alpha_n)d(x_n, x_{n+1}) \\
&= \alpha_n M + \frac{1}{2}(1-\alpha_n)d(x_n, x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where $M = \sup_{n\geqslant 0} d(f(x_n), Tx_{n+1})$.

4. Application to equilibrium problem

First, we present an example of a nonexpansive mapping.

Example 4.1 ([8]). Let H be a real Hilbert space, D be a nonempty closed and convex subset of H and $\phi : D \times D \to R$ be a bifunction satisfying the conditions:

(A1) $\phi(x, x) = 0, \forall x \in D$;

(A2) $\phi(x, y) + \phi(y, x) \leqslant 0,\ \forall x, y \in D$;

(A3) for each $x, y, z \in D$, $\lim_{t\to 0} \phi(tz + (1-t)x, y) \leqslant \phi(x, y)$;

(A4) for each given $x \in D$, the function $y \longmapsto \phi(x, y)$ is convex and lower semicontinuous. The "so-called" equilibrium problem for ϕ is to find an $x^* \in D$ such that $\phi(x^*, y) \geqslant 0$, $\forall y \in D$. The set of its solutions is denoted by $EP(\phi)$.

Let $r > 0$, $x \in H$ and define a mapping $T_r : D \to D \subset H$ as follows

$$T_r(x) = \{z \in D, \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0,\ \forall y \in D\}, \quad \forall x \in D \subset H. \tag{4.1}$$

Then

(1) T_r is single-valued, and so $z = T_r(x)$;

(2) T_r is a relatively nonexpansive mapping. Therefore, it is a closed nonexpansive mapping;

(3) $F(T_r) = EP(\phi)$ and $F(T_r)$ is a nonempty and closed convex subset of D;

(4) $T_r : D \to D$ is a nonexpansive.

Since every real Hilbert space is a complete CAT(0) space, using Theorem 3.3 to study the implicit midpoint rule of a modified nonexpansive mapping for a system of equilibrium problems, we have the following result.

Theorem 4.2. *Let* H *be a real Hilbert space,* D *be a nonempty closed and convex subset of* H*.* $\{\alpha_n\}$ *and* f *are the same as in Theorem 3.3. Let* $\phi : D \times D \to R$ *be a bifunction satisfying conditions* (A1)–(A4) *as given in example above. Let* $T_r : D \to D \subset H$ *be mapping defined by* (4.1)*, i.e.,*

$$T_r(x) = \{z \in D, \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0,\ \forall y \in D\}, \quad \forall x \in D \subset H.$$

Let $\{x_n\}$ *be the sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_r(\frac{x_n + x_{n+1}}{2}), \quad n \geqslant 0.$$

If $F(T_r) \neq \emptyset$*, then* $\{x_n\}$ *converges strongly to* $q = P_{F(T_r)}f(q)$*, which is a common solution of the system of equilibrium problems* $EP(\phi)$*.*

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