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Well-posedness for a class of generalized Zakharov system

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Abstract

In this paper, we study the existence and uniqueness of the global smooth solution for the initial value problem of generalized Zakharov equations in dimension two. By means of a priori integral estimates and Galerkin method, we first construct the existence of global solution with some conditions. Furthermore, we prove that the global solution is unique. ©2017 All rights reserved.

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1. Introduction

At the classical level, a set of coupled nonlinear wave equations describing the interaction between high frequency Langmuir waves and low frequency ion-acoustic waves was first derived by Zakharov [17]. This system has been the subject of a large number of studies [3, 5–9, 11–16]. In addition to the energy method, Zakharov type systems have also been studied by others using different approaches. Aslan dealt with the generalized Zakharov system and derived some further results using the so-called first integral method [2]. In [1], the Exp-function method was employed to the Zakharov-Kuznetsov equation as a $(2+1)$-dimensional model for nonlinear Rossby waves.

In this paper, we are interested in studying the following generalized Zakharov system in dimension two.

$$i\varepsilon_t + \Delta\varepsilon - n\varepsilon + \alpha|\varepsilon|^p\varepsilon = 0, \tag{1.1}$$

$$n_{tt} - \Delta n = \Delta|\varepsilon|^2, \tag{1.2}$$

with initial data

$$\varepsilon(x,0) = \varepsilon_0(x), \quad n(x,0) = n_0(x), \quad n_t(x,0) = n_1(x), \tag{1.3}$$

where $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_N)$, $x = (x_1, x_2) \in \mathbb{R}^2$.

Now we state the main results of the paper.

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Theorem 1.1. *Assume that* $\varepsilon_0(x) \in H^{l+2}$, $n_0(x) \in H^{l+1}$, $n_1(x) \in H^l$, $l \geqslant 1$ *and* $0 < p \leqslant 2$ *with* $\|\varepsilon_0(x)\|_{L^2}$ *small. Then there exists unique global smooth solution of the initial value problem* (1.1), (1.2), (1.3),

$$\begin{aligned}&\varepsilon(x,t) \in L^\infty(0,T;H^{l+2}) \cap W^{1,\infty}(0,T;H^l), \quad n(x,t) \in L^\infty(0,T;H^{l+1}),\\ &n_t(x,t) \in L^\infty(0,T;H^l) \cap W^{1,\infty}(0,T;H^{l-1}).\end{aligned}$$

To study smooth solution of the generalized Zakharov systems, we transform it into the following form:

$$i\varepsilon_t + \Delta\varepsilon - n\varepsilon + \alpha|\varepsilon|^p\varepsilon = 0, \tag{1.4}$$

$$\varphi_t - (n + |\varepsilon|^2) = 0, \tag{1.5}$$

$$n_t - \Delta\varphi = 0, \ x \in \mathbb{R}^2, \tag{1.6}$$

with initial data

$$\varepsilon(x,0) = \varepsilon_0(x), \quad n(x,0) = n_0(x), \quad \varphi(x,0) = \varphi_0(x), \tag{1.7}$$

where φ_0 satisfies $\nabla^2\varphi_0 = n_1$.

We use C to represent various positive constants that can depend on initial data.

2. A priori estimations

In this section, we will derive a priori estimations for the solution of the system (1.4), (1.5), (1.6), (1.7). For the smooth solution of system (1.4)-(1.7), we have the following conserved results

$$\|\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} = \|\varepsilon_0(x)\|^2_{L^2(\mathbb{R}^2)},$$

$$\begin{aligned}\mathcal{H}(t) :&= \|\nabla\varepsilon\|^2_{L^2} + \int_{\mathbb{R}^2} n|\varepsilon|^2 dx + \frac{1}{2}\|\nabla\varphi\|^2_{L^2} + \frac{1}{2}\|n\|^2_{L^2} - \frac{2\alpha}{p+2}\|\varepsilon\|^{p+2}_{L^{p+2}}\\ &= \mathcal{H}(0).\end{aligned} \tag{2.1}$$

The conservation of $\|\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}$ is obtained by taking the inner product of (1.4) and ε and then taking the imaginary part. The conservation of $\mathcal{H}(t)$ can be obtained by taking the inner product of (1.4) and ε_t and taking the real part, and taking the inner product of (1.5), (1.6) by n_t, φ_t, respectively.

Lemma 2.1 (Gagliardo-Nirenberg inequality)**.** *Assume that* $u \in L^q(\mathbb{R}^n)$, $D^m u \in L^r(\mathbb{R}^n)$, $1 \leqslant q,\ r \leqslant \infty$, $0 \leqslant j \leqslant m$, *we have the estimations*

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \leqslant C\|D^m u\|^\alpha_{L^r(\mathbb{R}^n)}\|u\|^{1-\alpha}_{L^q(\mathbb{R}^n)},$$

where C *is a positive constant,* $0 \leqslant \frac{j}{m} \leqslant \alpha \leqslant 1$,

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}.$$

Lemma 2.2. *Assume that* $\varepsilon_0 \in H^1$, $n_0 \in L^2$, $\varphi_0 \in H^1$ *and* $0 < p \leqslant 2$ *with* $\|\varepsilon_0(x)\|_{L^2}$ *small. Then*

$$\sup_{0\leqslant t\leqslant T}(\|\varepsilon\|_{H^1} + \|\varphi\|_{H^1} + \|n\|_{L^2}) \leqslant C.$$

Proof. By Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, there holds

$$\begin{aligned}\left|\int_{\mathbb{R}^2} n|\varepsilon|^2 dx\right| &\leqslant \|n\|_{L^2}\|\varepsilon\|^2_{L^4} \leqslant \frac{1}{4}\|n\|^2_{L^2} + \|\varepsilon\|^4_{L^4}\\ &\leqslant \frac{1}{4}\|n\|^2_{L^2} + C\|\nabla\varepsilon\|^2_{L^2}\|\varepsilon\|^2_{L^2}.\end{aligned} \tag{2.2}$$

Using Gagliardo-Nirenberg inequality, we write

$$\frac{2|\alpha|}{p+2}\|\varepsilon\|_{L^{p+2}}^{p+2} \leqslant C\|\nabla\varepsilon\|_{L^2}^{p}\|\varepsilon\|_{L^2}^{2}. \tag{2.3}$$

From (2.1), (2.2), and (2.3), one has

$$\|\nabla\varepsilon\|_{L^2}^2 + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{4}\|n\|_{L^2}^2 \leqslant |\mathcal{H}(0)| + C\|\nabla\varepsilon\|_{L^2}^2\|\varepsilon\|_{L^2}^2 + C\|\nabla\varepsilon\|_{L^2}^{p}\|\varepsilon\|_{L^2}^2.$$

Note that $0 < p \leqslant 2$ and $\|\varepsilon_0\|_{L^2}$ small, we get

$$\|\nabla\varepsilon\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 + \|n\|_{L^2}^2 \leqslant C.$$

