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A novel approach for obtaining new identities for the λ extension of q-Euler polynomials arising from the q-umbral calculus

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Abstract

In this article, a new q-generalization of the Apostol-Euler polynomials is introduced using the usual q-exponential function. We make use of such a generalization to derive several properties arising from the q-umbral calculus. ©2017 All rights reserved.

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1. Introduction

Let $q \in (0, 1)$. In the theory of q-calculus, a q-number $[x]_q$ is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Similarly, for $z \in \mathbb{C}$ with $|z| < 1$, the q-exponential function $e_q(z)$ is defined by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}.$$

In this paper, we use the following notations:

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad \text{and} \quad \binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

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The q-integral of a function f is defined by

$$\int_0^x f(\xi) d_q \xi = x(1-q) \sum_{\xi=0}^{\infty} f(q^\xi x) q^\xi.$$

For more on this and related issues, see, e.g., [4, 10–13, 15].

The q-derivative D_q is defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}, \qquad \text{and} \qquad \lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}.$$

For a systematic study of q-derivatives, we refer the reader to [4, 10, 12, 13, 15] and [11].

Recently, Kim [11] introduced q-Euler polynomials by means of the following generating function:

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(t) + 1} e_q(xt),$$

with $E_{n,q}(0) = E_{n,q}$ (called the n-th q-Euler numbers).

Let us now consider λ extension of q-Euler polynomials (or can be called (λ, q)-Euler polynomials) given by

$$\sum_{n=0}^{\infty} E_{n,q}(x \mid \lambda) \frac{t^n}{[n]_q!} = \frac{[2]_q}{\lambda e_q(t) + 1} e_q(xt), \tag{1.1}$$

where $\lambda \in \mathbb{R}^+$.

Letting $q \to 1$ in (1.1), one obtains the Apostol-Euler polynomial. Consequently, the polynomials defined by (1.1) are a new q-generalization of Euler polynomial of Apostol type. For more information on the Apostol-type polynomials, we refer the reader to recent works such as [6, 7, 16, 18] and [28].

Recent investigations on this topic include an elementary and real approach to values of the Riemann zeta function [5], Apostol-Euler polynomials arising from the umbral calculus [16], a new generalization of q-Bernoulli polynomial [17], a new q-generalization of Euler numbers and polynomials using the method of Kupershmidt [11], additional theorems for the Appell polynomials and associated classes of polynomial expansions [21], theorems on Apostol-Euler polynomials arising from Euler basis [27], a new class of q-Euler and q-Bernoulli polynomials [19, 20], a modified q-Euler numbers of higher order with weight [22], the zeta and the q-zeta functions and associated series and integrals [29].

Let $\mathcal{F}$ be the space of all formal power series in variable t over the complex number field $\mathbb{C}$, namely

$$\mathcal{F} = \left\{ f : f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{c_k}, \quad (a_k \in \mathbb{C}) \right\},$$

where c_k is an admissible sequence, i.e., $c_n \neq 0$ for all $n \geqslant 0$, and $\mathbb{P} = \mathbb{C}[t]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. We also denote by $\langle L | p(x) \rangle$ the action of a linear functional L on the polynomial $p(x)$, which obviously satisfies the following properties:

$$\langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle,$$

and

$$\langle \beta L | p(x) \rangle = \beta \langle L | p(x) \rangle,$$

where β is a complex constant (see, for details, [1, 9–16, 26] and [25]).

Roman [23, 26] defined linear functionals and operators as follows:

$$\langle t^k | x^n \rangle = c_n \delta_{n,k}, \qquad (n, k \geqq 0), \tag{1.2}$$

$$\langle f(t)|x^n \rangle = a_n, \qquad (n \geqq 0),$$ (1.3)

and

$$t^k x^n = \begin{cases} \dfrac{c_n}{c_{n-k}} x^{n-k} & k \leqslant n, \\ 0 & n < k. \end{cases}$$

Roman also defined the following equivalent conditions in which $s_n(x)$ is known as Sheffer for $(g(t), t)$, where $g(t)$ is an invertible formal power series:

(S1) $\langle g(t) t^k \mid s_n(x) \rangle = c_n \delta_{n,k}$;

(S2) $t s_n(x) = \dfrac{c_n}{c_{n-1}} s_{n-1}(x)$;

(S3) $\sum_{k=0}^{\infty} s_k(x) \dfrac{t^k}{c_k} = \dfrac{\xi(xt)}{g(t)}$, where $\xi(t) = \sum_{k=0}^{\infty} \dfrac{t^k}{c_k}$.

Recently, Kim and Kim [12, 13] considered q-umbral calculus and derived some new interesting identities for q-Bernoulli and q-Euler polynomials. The following tools about q-umbral calculus are taken from Kim and Kim's works [12, 13].

