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Differential equations for Daehee polynomials and their applications

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Abstract

Recently, differential equations for Changhee polynomials and their applications were introduced by Kim et al. and by using their differential equations, they derived some new identities on Changhee polynomials. Specially, they presented Changhee polynomials $Ch_{n+N}(x)$ by sums of lower terms of Changhee polynomials $Ch_n(x)$. Compare to the result, Kim et al. described Changhee polynomials $Ch_{n+N}(x)$ via lower term of higher order Chaghee polynomials by using non-linear differential equations arising from generating function of Changhee polynomials. In the first part of this paper, the author uses the idea of Kim et al. to apply to generating function for Daehee polynomials. From differential equations associated with the generating function of those polynomials, we derive some formulae and combinatorial identities.

Also, Kwon et al. developed the method of differential equations from the generating function of Daehee numbers and investigated new explicit identities of Daehee numbers. In the second part of the present paper, the author applies their methods to generating function of Daehee polynomials, and get the explicit representations of Daehee polynomials. And specially we put $x = 0$ in our results, we can get new representations of Daehee numbers compare to the above results.

Keywords: Daehee polynomial, Daehee number, differential equations.
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1. Introduction

It is common knowledge that the Bernoulli polynomials $B_n(x)$ for $n \geqslant 0$ can be generated by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

(see [2, 7, 11]).

With the viewpoint of deformed Bernoulli polynomials, the Daehee polynomials $D_n(x)$ for $n \geqslant 0$ are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}. \tag{1.1}$$

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It is easy to see that the generating function of the Daehee polynomials $D_n(x)$ can be reformed as

$$\frac{\log(1+t)}{t}(1+t)^x = \frac{\log(1+t)}{e^{\log(1+t)}-1}e^{x\log(1+t)}.$$

From (1.1), we note that

$$\begin{aligned}\frac{\log(1+t)}{e^{\log(1+t)}-1}e^{x\log(1+t)} &= \sum_{n=0}^{\infty} B_n(x)\frac{1}{n!}(\log(1+t))^n \\ &= \sum_{n=0}^{\infty} B_n(x)\sum_{m=n}^{\infty} S_1(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} B_n(x)S_1(m,n)\right)\frac{t^m}{m!},\end{aligned} \tag{1.2}$$

where $S_1(m,n)$ stands for the Stirling number of the first kind which is defined as

$$(x)_0 = 1,\ \ (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l,\ \ (n \geqslant 1).$$

Combining (1.1) with (1.2) yields the following relation

$$D_m(x) = \sum_{n=0}^{m} B_n(x)S_1(m,n),\ \ (m \geqslant 0).$$

By replacing t by e^t-1 in (1.1), we can derive

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt} &= \sum_{m=0}^{\infty} D_m(x)\frac{1}{m!}(e^t-1)^m \\ &= \sum_{m=0}^{\infty} D_m(x)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} D_n(x)S_2(n,m)\right)\frac{t^n}{n!},\end{aligned} \tag{1.3}$$

where $S_2(n,m)$ is the Stirling number of the second kind which is given by $x^n = \sum_{l=0}^{\infty} S_2(n,l)(x)_l$, $(n \geqslant 0)$.

Comparing the coefficients on the both sides of (1.3), we obtain

$$B_n(x) = \sum_{m=0}^{n} D_m(x)S_2(n,m),\ \ (n \geqslant 0). \tag{1.4}$$

In recent decades, many mathematicians have investigated some interesting extensions and modifications of Daehee polynomials related combinatorial identities and their applications (see [1, 2, 5, 6, 9, 11–13]).

In [4], differential equations for Changhee polynomials and their applications were introduced by Kim et al.. By using their differential equations, they derived some new identities on Changhee polynomials. Specially, they presented Changhee polynomials $Ch_{n+N}(x)$ by sums of lower terms of Changhee polynomials $Ch_n(x)$. Compared to [4], Kim et al. described Changhee polynomials $Ch_{n+N}(x)$ via lower term of higher order Chaghee polynomials by using non-linear differential equations arising from generating function of Changhee polynomials in [8]. Both papers [4] and [8] treated the inversion problem for the results in Kim's work (see [3]). In the first part of this paper, we use the idea of [4] to apply to generating function for Daehee polynomials. From differential equations associated with the generating function of those polynomials, we derive some formulae and combinatorial identities.

Also in [10], Kwon et al. developed the method of differential equations from the generating function of Daehee numbers and investigated new explicit identities of Daehee numbers. In the second part of the present paper, we apply their methods to generating function of Daehee polynomials, and get the explicit representations of Daehee polynomials. And specially we put $x = 0$ in our results, we can get new representations of Daehee numbers compare to the above results.