Taking the inner products of (1.5) and φ, since

$$(\varphi_t, \varphi) = \frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2,$$

$$\left|(n+|\varepsilon|^2, \varphi)\right| \leqslant \left(\|n\|_{L^2} + \|\varepsilon\|_{L^4}^2\right)\|\varphi\|_{L^2} \leqslant C\|\varphi\|_{L^2} \leqslant \frac{1}{2}\|\varphi\|_{L^2}^2 + C,$$

thus it follows that

$$\frac{d}{dt}\|\varphi\|_{L^2}^2 \leqslant \|\varphi\|_{L^2}^2 + C.$$

Using Gronwall's inequality, it follows that

$$\sup_{0\leqslant t\leqslant T}\|\varphi\|_{L^2}^2 \leqslant C.$$

We thus get Lemma 2.2. □

Lemma 2.3 (Brezis-Wainger inequality [4]). *Let* $u \in W^{k,p}(\mathbb{R}^d) \cap W^{s,q}(\mathbb{R}^d)$, $k, s > 0$, $p > 1$, $q \geqslant 1$ *and* $kp = d < sq$. *Then*

$$\|u\|_{L^\infty} \leqslant C\|u\|_{W^{k,p}}\left(1 + \ln\left(1 + \frac{\|u\|_{W^{s,q}}}{\|u\|_{W^{k,p}}}\right)\right)^{1-\frac{1}{p}},$$

where C *is a constant that depends only on* k, p, s, q, d.

Applying Lemma 2.3 in the case $d = 2$, $k = 1, p = 2$, $s = 2, q = 2$, hence for $u \in H^2(\mathbb{R}^2)$, we have

$$\|u\|_{L^\infty} \leqslant C\|u\|_{H^1}\left(1 + \ln\left(1 + \frac{\|u\|_{H^2}}{\|u\|_{H^1}}\right)\right)^{\frac{1}{2}}. \tag{2.4}$$

Lemma 2.4. *Assume that* $\varepsilon_0(x) \in H^2$, $n_0(x) \in H^1$, $\varphi_0(x) \in H^2$ *and* $0 < p \leqslant 2$ *with* $\|\varepsilon_0(x)\|_{L^2}$ *small. Then we have*

$$\sup_{0\leqslant t\leqslant T}\left(\|\varepsilon\|_{H^2} + \|n\|_{H^1} + \|\varphi\|_{H^2}\right) \leqslant C,$$

$$\sup_{0\leqslant t\leqslant T}\left(\|\varepsilon_t\|_{L^2} + \|n_t\|_{L^2} + \|\varphi_t\|_{H^1}\right) \leqslant C.$$

Proof. Differentiating (1.4) with respect to t, then taking the inner product of the resulting equation and ε_t, we get

$$(i\varepsilon_{tt} + \Delta\varepsilon_t - n_t\varepsilon - n\varepsilon_t + (\alpha|\varepsilon|^p\varepsilon)_t, \varepsilon_t) = 0, \tag{2.5}$$

Since

$$\mathrm{Im}\,(i\varepsilon_{tt},\varepsilon_t)=\frac{1}{2}\frac{d}{dt}\|\varepsilon_t\|_{L^2}^2,\quad \mathrm{Im}\,(\Delta\varepsilon_t-n\varepsilon_t,\varepsilon_t)=0,$$
$$|\mathrm{Im}\,(-n_t\varepsilon,\varepsilon_t)|\leqslant\|\varepsilon\|_{L^\infty}\|n_t\|_{L^2}\|\varepsilon_t\|_{L^2},$$
$$|\mathrm{Im}\,((\alpha|\varepsilon|^p\varepsilon)_t,\varepsilon_t)|\leqslant\frac{|\alpha|p}{2}\int_{\mathbb{R}^2}|\varepsilon|^p|\varepsilon_t|^2dx\leqslant\frac{|\alpha|p}{2}\|\varepsilon\|_{L^\infty}^p\|\varepsilon_t\|_{L^2}^2,$$

thus from (2.5) we get

$$\frac{d}{dt}\|\varepsilon_t\|_{L^2}^2\leqslant 2\|\varepsilon\|_{L^\infty}\|n_t\|_{L^2}\|\varepsilon_t\|_{L^2}+|\alpha|p\|\varepsilon\|_{L^\infty}^p\|\varepsilon_t\|_{L^2}^2. \tag{2.6}$$

Differentiating (1.6) with respect to t, then taking the inner product of the resulting equation and n_t, we get

$$(n_{tt}-\Delta\varphi_t,n_t)=0. \tag{2.7}$$

Since

$$(n_{tt},n_t)=\frac{1}{2}\frac{d}{dt}\|n_t\|_{L^2}^2,$$
$$(-\Delta\varphi_t,n_t)=(-\Delta n-\Delta|\varepsilon|^2,n_t),$$

where

$$(-\Delta n,n_t)=\frac{1}{2}\frac{d}{dt}\|\nabla n\|_{L^2}^2,$$

$$\begin{aligned}\left|(-\Delta|\varepsilon|^2,n_t)\right|&\leqslant 2\left(\|\varepsilon\|_{L^\infty}\|\Delta\varepsilon\|_{L^2}+\|\nabla\varepsilon\|_{L^4}^2\right)\|n_t\|_{L^2}\\&\leqslant 2\left(\|\varepsilon\|_{L^\infty}\|\Delta\varepsilon\|_{L^2}+C\|\Delta\varepsilon\|_{L^2}\right)\|n_t\|_{L^2}.\end{aligned}$$

Thus from (2.7) we get

$$\frac{d}{dt}\left(\|n_t\|_{L^2}^2+\|\nabla n\|_{L^2}^2\right)\leqslant 4\left(\|\varepsilon\|_{L^\infty}\|\Delta\varepsilon\|_{L^2}+C\|\Delta\varepsilon\|_{L^2}\right)\|n_t\|_{L^2}. \tag{2.8}$$

From (1.4) we have

$$\begin{aligned}\|\Delta\varepsilon\|_{L^2}&\leqslant\|\varepsilon_t\|_{L^2}+\|n\|_{L^4}\|\varepsilon\|_{L^4}+|\alpha|\,\|\varepsilon\|_{L^{2p+2}}^{p+1}\\&\leqslant C\left(\|\varepsilon_t\|_{L^2}+\|\nabla n\|_{L^2}^{\frac{1}{2}}+1\right).\end{aligned} \tag{2.9}$$

Using (2.4), one has

$$\begin{aligned}\|\varepsilon\|_{L^\infty}&\leqslant C\left(1+\ln\left(1+\|\varepsilon\|_{H^2}\right)\right)^{\frac{1}{2}}\\&\leqslant C\left(1+\ln\left(1+\|\Delta\varepsilon\|_{L^2}\right)\right)^{\frac{1}{2}}.\end{aligned} \tag{2.10}$$

Set $\mathcal{F}(t)=\|\varepsilon_t\|_{L^2}^2+\|n_t\|_{L^2}^2+\|\nabla n\|_{L^2}^2+1$, then (2.6) and (2.8), (2.9), (2.10) together yield

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t)&\leqslant C\mathcal{F}(t)\left(\|\varepsilon\|_{L^\infty}+\|\varepsilon\|_{L^\infty}^p+1\right)\\&\leqslant C\mathcal{F}(t)\left(1+\ln\mathcal{F}(t)\right).\end{aligned}$$