Let

$$f_L(t) = \sum_{k=0}^{\infty} \langle L|x^k \rangle \frac{t^k}{[k]_q!}.$$

Then, (1.2) gives us $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$, that is, $f_L(t) = L$. Moreover, the map $L \longmapsto f_L(t)$ is an isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$. Thus an element $f(t)$ of $\mathcal{F}$ will be seen as a formal power series and a linear functional. Kim and Kim [12, 13] called it q-umbral algebra which is the study of the q-umbral calculus. From (1.3), we notice that

$$\langle e_q(yt)|x^n \rangle = y^n,$$

and so

$$\langle e_q(yt)|p(x) \rangle = p(y), \qquad (p(x) \in \mathbb{P}).$$

The order $o(f(t))$ of the power series $f(t) \neq 0$ is the smallest integer for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series. If $o(f(t)) = 1$, then $f(t)$ is called a delta series (see [1–4, 8–16, 23–26]). For $f(t), g(t) \in \mathcal{F}$, we have

$$\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle = \langle g(t)|f(t)p(x) \rangle.$$

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have (see, e.g., [10–15, 26] and [25]).

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k \rangle \frac{t^k}{[k]_q!}, \qquad \text{and} \qquad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x) \rangle \frac{x^k}{[k]_q!}.$$ (1.4)

Using (1.4), we obtain

$$p^{(k)}(x) = D_q^k p(x) = \sum_{l=k}^{\infty} \frac{\langle t^l|p(x) \rangle}{[l]_q!} x^{l-k} \prod_{s=1}^{k} [l-s+1]_q,$$

which in turn yields

$$p^{(k)}(0) = \langle t^k|p(x) \rangle, \qquad \text{and} \qquad \langle 1|p^{(k)}(x) \rangle = p^{(k)}(0).$$ (1.5)

Thus from (1.5), we note that

$$t^k p(x) = p^{(k)}(x) = D_q^k p(x).$$

Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ (deg $s_n(x) = n$) of polynomials such that

$$\langle g(t)f(t)^k | s_n(x) \rangle = [n]_q! \delta_{n,k}, \qquad (n, k \geqq 0).$$

The sequence $s_n(x)$ is called the q-Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$. Let $s_n(x) \sim (g(t), f(t))$. For $h(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | s_k(x) \rangle}{[k]_q!} g(t)f(t)^k, \qquad \text{and} \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k | p(x) \rangle}{[k]_q!} s_k(x).$$

The sequence $s_n(x)$ is Appell for $g(t)$ that

$$s_n(x) = \frac{1}{g(t)} x^n, \quad \text{if and only if,} \quad t s_n(x) = [n]_q s_{n-1}(x), \quad \text{(see, e.g., [25]).} \tag{1.6}$$

Furthermore, for all $y \in \mathbb{C}$, we get

$$\frac{1}{g(\bar{f}(t))} e_q(y\bar{f}(t)) = \sum_{k=0}^{\infty} \frac{s_k(y)}{[k]_q!} t^k,$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$. For further information about q-umbral calculus, we refer the reader to see Kim and Kim's works [12, 13].

Kim et al. (see [11–13, 15]) derived some interesting properties of the new family of q-Euler numbers and polynomials from the viewpoint of the theory of the q-calculus. On the other hand, by using the orthogonality type as defined in the umbral calculus, Kim et al. [16] derived explicit formulas for several well-known polynomials as a linear combination of the Apostol-Euler polynomials.

The main objective of this paper consists of introducing (λ, q)-Euler polynomials that seem to be a new q-generalization of Apostol-Euler polynomials. We derive several interesting properties and identities arising from the q-umbral calculus. These numbers and polynomials have interesting properties in the areas of both number theory and mathematical physics.

2. The (λ, q)-Euler Numbers and the (λ, q)-Euler Polynomials

Let us introduce the (λ, q)-extension of the Euler polynomials by means of the following generating function:

$$\frac{[2]_q}{\lambda e_q(t) + 1} e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}(x \mid \lambda) \frac{t^n}{[n]_q!}. \tag{2.1}$$

Clearly, if $x = 0$, then $E_{n,q}(0 \mid \lambda) = E_{n,q}(\lambda)$, which are known as the nth (λ, q)-Euler numbers. Using (2.1), we obtain the following,

$$E_{n,q}(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l}_q E_{l,q}(\lambda) x^{n-l}. \tag{2.2}$$

Using (2.2), the (λ, q)-Euler numbers can be found by means of the following identities:

$$E_{0,q}(\lambda) = \frac{2}{\lambda + 1}, \qquad \text{and} \qquad \lambda E_{n,q}(1 \mid \lambda) + E_{n,q}(\lambda) = [2]_q \delta_{0,n}.$$