2. Differential equations for Daehee polynomials

Let

$$F = F(t, x) = \frac{\log(1+t)}{t}(1+t)^x. \tag{2.1}$$

By taking the derivative with respect to t in (2.1), we can derive

$$F^{(1)} = \frac{d}{dt}F(t, x) = \left[\left(x + \frac{1}{\log(1+t)}\right)(t+1)^{-1} - t^{-1}\right] F \tag{2.2}$$

and

$$F^{(2)} = \frac{d}{dt}F^{(1)} = \left[2t^{-2} - 2\left(\frac{1}{\log(1+t)} + x\right)(1+t)^{-1}t^{-1} + \left(\frac{2x-1}{\log(1+t)} + x(x-1)\right)(1+t)^{-2}\right] F.$$

Similarly, we get

$$\begin{aligned} F^{(3)} = \frac{d}{dt}F^{(2)} = &\left[-6t^{-3} + 6\left(\frac{1}{\log(1+t)} + x\right)(1+t)^{-1}t^{-2} - 3\left(\frac{2x-1}{\log(1+t)} + x(x-1)\right)(1+t)^{-2}t^{-1}\right.\\ &\left. + \left(\frac{3x^2-6x+2}{\log(1+t)} + x(x-1)(x-2)\right)(1+t)^{-3}\right] F. \end{aligned}$$

Continuing this process, we are led to put

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t, x) = \left[\sum_{i=0}^{N}\left(a_i(N, x) + \frac{b_i(N, x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N}\right] F, \tag{2.3}$$

where $N = 0, 1, 2, \cdots$.

Moreover, from (2.3), we note that

$$\begin{aligned} F^{(N+1)} = \frac{d}{dt}F^{(N)} = &-\left[\sum_{i=0}^{N}(N-i)\left(a_i(N, x) + \frac{b_i(N, x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N-1}\right] F\\ &-\left[\sum_{i=0}^{N}\left(ia_i(N, x) + \frac{ib_i(N, x)}{\log(1+t)} + \frac{b_i(N, x)}{(\log(1+t))^2}\right)(t+1)^{-i-1}t^{i-N}\right] F\\ &+\left[\sum_{i=0}^{N}\left(a_i(N, x) + \frac{b_i(N, x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N}\right] F^{(1)}\\ = &-\left[\sum_{i=0}^{N}(N-i)\left(a_i(N, x) + \frac{b_i(N, x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N-1}\right] F\\ &-\left[\sum_{i=0}^{N}\left(ia_i(N, x) + \frac{ib_i(N, x)}{\log(1+t)} + \frac{b_i(N, x)}{(\log(1+t))^2}\right)(t+1)^{-i-1}t^{i-N}\right] F\\ &+\left[\sum_{i=0}^{N}\left(xa_i(N, x) + \frac{a_i(N, x) + xb_i(N, x)}{\log(1+t)} + \frac{b_i(N, x)}{(\log(1+t))^2}\right)(t+1)^{-i-1}t^{i-N}\right] F \end{aligned}$$

$$
\begin{aligned}
&-\left[\sum_{i=0}^{N}\left(a_i(N,x)+\frac{b_i(N,x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N-1}\right]F\\
&=-\left[\sum_{i=0}^{N}(N-i)\left(a_i(N,x)+\frac{b_i(N,x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N-1}\right.\\
&\quad+\sum_{i=1}^{N+1}\left((i-1)a_{i-1}(N,x)+\frac{(i-1)b_{i-1}(N,x)}{\log(1+t)}+\frac{b_{i-1}(N,x)}{(\log(1+t))^2}\right)(t+1)^{-i}t^{i-N-1}\\
&\quad-\sum_{i=1}^{N+1}\left(xa_{i-1}(N,x)+\frac{a_{i-1}(N,x)+xb_{i-1}(N,x)}{\log(1+t)}+\frac{b_{i-1}(N,x)}{(\log(1+t))^2}\right)(t+1)^{-i}t^{i-N-1}\\
&\quad\left.+\sum_{i=0}^{N}\left(a_i(N,x)+\frac{b_i(N,x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N-1}\right]F.
\end{aligned}
\tag{2.4}
$$

On the other hand, by replacing N by $N+1$ in (2.3), we obtain

$$
F^{(N+1)}=\left[\sum_{i=0}^{N+1}\left(a_i(N+1,x)+\frac{b_i(N+1,x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N-1}\right]F. \tag{2.5}
$$

By comparing the coefficients on both sides of (2.4) and (2.5), we have

$$
a_0(N+1,x)+\frac{b_0(N+1,x)}{\log(1+t)}=-(N+1)\left(a_0(N,x)+\frac{b_0(N,x)}{\log(1+t)}\right), \tag{2.6}
$$

$$
a_{N+1}(N+1,x)+\frac{b_{N+1}(N+1,x)}{\log(1+t)}=(x-N)\left(a_N(N,x)+\frac{b_N(N,x)}{\log(1+t)}\right)+\frac{a_N(N,x)}{\log(1+t)}, \tag{2.7}
$$

and

$$
\begin{aligned}
a_i(N+1,x)+\frac{b_i(N+1,x)}{\log(1+t)}&=(x-i+1)\left(a_{i-1}(N,x)+\frac{b_{i-1}(N,x)}{\log(1+t)}\right)+\frac{a_{i-1}(N,x)}{\log(1+t)}\\
&\quad-(N-i+1)\left(a_i(N,x)+\frac{b_i(N,x)}{\log(1+t)}\right),
\end{aligned}
\tag{2.8}
$$