By using Gronwall inequality, it follows that

$$\sup_{0\leqslant t\leqslant T}\left(\|\varepsilon_t\|_{L^2}^2+\|n_t\|_{L^2}^2+\|\nabla n\|_{L^2}^2\right)\leqslant C,$$

and with (2.9), (1.6) and (1.5), we arrive at

$$\sup_{0\leqslant t\leqslant T}(\|\Delta\varepsilon\|_{L^2}+\|\Delta\varphi\|_{L^2}+\|\nabla\varphi_t\|_{L^2})\leqslant C.$$

We thus get Lemma 2.4. □

Lemma 2.5. *Assume that* $\varepsilon_0(x)\in H^3$, $n_0(x)\in H^2$, $\varphi_0(x)\in H^3$ *and* $0<p\leqslant 2$ *with* $\|\varepsilon_0(x)\|_{L^2}$ *small. Then we have*

$$\sup_{0\leqslant t\leqslant T}(\|\varepsilon\|_{H^3}+\|n\|_{H^2}+\|\varphi\|_{H^3})\leqslant C,$$
$$\sup_{0\leqslant t\leqslant T}(\|\varepsilon_t\|_{H^1}+\|n_t\|_{H^1}+\|\varphi_t\|_{H^2})\leqslant C.$$

Proof. Differentiating (1.4) with respect to t, then taking the inner product of the resulting equation and $-\Delta\varepsilon_t$, we get

$$(i\varepsilon_{tt}+\Delta\varepsilon_t-n_t\varepsilon-n\varepsilon_t+(\alpha|\varepsilon|^p\varepsilon)_t,-\Delta\varepsilon_t)=0. \tag{2.11}$$

Since

$$\mathrm{Im}(i\varepsilon_{tt},-\Delta\varepsilon_t)=\frac{1}{2}\frac{d}{dt}\|\nabla\varepsilon_t\|_{L^2}^2,\quad \mathrm{Im}(\Delta\varepsilon_t,-\Delta\varepsilon_t)=0,$$

$$\begin{aligned}|\mathrm{Im}(-n_t\varepsilon,-\Delta\varepsilon_t)|&\leqslant|(\nabla(n_t\varepsilon),\nabla\varepsilon_t)|\\&\leqslant C(\|\nabla n_t\|_{L^2}\|\varepsilon\|_{L^\infty}+\|n_t\|_{L^4}\|\nabla\varepsilon\|_{L^4})\|\nabla\varepsilon_t\|_{L^2}\\&\leqslant C(\|\nabla n_t\|_{L^2}^2+\|\nabla\varepsilon_t\|_{L^2}^2+1),\end{aligned}$$

$$\begin{aligned}|\mathrm{Im}(-n\varepsilon_t,-\Delta\varepsilon_t)|&=|\mathrm{Im}(\nabla(n\varepsilon_t),\nabla\varepsilon_t)|\\&=|\mathrm{Im}((\nabla n)\varepsilon_t,\nabla\varepsilon_t)|\\&\leqslant\|\nabla n\|_{L^4}\|\varepsilon_t\|_{L^4}\|\nabla\varepsilon_t\|_{L^2}\\&\leqslant C(\|\Delta n\|_{L^2}^2+\|\nabla\varepsilon_t\|_{L^2}^2),\end{aligned}$$

$$\begin{aligned}|\mathrm{Im}\,((\alpha|\varepsilon|^p\varepsilon)_t,-\Delta\varepsilon_t)|&=|\mathrm{Im}\,(\nabla\,(\alpha|\varepsilon|^p\varepsilon)_t\,,\nabla\varepsilon_t)|\\&\leqslant C\Big(\|\varepsilon\|_{L^\infty}^{p-1}\|\nabla\varepsilon\|_{L^4}\|\varepsilon_t\|_{L^4}+\|\varepsilon\|_{L^\infty}^p\|\nabla\varepsilon_t\|_{L^2}\Big)\|\nabla\varepsilon_t\|_{L^2}\\&\leqslant C(\|\nabla\varepsilon_t\|_{L^2}^2+1),\end{aligned}$$

thus from (2.11) we get

$$\frac{d}{dt}\|\nabla\varepsilon_t\|_{L^2}^2\leqslant C(\|\nabla n_t\|_{L^2}^2+\|\nabla\varepsilon_t\|_{L^2}^2+\|\Delta n\|_{L^2}^2+1). \tag{2.12}$$

Differentiating (1.6) with respect to t, then taking the inner product of the resulting equation and $-\Delta n_t$, we get

$$(n_{tt}-\Delta\varphi_t,-\Delta n_t)=0. \tag{2.13}$$

Since

$$\begin{aligned}(n_{tt},-\Delta n_t)&=\frac{1}{2}\frac{d}{dt}\|\nabla n_t\|_{L^2}^2,\\(-\Delta\varphi_t,-\Delta n_t)&=(\Delta n+\Delta|\varepsilon|^2,\Delta n_t),\\(\Delta n,\Delta n_t)&=\frac{1}{2}\frac{d}{dt}\|\Delta n\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}\left|(\Delta|\varepsilon|^2,\Delta n_t)\right| &\leqslant \left|(\nabla^3|\varepsilon|^2,\nabla n_t)\right|\\ &\leqslant C\left(\|\nabla^3\varepsilon\|_{L^2}\|\varepsilon\|_{L^\infty}+\|\Delta\varepsilon\|_{L^4}\|\nabla\varepsilon\|_{L^4}\right)\|\nabla n_t\|_{L^2}\\ &\leqslant C\left(\|\nabla^3\varepsilon\|_{L^2}^2+\|\nabla n_t\|_{L^2}^2+1\right),\end{aligned}$$

thus from (2.13) we get

$$\frac{d}{dt}\left[\|\nabla n_t\|_{L^2}^2+\|\Delta n\|_{L^2}^2\right]\leqslant C\left(\|\nabla^3\varepsilon\|_{L^2}^2+\|\nabla n_t\|_{L^2}^2+1\right). \tag{2.14}$$

Taking the inner product of (1.4) and $\Delta^2\varepsilon$, since

$$\left(\Delta\varepsilon,\,\Delta^2\varepsilon\right)=-\left\|\nabla^3\varepsilon\right\|_{L^2}^2,$$

$$\begin{aligned}\left|\left(i\varepsilon_t-n\varepsilon+\alpha|\varepsilon|^p\varepsilon,\,\Delta^2\varepsilon\right)\right| &=\left|\left(i\nabla\varepsilon_t-\nabla(n\varepsilon)+\alpha\nabla\left(|\varepsilon|^p\varepsilon\right),\,\nabla^3\varepsilon\right)\right|\\ &\leqslant C\left(\|\nabla\varepsilon_t\|_{L^2}+\|\nabla n\|_{L^2}\|\varepsilon\|_{L^\infty}\right)\left\|\nabla^3\varepsilon\right\|_{L^2}\\ &\quad+C\left(\|n\|_{L^4}\|\nabla\varepsilon\|_{L^4}+\|\varepsilon\|_{L^\infty}^p\|\nabla\varepsilon\|_{L^2}\right)\left\|\nabla^3\varepsilon\right\|_{L^2}\\ &\leqslant C\left(\|\nabla\varepsilon_t\|_{L^2}+1\right)\left\|\nabla^3\varepsilon\right\|_{L^2},\end{aligned}$$

thus we get

$$\left\|\nabla^3\varepsilon\right\|_{L^2}\leqslant C\left(\|\nabla\varepsilon_t\|_{L^2}+1\right). \tag{2.15}$$