We immediately derive the following consequences based on (S1)-(S2) for q-polynomial with a parameter λ defined in (2.1),

$$E_{n,q}(x \mid \lambda) \sim \left(\frac{\lambda e_q(t) + 1}{[2]_q}, t \right),$$

$$tE_{n,q}(x \mid \lambda) = [n]_q E_{n-1,q}(x \mid \lambda).$$

(2.3)

It follows from (2.3) that $E_{n,q}(x \mid \lambda)$ is Appell for $\frac{\lambda e_q(t)+1}{[2]_q}$. So, by the (1.6), we have

$$\frac{[2]_q}{\lambda e_q(t)+1} x^n = E_{n,q}(x \mid \lambda) \quad (n \geqq 0).$$

Using (2.2), we obtain

$$\int_x^{x+y} E_{n,q}(u \mid \lambda) d_q u = \sum_{l=0}^n \binom{n}{l}_q E_{n-l,q}(\lambda) \frac{1}{[l+1]_q} \left\{ (x+y)^{l+1} - x^{l+1} \right\}$$

$$= \frac{1}{[n+1]_q} \sum_{l=0}^n \binom{n+1}{l+1}_q E_{n-l,q}(\lambda) \left\{ (x+y)^{l+1} - x^{l+1} \right\}$$

(2.4)

$$= \frac{1}{[n+1]_q} \left(E_{n+1,q}(x+y \mid \lambda) - E_{n+1,q}(x \mid \lambda) \right).$$

Thus, by applying (2.4), we get

$$\left\langle \frac{e_q(t)-1}{t} \mid E_{n,q}(x \mid \lambda) \right\rangle = \frac{1}{[n+1]_q} \left\langle \frac{e_q(t)-1}{t} \mid tE_{n+1,q}(x \mid \lambda) \right\rangle$$

$$= \frac{1}{[n+1]_q} \left\{ E_{n+1,q}(1 \mid \lambda) - E_{n+1,q}(\lambda) \right\}$$

(2.5)

$$= \int_0^1 E_{n,q}(u \mid \lambda) d_q u.$$

Therefore, by (2.5), we obtain the following result.

Theorem 2.1. *Let* $n \geqq 0$. *Then*

$$\left\langle \frac{e_q(t)-1}{t} \mid E_{n,q}(x \mid \lambda) \right\rangle = \int_0^1 E_{n,q}(u \mid \lambda) d_q u.$$

Now also, by using (1.2), we have

$$\left\langle \frac{\lambda e_q(t)+1}{[2]_q} t^k \mid E_{n,q}(x \mid \lambda) \right\rangle = \frac{[k]_q!}{[2]_q} \binom{n}{k}_q \left\langle \lambda e_q(t)+1 \mid E_{n-k,q}(x \mid \lambda) \right\rangle$$

$$= [n]_q! \delta_{k,n}.$$

From the last identity, we have

$$\left\langle \left(\frac{\lambda e_q(t)+1}{[2]_q} \right) t^k \mid E_{n,q}(x \mid \lambda) \right\rangle = [n]_q! \delta_{n,k}.$$

(2.6)

Let

$$\mathbb{P}_n = \{ q(x) \in \mathbb{C}[x] \mid \deg q(x) \leqslant n \}.$$

Also, for $q(x) \in \mathbb{P}_n$, we assume that

$$q(x) = \sum_{k=0}^n b_{k,q} E_{k,q}(x \mid \lambda).$$

(2.7)

It follows from (2.6) and (2.7) that

$$\left\langle \frac{\lambda e_q(t)+1}{[2]_q} t^k \mid q(x) \right\rangle = \sum_{l=0}^{n} b_{l,q} \left\langle \frac{\lambda e_q(t)+1}{[2]_q} t^k \mid E_{l,q}(x \mid \lambda) \right\rangle$$

$$= \sum_{l=0}^{n} b_{l,q} [l]_q! \delta_{l,k} = [k]_q! b_{k,q}.$$

Furthermore

$$b_{k,q} = \frac{1}{[k]_q!} \left\langle \frac{\lambda e_q(t)+1}{[2]_q} t^k \mid q(x) \right\rangle = \frac{1}{[2]_q [k]_q!} \left\{ \lambda q^{(k)}(1) + q^{(k)}(0) \right\}, \tag{2.8}$$

where

$$q^{(k)}(x) = D_q^k q(x).$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.2. *For* $q(x) \in \mathbb{P}_n$, *let*

$$q(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x \mid \lambda).$$

Then

$$b_{k,q} = \frac{1}{[2]_q [k]_q!} \left\{ \lambda q^{(k)}(1) + q^{(k)}(0) \right\}.$$