where $1\leqslant i\leqslant N$. We also note that

$$
F=F^{(0)}=\left[a_0(0,x)+\frac{b_0(0,x)}{\log(1+t)}\right]F. \tag{2.9}
$$

Hence, by (2.9), we get

$$
a_0(0,x)+\frac{b_0(0,x)}{\log(1+t)}=1. \tag{2.10}
$$

From (2.10), it follows that

$$
a_0(0,x)=1,\quad b_0(0,x)=0.
$$

In addition, (2.2) and (2.3) lead to

$$
\begin{aligned}
&\left[\left(x+\frac{1}{\log(1+t)}\right)(t+1)^{-1}-t^{-1}\right]F=F^{(1)}\\
&=\left[\sum_{i=0}^{1}\left(a_i(1,x)+\frac{b_i(1,x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-1}\right]F\\
&=\left[\left(a_0(1,x)+\frac{b_0(1,x)}{\log(1+t)}\right)t^{-1}+\left(a_1(1,x)+\frac{b_1(1,x)}{\log(1+t)}\right)(t+1)^{-1}\right]F.
\end{aligned}
\tag{2.11}
$$

Comparing the coefficients on both sides of (2.11) results in

$$
a_0(1,x)=-1,\quad b_0(1,x)=0,\quad a_1(1,x)=x,\quad b_1(1,x)=1.
$$

Also by (2.6) and (2.7), we have

$$\begin{aligned}
a_0(N+1,x)+\frac{b_0(N+1,x)}{\log(1+t)} &= -(N+1)\left(a_0(N,x)+\frac{b_0(N,x)}{\log(1+t)}\right)\\
&= (-1)^2(N+1)N\left(a_0(N-1,x)+\frac{b_0(N-1,x)}{\log(1+t)}\right)\\
&\vdots\\
&= (-1)^N(N+1)N\cdots 2\left(a_0(1,x)+\frac{b_0(1,x)}{\log(1+t)}\right)=(-1)^{N+1}(N+1)!
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
a_{N+1}(N+1,x)+\frac{b_{N+1}(N+1,x)}{\log(1+t)} &= (x-N)\left(a_N(N,x)+\frac{b_N(N,x)}{\log(1+t)}\right)+\frac{a_N(N,x)}{\log(1+t)}\\
&= (x-N)(x-(N-1))\left(a_{N-1}(N-1,x)+\frac{b_{N-1}(N-1,x)}{\log(1+t)}\right)\\
&\quad+\frac{a_N(N,x)+(x-N)a_{N-1}(N-1,x)}{\log(1+t)}\\
&\vdots\\
&= (x-N)(x-(N-1))\cdots(x-1)\left(a_1(1,x)+\frac{b_1(1,x)}{\log(1+t)}\right)\\
&\quad+\frac{a_N(N,x)+(x-N)a_{N-1}(N-1,x)+\cdots+(x-N)\cdots(x-2)a_1(1,x)}{\log(1+t)}\\
&= (x-N)(x-(N-1))\cdots(x-1)\left(x+\frac{1}{\log(1+t)}\right)\\
&\quad+\frac{a_N(N,x)+(x-N)a_{N-1}(N-1,x)+\cdots+(x-N)\cdots(x-2)a_1(1,x)}{\log(1+t)}\\
&= (x)_{N+1}+\frac{\sum_{k=0}^{N}(x|N)_{N-k}a_k(k,x)}{\log(1+t)}.
\end{aligned}$$

From (2.8), we can derive

$$\begin{aligned}
&a_1(N+1,x)+\frac{b_1(N+1,x)}{\log(1+t)}\\
&= x\left(a_0(N,x)+\frac{b_0(N,x)}{\log(1+t)}\right)+\frac{a_0(N,x)}{\log(1+t)}-N\left(a_1(N,x)+\frac{b_1(N,x)}{\log(1+t)}\right)\\
&= x\left(a_0(N,x)+\frac{b_0(N,x)}{\log(1+t)}-N\left(a_0(N-1,x)+\frac{b_0(N-1,x)}{\log(1+t)}\right)\right)\\
&\quad+\frac{a_0(N,x)-Na_0(N-1,x)}{\log(1+t)}+(-1)^2N(N-1)\left(a_1(N-1,x)+\frac{b_1(N-1,x)}{\log(1+t)}\right)\\
&\vdots\\
&= \sum_{i=0}^{N-1}(-1)^i(N)_i\left[x\left(a_0(N-i,x)+\frac{b_0(N-i,x)}{\log(1+t)}\right)+\frac{a_0(N-i,x)}{\log(1+t)}\right]\\
&\quad+(-1)^NN!\left(a_1(1,x)+\frac{b_1(1,x)}{\log(1+t)}\right)\\
&= \sum_{i=0}^{N}(-1)^i(N)_i\left[x\left(a_0(N-i,x)+\frac{b_0(N-i,x)}{\log(1+t)}\right)+\frac{a_0(N-i,x)}{\log(1+t)}\right],
\end{aligned} \tag{2.13}$$