Hence from (2.12), (2.14) and (2.15) we get

$$\frac{d}{dt}\left[\|\nabla\varepsilon_t\|_{L^2}^2+\|\nabla n_t\|_{L^2}^2+\|\Delta n\|_{L^2}^2\right]\leqslant C(\|\nabla n_t\|_{L^2}^2+\|\nabla\varepsilon_t\|_{L^2}^2+\|\Delta n\|_{L^2}^2+1).$$

By using Gronwall inequality, it follows that

$$\|\nabla\varepsilon_t\|_{L^2}^2+\|\nabla n_t\|_{L^2}^2+\|\Delta n\|_{L^2}^2\leqslant C,$$

and with (2.15), (1.6) and (1.5) we arrive at

$$\|\nabla^3\varepsilon\|_{L^2}^2+\|\nabla^3\varphi\|_{L^2}+\|\Delta\varphi_t\|_{L^2}\leqslant C.$$

□

Lemma 2.6. *Assume that* $\varepsilon_0(x)\in H^{l+2}$, $n_0(x)\in H^{l+1}$, $\varphi_0(x)\in H^{l+2}$, $l\geqslant 1$ *and* $0<p\leqslant 2$ *with* $\|\varepsilon_0(x)\|_{L^2}$ *small. Then we have*

$$\sup_{0\leqslant t\leqslant T}\left[\|\varepsilon\|_{H^{l+2}}+\|n\|_{H^{l+1}}+\|\varphi\|_{H^{l+2}}\right]\leqslant C,$$
$$\sup_{0\leqslant t\leqslant T}\left[\|\varepsilon_t\|_{H^{l}}+\|n_t\|_{H^{l}}+\|\varphi_t\|_{H^{l+1}}\right]\leqslant C.$$

Proof. Lemma 2.6 is true when $l=1$ (Lemma 2.5). Suppose Lemma 2.6 is true when $l=k$, $(k\geqslant 1)$, i.e.,

$$\sup_{0\leqslant t\leqslant T}\left[\|\varepsilon\|_{H^{k+2}}+\|n\|_{H^{k+1}}+\|\varphi\|_{H^{k+2}}\right]\leqslant C,$$
$$\sup_{0\leqslant t\leqslant T}\left[\|\varepsilon_t\|_{H^{k}}+\|n_t\|_{H^{k}}+\|\varphi_t\|_{H^{k+1}}\right]\leqslant C.$$

Next, we will show that Lemma 2.6 is true when $l=k+1$.

Differentiating (1.4) with respect to t, then taking the inner product of the resulting equation and $(-1)^{k+1}\Delta^{k+1}\varepsilon_t$, we get

$$\left(i\varepsilon_{tt}+\Delta\varepsilon_t-n_t\varepsilon-n\varepsilon_t+\left(\alpha|\varepsilon|^p\varepsilon\right)_t,(-1)^{k+1}\Delta^{k+1}\varepsilon_t\right)=0. \tag{2.16}$$

Since

$$\begin{aligned}&\mathrm{Im}(i\varepsilon_{tt},(-1)^{k+1}\Delta^{k+1}\varepsilon_t)=\frac{1}{2}\frac{d}{dt}\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2,\\&\mathrm{Im}(\Delta\varepsilon_t,(-1)^{k+1}\Delta^{k+1}\varepsilon_t)=0,\end{aligned}$$

$$\begin{aligned}\left|\mathrm{Im}(-n_t\varepsilon,(-1)^{k+1}\Delta^{k+1}\varepsilon_t)\right|&\leqslant\left|(\nabla^{k+1}(n_t\varepsilon),\nabla^{k+1}\varepsilon_t)\right|\\&\leqslant C(\|\nabla^{k+1}n_t\|_{L^2}^2+\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2+1),\end{aligned}$$

$$\begin{aligned}\left|\mathrm{Im}(-n\varepsilon_t,(-1)^{k+1}\Delta^{k+1}\varepsilon_t)\right|&\leqslant\left|(\nabla^{k+1}(n\varepsilon_t),\nabla^{k+1}\varepsilon_t)\right|\\&\leqslant C(\|\nabla^{k+2}n\|_{L^2}^2+\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2+1),\end{aligned}$$

$$\begin{aligned}\left|\mathrm{Im}\left((\alpha|\varepsilon|^p\varepsilon)_t,(-1)^{k+1}\Delta^{k+1}\varepsilon_t\right)\right|&\leqslant\left|\left(\nabla^{k+1}\left(\alpha|\varepsilon|^p\varepsilon\right)_t,\nabla^{k+1}\varepsilon_t\right)\right|\\&\leqslant C(\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2+1),\end{aligned}$$

thus from (2.16) we get

$$\frac{d}{dt}\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2\leqslant C\left(\|\nabla^{k+1}n_t\|_{L^2}^2+\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2\right)C\left(\|\nabla^{k+2}n\|_{L^2}^2+1\right). \tag{2.17}$$

By differentiating (1.6) with respect to t, then taking the inner product of the resulting equation and $(-1)^{k+1}\Delta^{k+1}n_t$, we get

$$(n_{tt}-\Delta\varphi_t,(-1)^{k+1}\Delta^{k+1}n_t)=0. \tag{2.18}$$

Since

$$\begin{aligned}(n_{tt},(-1)^{k+1}\Delta^{k+1}n_t)&=\frac{1}{2}\frac{d}{dt}\|\nabla^{k+1}n_t\|_{L^2}^2,\\(-\Delta\varphi_t,(-1)^{k+1}\Delta^{k+1}n_t)&=(\Delta n+\Delta|\varepsilon|^2,(-1)^k\Delta^{k+1}n_t),\\(\Delta n,(-1)^k\Delta^{k+1}n_t)&=\frac{1}{2}\frac{d}{dt}\|\nabla^{k+2}n\|_{L^2}^2,\\\left|(\Delta|\varepsilon|^2,(-1)^k\Delta^{k+1}n_t)\right|&\leqslant\left|(\nabla^{k+3}|\varepsilon|^2,\nabla^{k+1}n_t)\right|\\&\leqslant C\left(\|\nabla^{k+3}\varepsilon\|_{L^2}^2+\|\nabla^{k+1}n_t\|_{L^2}^2+1\right),\end{aligned}$$

thus from (2.18) we get

$$\frac{d}{dt}\left[\|\nabla^{k+1}n_t\|_{L^2}^2+\|\nabla^{k+2}n\|_{L^2}^2\right]\leqslant C\left(\|\nabla^{k+3}\varepsilon\|_{L^2}^2+\|\nabla^{k+1}n_t\|_{L^2}^2+1\right). \tag{2.19}$$