$$
\begin{aligned}
&a_2(N+1,x)+\frac{b_2(N+1,x)}{\log(1+t)}\\
&= (x-1)\left(a_1(N,x)+\frac{b_1(N,x)}{\log(1+t)}\right)+\frac{a_1(N,x)}{\log(1+t)}+(1-N)\left(a_2(N,x)+\frac{b_2(N,x)}{\log(1+t)}\right)\\
&= (x-1)\left(a_1(N,x)+\frac{b_1(N,x)}{\log(1+t)}+(-1)(N-1)\left(a_1(N-1,x)+\frac{b_1(N-1,x)}{\log(1+t)}\right)\right)\\
&\quad+\frac{a_1(N,x)-(N-1)a_1(N-1,x)}{\log(1+t)}\\
&\quad+(-1)^2(N-1)(N-2)\left(a_2(N-1,x)+\frac{b_2(N-1,x)}{\log(1+t)}\right)\\
&\ \vdots\\
&= \sum_{i=0}^{N-2}(-1)^i(N-1)_i\left[(x-1)\left(a_1(N-i,x)+\frac{b_1(N-i,x)}{\log(1+t)}\right)+\frac{a_1(N-i,x)}{\log(1+t)}\right]\\
&\quad+(-1)^{N-1}(N-1)!\left(a_2(2,x)+\frac{b_2(2,x)}{\log(1+t)}\right)\\
&= \sum_{i=0}^{N-1}(-1)^i(N-1)_i\left[(x-1)\left(a_1(N-i,x)+\frac{b_1(N-i,x)}{\log(1+t)}\right)+\frac{a_1(N-i,x)}{\log(1+t)}\right],
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
&a_3(N+1,x)+\frac{b_3(N+1,x)}{\log(1+t)}\\
&= (x-2)\left(a_2(N,x)+\frac{b_2(N,x)}{\log(1+t)}\right)+\frac{a_2(N,x)}{\log(1+t)}+(2-N)\left(a_3(N,x)+\frac{b_3(N,x)}{\log(1+t)}\right)\\
&= (x-2)\left(a_2(N,x)+\frac{b_2(N,x)}{\log(1+t)}+(-1)(N-2)\left(a_2(N-1,x)+\frac{b_2(N-1,x)}{\log(1+t)}\right)\right)\\
&\quad+\frac{a_2(N,x)-(N-2)a_2(N-1,x)}{\log(1+t)}\\
&\quad+(-1)^2(N-2)(N-3)\left(a_3(N-1,x)+\frac{b_3(N-1,x)}{\log(1+t)}\right)\\
&\ \vdots\\
&= \sum_{i=0}^{N-3}(-1)^i(N-2)_i\left[(x-2)\left(a_2(N-i,x)+\frac{b_2(N-i,x)}{\log(1+t)}\right)+\frac{a_2(N-i,x)}{\log(1+t)}\right]\\
&\quad+(-1)^{N-2}(N-2)!\left(a_3(3,x)+\frac{b_3(3,x)}{\log(1+t)}\right)\\
&= \sum_{i=0}^{N-2}(-1)^i(N-2)_i\left[(x-2)\left(a_2(N-i,x)+\frac{b_2(N-i,x)}{\log(1+t)}\right)+\frac{a_2(N-i,x)}{\log(1+t)}\right].
\end{aligned}
\tag{2.15}
$$

Continuing in this fashion, we can find that

$$
\begin{aligned}
a_j(N+1,x)+\frac{b_j(N+1,x)}{\log(1+t)} = &\sum_{i=0}^{N-j+1}(-1)^i(N-j+1)_i\\
&\times\left[(x-j+1)\left(a_{j-1}(N-i,x)+\frac{b_{j-1}(N-i,x)}{\log(1+t)}\right)+\frac{a_{j-1}(N-i,x)}{\log(1+t)}\right],
\end{aligned}
\tag{2.16}
$$

where $i \leqslant j \leqslant N$.

We remark that the $(N+1)\times(N+1)$ matrix with the (i,j) entry given by $(a_i(j,x))_{0\leqslant i,j\leqslant N}$ is given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & \dots & N \end{array} \\
\begin{pmatrix}
1 & -1 & (-1)^2 2! & \dots & (-1)^N N! \\
0 & x & \dots & \dots & \vdots \\
0 & 0 & (x)_2 & \dots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & (x)_N
\end{pmatrix}
\end{array}.$$

Now, we give explicit expressions for $a_i(j,x)$. From (2.12), (2.13), (2.14), (2.15), and (2.16), we can derive

$$a_1(N+1,x) = x\sum_{i=0}^{N}(-1)^i(N)_i a_0(N-i,x) = x(-1)^N(N+1)!,$$

$$a_2(N+1,x) = (x-1)\sum_{i_1=0}^{N-1}(-1)^{i_1}(N-1)_{i_1}a_1(N-i_1,x) = (x)_2(-1)^{N-1}(N-1)!\sum_{i_1=0}^{N-1}(N-i_1),$$

$$a_3(N+1,x) = (x-2)\sum_{i_2=0}^{N-2}(-1)^{i_2}(N-2)_{i_2}a_2(N-i_2,x) = (x)_3(-1)^{N-2}(N-2)!\sum_{i_2=0}^{N-2}\sum_{i_1=0}^{N-2-i_2}(N-i_2-i_1-1),$$

and

$$\begin{aligned} a_4(N+1,x) &= (x-3)\sum_{i_3=0}^{N-3}(-1)^{i_3}(N-3)_{i_3}a_3(N-i_3,x) \\ &= (x)_4(-1)^{N-3}(N-3)!\sum_{i_3=0}^{N-3}\sum_{i_2=0}^{N-3-i_3}\sum_{i_1=0}^{N-3-i_3-i_2}(N-i_3-i_2-i_1-2). \end{aligned}$$

Continuing in this manner, we get

$$\begin{aligned} a_j(N+1,x) &= (x)_j(-1)^{N-j+1}(N-j+1)! \\ &\quad\times\sum_{i_{j-1}=0}^{N-j+1}\sum_{i_{j-2}=0}^{N-j+1-i_{j-1}}\cdots\sum_{i_1=0}^{N-j+1-i_{j-1}\cdots-i_2}(N-i_{j-1}-\cdots-i_1-j+2), \end{aligned} \tag{2.17}$$

where $1 \leqslant j \leqslant N+1$.

On the other hand, we now turn our attention to $b_i(j,x)$. From (2.12), (2.13), (2.14), (2.15), and (2.16), we can obtain

$$b_1(N+1,x) = \sum_{i=0}^{N-1}(-1)^i(N)_i\Big[xb_0(N-i,x)+a_0(N-i,x)\Big] = (-1)^N(N+1)!,$$

$$\begin{aligned} b_2(N+1,x) &= \sum_{i_1=0}^{N-1}(-1)^{i_1}(N-1)_{i_1}\Big[(x-1)b_1(N-i_1,x)+a_1(N-i_1,x)\Big] \\ &= (x-1)(-1)^{N-1}(N-1)!\sum_{i_1=0}^{N-1}(N-i_1)+x(-1)^{N-1}(N-1)!\sum_{i_1=0}^{N-1}(N-i_1) \end{aligned}$$

$$= (2x-1)(-1)^{N-1}(N-1)!\sum_{i_1=0}^{N-1}(N-i_1),$$

$$b_3(N+1,x) = \sum_{i_2=0}^{N-2}(-1)^{i_2}(N-2)_{i_2}\Big[(x-2)b_2(N-i_2,x)+a_2(N-i_2,x)\Big]$$
$$= [(2x-1)(x-2)+(x)_2](-1)^{N-2}(N-2)! \times \sum_{i_2=0}^{N-2}\sum_{i_1=0}^{N-2-i_2}(N-i_2-i_1-1),$$

and

$$b_4(N+1,x) = \sum_{i_3=0}^{N-3}(-1)^{i_3}(N-3)_{i_3}\Big[(x-3)b_3(N-i_3,x)+a_3(N-i_3,x)\Big]$$
$$= [(2x-1)(x-2)(x-3)+(x-3)(x_2)+(x)_3](-1)^{N-3}(N-3)!$$
$$\times \sum_{i_3=0}^{N-3}\sum_{i_2=0}^{N-3-i_3}\sum_{i_1=0}^{N-3-i_3-i_2}(N-i_3-i_2-i_1-2).$$

By continuing this process, we get

$$b_j(N+1,x) = \sum_{k=0}^{j-1}\frac{\prod_{i=0}^{j-1}(x-i)}{x-k}(-1)^{N-j+1}(N-j+1)!$$
$$\times \sum_{i_{j-1}=0}^{N-j+1}\sum_{i_{j-2}=0}^{N-j+1-i_{j-1}}\cdots\sum_{i_1=0}^{N-j+1-i_{j-1}\cdots-i_2}(N-i_{j-1}-\cdots-i_1-j+2), \tag{2.18}$$

where $1 \leqslant j \leqslant N+1$.

Therefore, by combining (2.17) and (2.18), we obtain the following theorem.

Theorem 2.1. *For $N = 0, 1, 2, \cdots$, let us consider the following family of differential equations:*

$$F^{(N)} = \left[\sum_{i=0}^{N}\left(a_i(N,x)+\frac{b_i(N,x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N}\right]F,$$

where

$$a_0(N,x) = (-1)^N N!, \quad b_0(N,x) = 0, \text{ for all } N,$$
$$a_j(N,x) = (x)_j(-1)^{N-j}(N-j)! \times \sum_{i_{j-1}=0}^{N-j}\sum_{i_{j-2}=0}^{N-j-i_{j-1}}\cdots\sum_{i_1=0}^{N-j-i_{j-1}\cdots-i_2}(N-i_{j-1}-\cdots-i_1-j+1),$$
$$b_j(N,x) = \sum_{k=0}^{j-1}\prod_{i=0}^{j-1}(x-i)(x-k)^{-1}(-1)^{N-j}(N-j)!$$
$$\times \sum_{i_{j-1}=0}^{N-j}\sum_{i_{j-2}=0}^{N-j-i_{j-1}}\cdots\sum_{i_1=0}^{N-j-i_{j-1}\cdots-i_2}(N-i_{j-1}-\cdots-i_1-j+1), \quad (1 \leqslant j \leqslant N+1).$$