Taking the inner product of (1.4) and $(-1)^{k+1}\Delta^{k+2}\varepsilon$, since

$$\left(\Delta\varepsilon,\,(-1)^{k+1}\Delta^{k+2}\varepsilon\right)=\left\|\nabla^{k+3}\varepsilon\right\|_{L^2}^2,$$

$$\begin{aligned}\left|\left(i\varepsilon_t-n\varepsilon+\alpha|\varepsilon|^p\varepsilon,\,(-1)^{k+1}\Delta^{k+2}\varepsilon\right)\right|&=\left|\left(i\nabla^{k+1}\varepsilon_t-\nabla^{k+1}(n\varepsilon)+\alpha\nabla^{k+1}\left(|\varepsilon|^p\varepsilon\right),\,\nabla^{k+3}\varepsilon\right)\right|\\&\leqslant C\left(\left\|\nabla^{k+1}\varepsilon_t\right\|_{L^2}+1\right)\left\|\nabla^{k+3}\varepsilon\right\|_{L^2},\end{aligned}$$

thus we get

$$\left\|\nabla^{k+3}\varepsilon\right\|_{L^2}\leqslant C\left(\left\|\nabla^{k+1}\varepsilon_t\right\|_{L^2}+1\right). \tag{2.20}$$

Hence from (2.17), (2.19) and (2.20) we get

$$\frac{d}{dt}\left[\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2+\|\nabla^{k+1}n_t\|_{L^2}^2+\|\nabla^{k+2}n\|_{L^2}^2\right]\leqslant C\left(\|\nabla^{k+1}n_t\|_{L^2}^2+\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2+\|\nabla^{k+2}n\|_{L^2}^2+1\right).$$

By using Gronwall inequality, it follows that

$$\|\nabla^{k+1}\varepsilon_t\|_{L^2}^2+\|\nabla^{k+1}n_t\|_{L^2}^2+\|\nabla^{k+2}n\|_{L^2}^2\leqslant C,$$

and with (2.20), (1.6) and (1.5), we arrive at

$$\|\nabla^{k+3}\varepsilon\|_{L^2}^2+\|\nabla^{k+3}\varphi\|_{L^2}+\|\nabla^{k+2}\varphi_t\|_{L^2}\leqslant C.$$

Hence

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}&\left[\|\varepsilon\|_{H^{k+3}}+\|n\|_{H^{k+2}}+\|\varphi\|_{H^{k+3}}+\|\varepsilon_t\|_{H^{k+1}}\right]\\&+\sup_{0\leqslant t\leqslant T}\left[\|n_t\|_{H^{k+1}}+\|\varphi_t\|_{H^{k+2}}\right]\leqslant C.\end{aligned}$$

Lemma 2.6 is proved completely. □

3. The existence and uniqueness of solution

In this section, we formulate the proof of Theorem 1.1. First we give the definition of generalized solution for problem (1.4)-(1.7).

Definition 3.1. The functions

$$\begin{aligned}&\varepsilon(x,t)\in L^\infty(0,T;H^2)\cap W^{1,\infty}(0,T;L^2),\\&n(x,t)\in L^\infty(0,T;H^1)\cap W^{1,\infty}(0,T;L^2),\\&\varphi(x,t)\in L^\infty(0,T;H^2)\cap W^{1,\infty}(0,T;H^1),\end{aligned}$$

are called generalized solution of problem (1.4)-(1.7), if for any $v\in L^2$ they satisfy the integral equality:

$$\begin{aligned}&(i\varepsilon_{mt},v)+(\Delta\varepsilon_m,v)-(n\varepsilon_m,v)+(\alpha|\varepsilon|^p\varepsilon_m,v)=0,\quad m=1,\cdots,N,\\&(\varphi_t,v)-(n,v)-(|\varepsilon|^2,v)=0,\\&(n_t,v)-(\Delta\varphi,v)=0,\end{aligned}$$

with initial data

$$\varepsilon(x,0)=\varepsilon_0(x),\quad n(x,0)=n_0(x),\quad \varphi(x,0)=\varphi_0(x).$$

Next, we give two lemmas recalled in [10].

Lemma 3.2. *Let B_0, B, B_1 be three reflexive Banach spaces and assume that the embedding $B_0\to B$ is compact. Let*

$$W=\left\{V\in L^{p_0}((0,T);B_0),\ \frac{\partial V}{\partial t}\in L^{p_1}((0,T);B_1)\right\},\quad T<\infty,\quad 1<p_0,p_1<\infty.$$

W is a Banach space with norm

$$\|V\|_W=\|V\|_{L^{p_0}((0,T);B_0)}+\|V_t\|_{L^{p_1}((0,T);B_1)}.$$

Then the embedding $W\to L^{p_0}((0,T);B)$ is compact.

Lemma 3.3. *Let Ω be an open set of $\mathbb{R}^n$ and let $g, g_\varepsilon \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, such that*

$$g_\varepsilon \to g \quad \text{a.e. in } \Omega \qquad \text{and} \qquad \|g_\varepsilon\|_{L^p(\Omega)} \leqslant C.$$

Then $g_\varepsilon \to g$ weakly in $L^p(\Omega)$.

Now, one can estimate the following theorem.

Theorem 3.4. *Suppose that $\varepsilon_0(x) \in H^2$, $n_0(x) \in H^1$, $\varphi_0(x) \in H^2$ and $0 < p \leqslant 2$ with $\|\varepsilon_0(x)\|_{L^2}$ small. Then there exists global generalized solution of the initial value problem (1.4)-(1.7).*

$$\begin{aligned} &\varepsilon(x,t) \in L^\infty(0,T;H^2) \cap W^{1,\infty}(0,T;L^2),\\ &n(x,t) \in L^\infty(0,T;H^1) \cap W^{1,\infty}(0,T;L^2),\\ &\varphi(x,t) \in L^\infty(0,T;H^2) \cap W^{1,\infty}(0,T;H^1). \end{aligned}$$

Proof. By using Galerkin method, choose the basic periodic functions $\{\omega_j(x)\}$ as follows:

$$-\Delta\omega_j(x) = \lambda_j\omega_j(x), \quad \omega_j(x) \in H^2(\mathbb{R}^2), \quad j = 1, 2, \cdots, l.$$

The approximate solutions of problem (1.4)-(1.7) can be written as

$$\begin{aligned} &\varepsilon^l(x,t) = \sum_{j=1}^{l} \alpha_j^l(t)\omega_j(x), \quad n^l(x,t) = \sum_{j=1}^{l} \beta_j^l(t)\omega_j(x),\\ &\varphi^l(x,t) = \sum_{j=1}^{l} \gamma_j^l(t)\omega_j(x), \end{aligned}$$

where

$$\varepsilon^l = \left(\varepsilon_1^l, \cdots, \varepsilon_N^l\right), \quad \alpha_j^l(t) = (\alpha_{j1}^l(t), \cdots, \alpha_{jN}^l(t)),$$

and Ω is a 2-dimensional cube with 2D in each direction, that is, $\overline{\Omega} = \{x = (x_1, x_2) \| x_i| \leqslant 2D,\ i = 1, 2\}$.