Then the above family of differential equations has a solution

$$F = F(t,x) = \frac{\log(1+t)}{t}(1+t)^x.$$

From (2.1), we obtain

$$F^{(N)}=\sum_{k=N}^{\infty}D_k(x)(k)_N\frac{t^{k-N}}{k!}=\sum_{k=0}^{\infty}D_{N+k}(x)\frac{t^k}{k!}. \tag{2.19}$$

Recall from [7] that the Bernoulli numbers of the second kind $\mathfrak{b}_n$ for $n \geqslant 0$ are generated by

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}\mathfrak{b}_n\frac{t^n}{n!}. \tag{2.20}$$

The first few Bernoulli numbers $\mathfrak{b}_n$ of the second kind are

$$\mathfrak{b}_0=1,\quad \mathfrak{b}_1=\frac{1}{2},\quad \mathfrak{b}_2=-\frac{1}{12},\quad \mathfrak{b}_3=\frac{1}{24},\quad \mathfrak{b}_4=-\frac{19}{720},\quad \cdots.$$

Furthermore (2.20), we observe that

$$\frac{1}{\log(1+t)}=\sum_{n=0}^{\infty}\mathfrak{b}_n\frac{t^{n-1}}{n!}=\frac{\mathfrak{b}_0}{t}+\sum_{n=0}^{\infty}\mathfrak{b}_{n+1}\frac{t^n}{(n+1)!}=\frac{1}{t}+\sum_{n=0}^{\infty}\frac{\mathfrak{b}_{n+1}}{n+1}\frac{t^n}{n!}.$$

Therefore, by Theorem 2.1, we acquire that

$$\begin{aligned}
F^{(N)}&=\left[\sum_{i=0}^{N}\left(a_i(N,x)+\frac{b_i(N,x)}{\log(1+t)}\right)(t+1)^{-i}t^{i-N}\right]F\\
&=\sum_{i=0}^{N}\left(a_i(N,x)+b_i(N,x)\sum_{n=0}^{\infty}\frac{\mathfrak{b}_{n+1}}{n+1}\frac{t^n}{n!}+\frac{b_i(N,x)}{t}\right)\\
&\quad\times\left(\sum_{l=0}^{\infty}(-1)^l\binom{i+l-1}{l}t^l\right)t^{i-N}\sum_{p=0}^{\infty}D_p(x)\frac{t^p}{p!}\\
&=\sum_{i=0}^{N}a_i(N,x)\sum_{k=0}^{\infty}\sum_{l+p+i-N=k}k!(-1)^l\binom{i+l-1}{l}\frac{1}{p!}D_p(x)\frac{t^k}{k!}\\
&\quad+\sum_{i=0}^{N}b_i(N,x)\sum_{k=0}^{\infty}\sum_{l+s+i-N+p=k}k!(-1)^l\binom{i+l-1}{l}\frac{\mathfrak{b}_{s+1}}{p!(s+1)!}D_p(x)\frac{t^k}{k!}\\
&\quad+\sum_{i=0}^{N}b_i(N,x)\sum_{k=0}^{\infty}\sum_{l+p+i-N-1=k}k!(-1)^l\binom{i+l-1}{l}\frac{1}{p!}D_p(x)\frac{t^k}{k!}\\
&=\sum_{k=0}^{\infty}\left[k!\sum_{i=0}^{N}a_i(N,x)\sum_{l+p+i-N=k}(-1)^l\binom{i+l-1}{l}\frac{1}{p!}D_p(x)\right]\frac{t^k}{k!}\\
&\quad+\sum_{k=0}^{\infty}\left[k!\sum_{i=0}^{N}b_i(N,x)\sum_{l+s+i-N+p=k}(-1)^l\binom{i+l-1}{l}\frac{1}{p!(s+1)!}\mathfrak{b}_{s+1}D_p(x)\right]\frac{t^k}{k!}\\
&\quad+\sum_{k=0}^{\infty}\left[k!\sum_{i=0}^{N}b_i(N,x)\sum_{l+p+i-N-1=k}(-1)^l\binom{i+l-1}{l}\frac{1}{p!}D_p(x)\right]\frac{t^k}{k!}.
\end{aligned} \tag{2.21}$$

By equating coefficients of (2.19) and (2.21), we finally arrive at the following theorem.

Theorem 2.2. *For* $N = 0, 1, 2, \cdots$ *and* $k = 0, 1, 2, \cdots$, *we have*

$$\begin{aligned} D_{k+N}(x) &= k! \sum_{i=0}^{N} a_i(N, x) \sum_{l+p+i-N=k} (-1)^l \binom{i+l-1}{l} \frac{1}{p!} D_p(x) \\ &+ k! \sum_{i=0}^{N} b_i(N, x) \sum_{l+s+i-N+p=k} (-1)^l \binom{i+l-1}{l} \frac{1}{p!(s+1)!} \mathfrak{b}_{s+1} D_p(x) \\ &+ k! \sum_{i=0}^{N} b_i(N, x) \sum_{l+p+i-N-1=k} (-1)^l \binom{i+l-1}{l} \frac{1}{p!} D_p(x), \end{aligned}$$

where $a_i(N, x)'s$ *and* $b_i(N, x)'s$ *are as in Theorem* 2.1.

3. Daehee numbers associated with differential equations

Now we consider

$$G = G(t, x) = \log(1+t)(1+t)^x. \tag{3.1}$$

Then, by (3.1), we have

$$G^{(1)} = \frac{d}{dx} G(t, x) = (1+t)^{x-1} + x \log(1+t)(1+t)^{x-1} = (1+t)^{x-1} + xG(t, x-1)$$

and

$$G^{(2)} = \frac{d}{dx} G^{(1)} = (x-1)(1+t)^{x-2} + xG'(t, x-1) = (x-1)(1+t)^{x-2} + x(1+t)^{x-2} + x(x-1)G(t, x-2).$$

Continuing this differentiation N-times, we derive

$$\begin{aligned} G^{(N)} = \left(\frac{d}{dx}\right)^{(N)} G(t, x) &= \sum_{i=0}^{N-1} \left(\prod_{\substack{j=0 \\ j \neq i}}^{N-1} (x-j) \right) (1+t)^{x-N} + (x)_N G(t, x-N) \\ &= \sum_{i=0}^{N-1} \left(\prod_{\substack{j=0 \\ j \neq i}}^{N-1} (x-j) \right) (1+t)^{x-N} + (x)_N \log(1+t)(1+t)^{x-N}. \end{aligned} \tag{3.2}$$

On the other hand, the left hand side of (3.2) has the presentation as follows:

$$\begin{aligned} G^{(N)} &= \left(\frac{d}{dx}\right)^{(N)} (\log(1+t)(1+t)^x) \\ &= \left(\frac{d}{dx}\right)^{(N)} \left(\frac{\log(1+t)}{t} (1+t)^x t \right) \\ &= \left(\frac{d}{dx}\right)^{(N)} \left(\sum_{m=0}^{\infty} D_m(x) \frac{t^{m+1}}{m!} \right) \\ &= \left(\frac{d}{dx}\right)^{(N)} \left(\sum_{m=0}^{\infty} m D_{m-1}(x) \frac{t^m}{m!} \right) \\ &= \sum_{m=0}^{\infty} (m+N) D_{m+N-1}(x) \frac{t^m}{m!}. \end{aligned} \tag{3.3}$$

For the right hand side of (3.3), we observe that

$$
\begin{aligned}
(x)_N G(t, x-N) = (x)_N \log(1+t)(1+t)^{x-N} t &= (x)_N \sum_{m=0}^{\infty} D_m(x-N) \frac{t^{m+1}}{m!} \\
&= \sum_{m=0}^{\infty} (x)_N m D_{m-1}(x-N) \frac{t^m}{m!}.
\end{aligned}
\tag{3.4}
$$

In addition, we note that

$$
\begin{aligned}
(1+t)^{x-N} = \frac{t}{\log(1+t)} \frac{\log(1+t)}{t} (1+t)^{x-N} &= \left(\sum_{n=0}^{\infty} \mathfrak{b}_n \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} D_l(x-N) \frac{t^l}{l!} \right) \\
&= \sum_{m=0}^{\infty} \left[\sum_{n=0}^{m} \binom{m}{n} \mathfrak{b}_n D_l(x-N) \right] \frac{t^m}{m!}.
\end{aligned}
\tag{3.5}
$$

Also we can expand $(1+t)^{x-N}$ as follows:

$$
(1+t)^{x-N} = \sum_{m=0}^{\infty} (x-N)_m \frac{t^m}{m!}. \tag{3.6}
$$

Now by (3.2), (3.3), (3.4), and (3.5), we have the following results

$$
(m+N) D_{m+N-1}(x) = \sum_{i=0}^{N-1} \left(\prod_{\substack{j=0 \\ j \neq i}}^{N-1} (x-j) \right) \sum_{n=0}^{m} \binom{m}{n} \mathfrak{b}_n D_l(x-N) \tag{3.7}
$$

and

$$
(m+N) D_{m+N-1}(x) = \sum_{i=0}^{N-1} \left(\prod_{\substack{j=0 \\ j \neq i}}^{N-1} (x-j) \right) (x-N)_m + (x)_N m D_{m-1}(x-N). \tag{3.8}
$$

Therefore, we obtain the following theorem.