These undetermined coefficients $\alpha_j^l(t)$, $\beta_j^l(t)$ and $\gamma_j^l(t)$ need to satisfy the following initial value problem of the system of ordinary differential equations

$$\left(i\varepsilon_{mt}^l, \omega_\kappa\right) + \left(\Delta\varepsilon_m^l, \omega_\kappa\right) - \left(n^l\varepsilon_m^l, \omega_\kappa\right) + \left(\alpha|\varepsilon^l|^p\varepsilon_m^l, \omega_\kappa\right) = 0, \quad m = 1, \cdots, N, \tag{3.1}$$

$$\left(\varphi_t^l, \omega_\kappa\right) - \left(n^l, \omega_\kappa\right) - \left(|\varepsilon^l|^2, \omega_\kappa\right) = 0, \tag{3.2}$$

$$\left(n_t^l, \omega_\kappa\right) - \left(\Delta\varphi^l, \omega_\kappa\right) = 0, \tag{3.3}$$

with initial data

$$\begin{aligned} &\varepsilon^l(x,0) = \varepsilon_0^l(x),\ n^l(x,0) = n_0^l(x),\\ &\varphi^l(x,0) = \varphi_0^l(x), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} &\varepsilon_0^l(x) \xrightarrow{H^2} \varepsilon_0(x), \quad n_0^l(x) \xrightarrow{H^1} n_0(x),\\ &\varphi_0^l(x) \xrightarrow{H^2} \varphi_0(x), \quad l \to \infty. \end{aligned}$$

Similar to the proof of Lemmas 2.2 and 2.4, for the solution $\varepsilon^l(x,t)$, $n^l(x,t)$, $\varphi^l(x,t)$ of problem (3.1), (3.2), (3.3), (3.4), we can establish the following estimations

$$\sup_{0\leqslant t\leqslant T}\left[\|\varepsilon^l(x,t)\|_{H^2} + \|n^l(x,t)\|_{H^1} + \|\varphi^l(x,t)\|_{H^2}\right] \leqslant C,$$

$$\sup_{0\leqslant t\leqslant T}\left[\|\varepsilon_t^l(x,t)\|_{L^2}+\|n_t^l\|_{L^2}+\|\varphi_t^l\|_{H^1}\right]\leqslant C, \tag{3.5}$$

where the constant C is independent of l and D. By compact argument, some subsequence of $(\varepsilon^l, n^l, \varphi^l)$, also labeled by l, has a weak limit $(\varepsilon, n, \varphi)$. More precisely

$$\begin{aligned}
&\varepsilon^l(x,t)\to\varepsilon(x,t) \text{ in } L^\infty(0,T;H^2) \text{ weakly star}, &(3.6)\\
&n^l(x,t)\to n(x,t) \text{ in } L^\infty(0,T;H^1) \text{ weakly star}, &(3.7)\\
&\varphi^l(x,t)\to\varphi(x,t) \text{ in } L^\infty(0,T;H^2) \text{ weakly star}.
\end{aligned}$$

Equation (3.5) implies that

$$\begin{aligned}
&\varepsilon_t^l\to\varepsilon_t \text{ in } L^\infty(0,T;L^2) \text{ weakly star}, &(3.8)\\
&n_t^l\to n_t \text{ in } L^\infty(0,T;L^2) \text{ weakly star},\\
&\varphi_t^l\to\varphi_t \text{ in } L^\infty(0,T;H^1) \text{ weakly star}.
\end{aligned}$$

Using Sobolev embedding theorem, Lemmas 3.2, 3.3 and (3.6), (3.7), (3.8), we deduce that

$$\begin{aligned}
&\left|\varepsilon^l\right|^2\to|\varepsilon|^2 \text{ in } L^\infty(0,T;H^1) \text{ weakly star},\\
&n^l\varepsilon^l\to n\varepsilon \text{ in } L^\infty(0,T;L^2) \text{ weakly star},\\
&\left|\varepsilon^l\right|^p\varepsilon^l\to|\varepsilon|^p\,\varepsilon \text{ in } L^\infty(0,T;L^2) \text{ weakly star}.
\end{aligned}$$

Hence taking $l\to\infty$ from (3.1)-(3.4), by using the density of ω_j in $L^2(\Omega)$ we get the existence of local generalized solution for the periodic initial value problem (1.4)-(1.7). Letting $D\to\infty$, the existence of local solution for the initial value problem (1.4)-(1.7) can be obtained. By the continuation extension principle and a prior estimates, we can get the existence of global generalized solution for problem (1.4)-(1.7). Theorem 3.4 is proved. □

Next, we prove the uniqueness of solution.

Theorem 3.5. *Suppose that* $\varepsilon_0(x)\in H^3$, $n_0(x)\in H^2$, $\varphi_0(x)\in H^3$ *and* $0<p\leqslant 2$ *with* $\|\varepsilon_0(x)\|_{L^2}$ *small. Then the global generalized solution of the initial value problem* (1.4)-(1.7) *is unique.*

Proof. Assume that there are two solutions ε_1, n_1, φ_1 and ε_2, n_2, φ_2. Let

$$\varepsilon=\varepsilon_1-\varepsilon_2,\quad n=n_1-n_2,\quad \varphi=\varphi_1-\varphi_2.$$

From (1.4)-(1.7) we get

$$\begin{aligned}
&i\varepsilon_t+\Delta\varepsilon-(n_1\varepsilon_1-n_2\varepsilon_2)+\alpha(|\varepsilon_1|^p\varepsilon_1-|\varepsilon_2|^p\varepsilon_2)=0, &(3.9)\\
&\varphi_t-n-\left[|\varepsilon_1|^2-|\varepsilon_2|^2\right]=0, &(3.10)\\
&n_t-\Delta\varphi=0, &(3.11)
\end{aligned}$$

with initial data

$$\varepsilon|_{t=0}=n|_{t=0}=\varphi|_{t=0}=0. \tag{3.12}$$

From Lemma 2.5, we know that

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\left[\|\varepsilon_\mu\|_{H^3}+\|n_\mu\|_{H^2}+\|\varphi_\mu\|_{H^3}\right]\leqslant C,\\
&\sup_{0\leqslant t\leqslant T}\left[\|\varepsilon_{\mu t}\|_{H^1}+\|n_{\mu t}\|_{H^1}+\|\varphi_{\mu t}\|_{H^2}\right]\leqslant C,\quad \mu=1,2.
\end{aligned}$$

Taking the inner product of (3.9) and ε, it follows that

$$(i\varepsilon_t + \Delta\varepsilon - (n_1\varepsilon_1 - n_2\varepsilon_2) + \alpha(|\varepsilon_1|^p\varepsilon_1 - |\varepsilon_2|^p\varepsilon_2), \varepsilon) = 0. \tag{3.13}$$