Theorem 3.1. *For* $N \in \mathbb{N}$ *and* $m \geqslant 0$, *we have*

$$
\begin{aligned}
(m+N) D_{m+N-1}(x) &= \sum_{i=0}^{N-1} \left(\prod_{\substack{j=0 \\ j \neq i}}^{N-1} (x-j) \right) \sum_{n=0}^{m} \binom{m}{n} \mathfrak{b}_n D_l(x-N) \\
&= \sum_{i=0}^{N-1} \left(\prod_{\substack{j=0 \\ j \neq i}}^{N-1} (x-j) \right) (x-N)_m + (x)_N m D_{m-1}(x-N).
\end{aligned}
$$

When $x = 0$ in (3.7), we have the following identity on Daehee numbers.

Corollary 3.2. *For* $N \in \mathbb{N}$ *and* $m \geqslant 0$, *we have*

$$
D_{m+N-1} = \frac{(-1)^{N-1}(N-1)!}{m+N} \sum_{n=0}^{m} \binom{m}{n} \mathfrak{b}_n D_l(-N).
$$

The above corollary can be compared to Theorem 2.1 of [10], which presents D_{m+N-1} via higher order Daehee numbers, we record the result here.

Theorem 3.3 ([10, Theorem 2.1]). *For* $N \in \mathbb{N}$ *and* $m \geqslant 0$, *we have*

$$D_{m+N-1} = \frac{(-1)^{N-1}(N-1)!}{m+N} \sum_{n=0}^{m} \binom{m}{n} (-1)^n N^n D_{m-n}^{(n)}.$$

When $x = 0$ in (3.8), we get the same identity on Daehee numbers, which appears in [10].

Corollary 3.4 ([10, Theorem 1]). *For* $N \in \mathbb{N}$ *and* $m \geqslant 0$, *we have*

$$D_{m+N-1} = \frac{(-1)^{N-1}(N-1)!}{m+N} \sum_{n=0}^{m} (-1)^n N^n S_1(m,n).$$

From (3.5), we replace t by $e^t - 1$, we get

$$\begin{aligned}(e^t)^{x-N} = \frac{e^t-1}{t}\frac{t}{e^t-1}(e^t)^{x-N} &= \left(\sum_{n=0}^{\infty} \mathfrak{b}_n \frac{(e^t-1)^n}{n!}\right)\left(\sum_{k=0}^{\infty} B_k(x-N)\frac{t^k}{k!}\right) \\ &= \left(\sum_{n=0}^{\infty} \mathfrak{b}_n \sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty} B_k(x-N)\frac{t^k}{k!}\right) \\ &= \left(\sum_{m=0}^{\infty}\sum_{n=0}^{m} \mathfrak{b}_n S_2(m,n)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty} B_k(x-N)\frac{t^k}{k!}\right) \\ &= \sum_{m=0}^{\infty}\left[\sum_{l=0}^{m}\binom{m}{l}\sum_{n=0}^{l} \mathfrak{b}_n S_2(l,n) B_{m-l}(x-N)\right]\frac{t^m}{m!}.\end{aligned} \tag{3.9}$$

On the other hand,

$$(e^t)^{x-N} = \sum_{m=0}^{\infty} (x-N)^m \frac{t^m}{m!}. \tag{3.10}$$

Comparing the coefficients of (3.9) and (3.10), we have

$$(x-N)^m = \sum_{l=0}^{m}\binom{m}{l}\sum_{n=0}^{l} \mathfrak{b}_n S_2(l,n) B_{m-l}(x-N). \tag{3.11}$$

Thus if we take $x = 0$ in (3.11), we can obtain

$$(-N)^m = \sum_{l=0}^{m}\sum_{n=0}^{l}\binom{m}{l} \mathfrak{b}_n S_2(l,n) B_{m-l}(-N). \tag{3.12}$$

By (1.4), we observe

$$B_{m-l}(-N) = \sum_{k=0}^{m-l} D_k(-N) S_2(m-l,k). \tag{3.13}$$

From (3.12) and (3.13), we know that

$$(-N)^m = \sum_{l=0}^{m}\sum_{n=0}^{l}\sum_{k=0}^{m-l}\binom{m}{l} \mathfrak{b}_n S_2(l,n) D_k(-N) S_2(m-l,k).$$

Theorem 3.5. *For* $N \in \mathbb{N}$ *and* $m \geqslant 0$, *we have*

$$N^m = \sum_{l=0}^{m}\sum_{n=0}^{l}\sum_{k=0}^{m-l}\binom{m}{l} (-1)^m \mathfrak{b}_n S_2(l,n) D_k(-N) S_2(m-l,k).$$

From (3.6), we replace t by $e^t - 1$, we have

$$\begin{aligned} (e^t)^{x-N} = \sum_{m=0}^{\infty} (x-N)_m \frac{(e^t-1)^m}{m!} &= \sum_{m=0}^{\infty} (x-N)_m \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} (x-N)_m S_2(n,m) \right] \frac{t^n}{n!}. \end{aligned} \tag{3.14}$$

Thus we have

$$(x-N)^n = \sum_{m=0}^{n} (x-N)_m S_2(n,m).$$

If we take $x = 0$ in (3.14), we have the following result, which is the same that of (2.16) in [10]

$$e^{-Nt} = \sum_{n=0}^{\infty} (-1)^n N^n \frac{t^n}{n!}.$$

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