Since

$$\mathrm{Im}(i\varepsilon_t, \varepsilon) = \frac{1}{2}\frac{d}{dt}\|\varepsilon\|_{L^2}^2, \quad \mathrm{Im}(\Delta\varepsilon, \varepsilon) = 0,$$

$$\begin{aligned} |\mathrm{Im}(n_1\varepsilon_1 - n_2\varepsilon_2, \varepsilon)| &\leqslant |(n\varepsilon_1 + n_2\varepsilon, \varepsilon)| \leqslant (\|n\|_{L^2}\|\varepsilon_1\|_{L^\infty} + \|n_2\|_{L^\infty}\|\varepsilon\|_{L^2})\,\|\varepsilon\|_{L^2} \\ &\leqslant C\left(\|n\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2\right), \end{aligned}$$

$$\begin{aligned} |\mathrm{Im}\,\alpha\left(|\varepsilon_1|^p\varepsilon_1 - |\varepsilon_2|^p\varepsilon_2, \varepsilon\right)| &\leqslant |\alpha\left(|\varepsilon_1|^p\varepsilon + \varepsilon_2\left(|\varepsilon_1|^p - |\varepsilon_2|^p\right), \varepsilon\right)| \\ &\leqslant C\left(\|\varepsilon_1\|_{L^\infty}^p\|\varepsilon\|_{L^2} + \|\varepsilon_2\|_{L^\infty}\|\varepsilon\|_{L^2}\right)\|\varepsilon\|_{L^2} \\ &\leqslant C\|\varepsilon\|_{L^2}^2, \end{aligned}$$

where the second inequality used

$$\begin{aligned} ||\varepsilon_1|^p - |\varepsilon_2|^p| &= \left|p\left[\theta|\varepsilon_1| + (1-\theta)|\varepsilon_2|\right]^{p-1}(|\varepsilon_1| - |\varepsilon_2|)\right| \\ &\leqslant p\left[\theta\|\varepsilon_1\|_{L^\infty} + (1-\theta)\|\varepsilon_2\|_{L^\infty}\right]^{p-1}|\varepsilon| \\ &\leqslant C|\varepsilon|, \quad 0 < \theta < 1, \end{aligned} \tag{3.14}$$

thus from (3.13) we get

$$\frac{d}{dt}\|\varepsilon\|_{L^2}^2 \leqslant C\left(\|n\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2\right). \tag{3.15}$$

Taking the inner product of (3.10) and φ it follows that

$$\left(\varphi_t - n - \left(|\varepsilon_1|^2 - |\varepsilon_2|^2\right), \varphi\right) = 0. \tag{3.16}$$

Since

$$\begin{aligned} (\varphi_t, \varphi) &= \frac{1}{2}\frac{d}{dt}\|\varphi\|_{L^2}^2, \\ |(-n, \varphi)| &\leqslant C(\|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2), \end{aligned}$$

$$\begin{aligned} \left|\left(-\left(|\varepsilon_1|^2 - |\varepsilon_2|^2\right), \varphi\right)\right| &= |(\varepsilon\cdot\overline{\varepsilon}_1 + \varepsilon_2\cdot\overline{\varepsilon}, \varphi)| \\ &\leqslant C(\|\varepsilon_1\|_{L^\infty}\|\varepsilon\|_{L^2} + \|\varepsilon_2\|_{L^\infty}\|\varepsilon\|_{L^2})\|\varphi\|_{L^2} \\ &\leqslant C(\|\varepsilon\|_{L^2}^2 + \|\varphi\|_{L^2}^2), \end{aligned}$$

thus from (3.16) we obtain

$$\frac{d}{dt}\|\varphi\|_{L^2}^2 \leqslant \left(\|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2\right). \tag{3.17}$$

Differentiating (3.11) with respect to t, then taking the inner product of the resulting equation and n_t, we get

$$(n_{tt} - \Delta\varphi_t, n_t) = 0. \tag{3.18}$$

Since

$$(n_{tt}, n_t) = \frac{1}{2}\frac{d}{dt}\|n_t\|_{L^2}^2,$$

$$(-\Delta\varphi_t, n_t) = (-\Delta n - \Delta(|\varepsilon_1|^2 - |\varepsilon_2|^2), n_t),$$

$$(-\Delta n, n_t) = \frac{1}{2}\frac{d}{dt}\|\nabla n\|_{L^2}^2,$$

$$\begin{aligned}\left|\left(\Delta\left(|\varepsilon_1|^2 - |\varepsilon_2|^2\right), n_t\right)\right| &= |(\Delta(\varepsilon\cdot\bar{\varepsilon}_1 + \varepsilon_2\cdot\bar{\varepsilon}), n_t)| \\ &\leqslant C\big(\|\Delta\varepsilon\|_{L^2}\|\varepsilon_1\|_{L^\infty} + \|\varepsilon\|_{L^\infty}\|\Delta\varepsilon_1\|_{L^2}\big)\|n_t\|_{L^2} \\ &\quad + C\big(\|\nabla\varepsilon\|_{L^4}\|\nabla\varepsilon_1\|_{L^4} + \|\Delta\varepsilon_2\|_{L^2}\|\varepsilon\|_{L^\infty}\big)\|n_t\|_{L^2} \\ &\quad + C\big(\|\Delta\varepsilon\|_{L^2}\|\varepsilon_2\|_{L^\infty} + \|\nabla\varepsilon_2\|_{L^4}\|\nabla\varepsilon\|_{L^4}\big)\|n_t\|_{L^2} \\ &\leqslant C\left(\|\Delta\varepsilon\|_{L^2}^2 + \|\nabla\varepsilon\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 + \|n_t\|_{L^2}^2\right),\end{aligned}$$

thus from (3.18) we get

$$\frac{d}{dt}\left[\|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2\right] \leqslant C\left(\|\Delta\varepsilon\|_{L^2}^2 + \|\nabla\varepsilon\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 + \|n_t\|_{L^2}^2\right). \tag{3.19}$$

Taking the inner product of (3.9) and $\Delta\varepsilon$, it follows that

$$(i\varepsilon_t + \Delta\varepsilon - (n_1\varepsilon_1 - n_2\varepsilon_2) + \alpha(|\varepsilon_1|^p\varepsilon_1 - |\varepsilon_2|^p\varepsilon_2), \Delta\varepsilon) = 0.$$

Since

$$|(i\varepsilon_t, \Delta\varepsilon)| \leqslant \|\varepsilon_t\|_{L^2}\|\Delta\varepsilon\|_{L^2}, \quad (\Delta\varepsilon, \Delta\varepsilon) = \|\Delta\varepsilon\|_{L^2}^2,$$

$$\begin{aligned}|(n_1\varepsilon_1 - n_2\varepsilon_2, \Delta\varepsilon)| = |(n\varepsilon_1 + n_2\varepsilon, \Delta\varepsilon)| &\leqslant \left(\|n\|_{L^2}\|\varepsilon_1\|_{L^\infty} + \|n_2\|_{L^4}\|\varepsilon\|_{L^4}\right)\|\Delta\varepsilon\|_{L^2} \\ &\leqslant C\left(\|n\|_{L^2} + \|\nabla\varepsilon\|_{L^2}^{\frac{1}{2}}\|\varepsilon\|_{L^2}^{\frac{1}{2}}\right)\|\Delta\varepsilon\|_{L^2} \\ &\leqslant C\left(\|n\|_{L^2} + \|\varepsilon\|_{L^2} + \|\nabla\varepsilon\|_{L^2}\right)\|\Delta\varepsilon\|_{L^2},\end{aligned}$$

$$\begin{aligned}|\alpha\left(|\varepsilon_1|^p\varepsilon_1 - |\varepsilon_2|^p\varepsilon_2, \Delta\varepsilon\right)| &\leqslant |\alpha\left(|\varepsilon_1|^p\varepsilon + \varepsilon_2\left(|\varepsilon_1|^p - |\varepsilon_2|^p\right), \Delta\varepsilon\right)| \\ &\leqslant C\left(\|\varepsilon_1\|_{L^\infty}^p\|\varepsilon\|_{L^2} + \|\varepsilon_2\|_{L^\infty}\|\varepsilon\|_{L^2}\right)\|\Delta\varepsilon\|_{L^2} \\ &\leqslant C\|\varepsilon\|_{L^2}\|\Delta\varepsilon\|_{L^2}.\end{aligned}$$

It follows that

$$\|\Delta\varepsilon\|_{L^2} \leqslant C\left(\|\varepsilon_t\|_{L^2} + \|n\|_{L^2}\right) C\left(\|\varepsilon\|_{L^2} + \|\nabla\varepsilon\|_{L^2}\right). \tag{3.20}$$

Differentiating (3.9) with respect to t, then taking the inner product of the resulting equation and ε_t, we get

$$(i\varepsilon_{tt} + \Delta\varepsilon_t - (n_1\varepsilon_1 - n_2\varepsilon_2)_t, \varepsilon_t) + (\alpha\left(|\varepsilon_1|^p\varepsilon_1 - |\varepsilon_2|^p\varepsilon_2\right)_t, \varepsilon_t) = 0. \tag{3.21}$$

Since

$$\mathrm{Im}(i\varepsilon_{tt}, \varepsilon_t) = \frac{1}{2}\frac{d}{dt}\|\varepsilon_t\|_{L^2}^2, \quad \mathrm{Im}(\Delta\varepsilon_t, \varepsilon_t) = 0,$$

$$\begin{aligned}|\mathrm{Im}\left(-(n_1\varepsilon_1 - n_2\varepsilon_2)_t, \varepsilon_t\right)| &= |\mathrm{Im}(n_t\varepsilon_1 + n\varepsilon_{1t} + n_{2t}\varepsilon, \varepsilon_t)| \\ &\leqslant C(\|\varepsilon_1\|_{L^\infty}\|n_t\|_{L^2} + \|\varepsilon_{1t}\|_{L^4}\|n\|_{L^4})\|\varepsilon_t\|_{L^2} + C\|n_{2t}\|_{L^4}\|\varepsilon\|_{L^4}\|\varepsilon_t\|_{L^2} \\ &\leqslant C(\|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\nabla\varepsilon\|_{L^2}^2) + C(\|\varepsilon\|_{L^2}^2 + \|\varepsilon_t\|_{L^2}^2),\end{aligned}$$

$$\begin{aligned}|\mathrm{Im}\left(\alpha\left(|\varepsilon_1|^p\varepsilon_1 - |\varepsilon_2|^p\varepsilon_2\right)_t, \varepsilon_t\right)| &= |\mathrm{Im}\left(\alpha\left(|\varepsilon_1|^p\varepsilon + \varepsilon_2(|\varepsilon_1|^p - |\varepsilon_2|^p)\right)_t, \varepsilon_t\right)| \\ &\leqslant \left|\alpha\left(|\varepsilon_1|_t^p\varepsilon + \varepsilon_{2t}(|\varepsilon_1|^p - |\varepsilon_2|^p), \varepsilon_t\right)\right| + |\alpha\left(\varepsilon_2(|\varepsilon_1|^p - |\varepsilon_2|^p)_t, \varepsilon_t\right)| \\ &\leqslant C\big(\|\varepsilon_1\|_{L^\infty}^{p-1}\|\varepsilon_{1t}\|_{L^4}\|\varepsilon\|_{L^4} + \|\varepsilon_{2t}\|_{L^4}\|\varepsilon\|_{L^4}\big)\|\varepsilon_t\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+ C\big(\|\varepsilon_2\|_{L^\infty}\|\varepsilon_t\|_{L^2} + +\|\varepsilon_2\|_{L^\infty}\|\varepsilon\|_{L^4}\|\varepsilon_{1t}\|_{L^4}\big)\|\varepsilon_t\|_{L^2} \\
&+ C\|\varepsilon_2\|_{L^\infty}\|\varepsilon\|_{L^4}\|\varepsilon_{2t}\|_{L^4}\|\varepsilon_t\|_{L^2} \\
&\leqslant C\left(\|\nabla\varepsilon\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 + \|\varepsilon_t\|_{L^2}^2\right),
\end{aligned}
$$

where the second inequality used (3.14) and

$$
\begin{aligned}
|(|\varepsilon_1|^p - |\varepsilon_2|^p)_t| &\leqslant \frac{p}{2}|\varepsilon_1|^{p-2}\,|\varepsilon_t\cdot\overline{\varepsilon}_1 + \varepsilon\cdot\overline{\varepsilon}_{1t} + \varepsilon_{2t}\cdot\overline{\varepsilon} + \varepsilon_2\cdot\overline{\varepsilon}_t| + \\
&\quad + \frac{p}{2}\left||\varepsilon_1|^{p-2} - |\varepsilon_2|^{p-2}\right|\,|\varepsilon_{2t}\cdot\overline{\varepsilon}_2 + \varepsilon_2\cdot\overline{\varepsilon}_{2t}| \\
&\leqslant C\left(|\varepsilon_t| + |\varepsilon||\varepsilon_{1t}| + |\varepsilon_{2t}||\varepsilon|\right),
\end{aligned}
$$

thus from (3.21) we get

$$
\frac{d}{dt}\|\varepsilon_t\|_{L^2}^2 \leqslant C\left(\|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + \|n\|_{L^2}^2\right) + C\left(\|\nabla\varepsilon\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 + \|\varepsilon_t\|_{L^2}^2\right). \tag{3.22}
$$

Obviously

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\varepsilon\|_{L^2}^2 &= -2\,\mathrm{Re}\,(\Delta\varepsilon, \varepsilon_t) \\
&\leqslant C\left(\|\Delta\varepsilon\|_{L^2}^2 + \|\varepsilon_t\|_{L^2}^2\right),
\end{aligned} \tag{3.23}
$$

$$
\frac{d}{dt}\|n\|_{L^2}^2 = 2(n_t, n) \leqslant C(\|n_t\|_{L^2}^2 + \|n\|_{L^2}^2). \tag{3.24}
$$

Hence from (3.15), (3.17), (3.19), (3.20) and (3.22), (3.23), (3.24) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\|\varepsilon\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2 + \|\nabla\varepsilon\|_{L^2}^2\right] + \frac{d}{dt}\left[\|\nabla n\|_{L^2}^2 + \|\varepsilon_t\|_{L^2}^2 + \|n_t\|_{L^2}^2\right] \\
&\leqslant C\left[\|\varepsilon\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2 + \|\nabla\varepsilon\|_{L^2}^2\right] \\
&\quad + C\left[\|\nabla n\|_{L^2}^2 + \|\varepsilon_t\|_{L^2}^2 + \|n_t\|_{L^2}^2\right].
\end{aligned}
$$

By using Gronwall inequality and noticing (3.12), it follows that

$$
\varepsilon = 0, \quad n = 0, \quad \varphi = 0.
$$

Theorem 3.5 is proved. □

Using Lemma 2.6 and the embedding theorems of Sobolev spaces, the result of Theorem 1.1 is obvious.

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