The q-Bernoulli polynomials $B_{n,q}(x)$ are defined by

$$\frac{t}{e_q(t)-1} e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!}.$$

From this, we have for $n \geqq 0$,

$$B_{n,q}(x) \sim \left(\frac{e_q(t)-1}{t}, t \right)$$

$$t B_{n,q}(x) = [n]_q B_{n-1,q}(x), \quad \text{(see [12] for details)}.$$

Let us take

$$q(x) = B_{n,q}(x) \in \mathbb{P}_n.$$

Then $B_{n,q}(x)$ can be generated as a linear combination of

$$\{E_{0,q}(x \mid \lambda), E_{1,q}(x \mid \lambda), \cdots, E_{n,q}(x \mid \lambda)\},$$

as follows:

$$B_{n,q}(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x \mid \lambda), \tag{2.9}$$

where

$$b_{k,q} = \frac{1}{[2]_q [k]_q!} \left\langle (\lambda e_q(t)+1) t^k \mid B_{n,q}(x) \right\rangle$$

$$= \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[2]_q [k]_q!} \left\langle \lambda e_q(t)+1 \mid B_{n-k,q}(x) \right\rangle$$

$$= \frac{1}{[2]_q} \binom{n}{k}_q \langle \lambda e_q(t) + 1 \mid B_{n-k,q}(x) \rangle$$

$$= \frac{1}{[2]_q} \binom{n}{k}_q \left(\lambda B_{n-k,q}(1) + B_{n-k,q} \right),$$

where $B_{n,q} := B_{n,q}(0)$ are called q-Bernoulli numbers, e.g., see [12]. Kim et al. [13] derived the following identity:

$$B_{0,q} = 1, \quad \text{and} \quad B_{n,q}(1) - B_{n,q} = \begin{cases} 1 & (n = 1), \\ 0 & (n > 1). \end{cases} \tag{2.10}$$

Since $B_{1,q} = -\frac{1}{[2]_q}$, by (2.9) and (2.10), we have

$$B_{n,q}(x) = b_{n,q} E_{n,q}(x \mid \lambda) + b_{n-1} E_{n-1,q}(x \mid \lambda) + \sum_{k=0}^{n-2} b_{k,q} E_{k,q}(x \mid \lambda)$$

$$= \frac{\lambda+1}{[2]_q} E_{n,q}(x \mid \lambda) + \frac{[n]_q}{[2]_q^2} (\lambda q - 1) E_{n-1,q}(x \mid \lambda) + \frac{\lambda+1}{[2]_q} \sum_{k=0}^{n-2} \binom{n}{k}_q B_{n-k,q} E_{k,q}(x \mid \lambda).$$

The previous facts can be formulated as follows:

Theorem 2.3. *Let* $n \geqq 2$. *Then*

$$B_{n,q}(x) = \frac{\lambda+1}{[2]_q} E_{n,q}(x \mid \lambda) + \frac{[n]_q}{[2]_q^2} (\lambda q - 1) E_{n-1,q}(x \mid \lambda)$$

$$+ \frac{\lambda+1}{[2]_q} \sum_{k=0}^{n-2} \binom{n}{k}_q B_{n-k,q} E_{k,q}(x \mid \lambda).$$

For $r \in \mathbb{Z}_{\geq 0}$, the higher-order (λ, q)-Euler polynomials $E_{n,q}^{(r)}(x \mid \lambda)$ are defined by the following q-Taylor expansion at $t = 0$:

$$\left(\frac{[2]_q}{\lambda e_q(t) + 1} \right)^r e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x \mid \lambda) \frac{t^n}{[n]_q!}. \tag{2.11}$$

In the case, $x = 0$, $E_{n,q}^{(r)}(0 \mid \lambda) = E_{n,q}^{(r)}(\lambda)$ are called the rth higher-order (λ, q)-Euler numbers.

Let

$$g^r(t \mid \lambda) = \left(\frac{\lambda e_q(t) + 1}{[2]_q} \right)^r.$$

It is clear that $g^r(t \mid \lambda)$ is an invertible series. It follows from (2.11) that $E_{n,q}(x \mid \lambda)$ is Appell for $\left(\frac{\lambda e_q(t)+1}{[2]_q} \right)^r$. So, by (1.6), we have

$$E_{n,q}^{(r)}(x \mid \lambda) = \frac{1}{g^r(t \mid \lambda)} x^n,$$

and

$$t E_{n,q}^{(r)}(x \mid \lambda) = [n]_q E_{n-1,q}^{(r)}(x \mid \lambda).$$

Thus, we have

$$E_{n,q}^{(r)}(x \mid \lambda) \sim \left(\left(\frac{\lambda e_q(t) + 1}{[2]_q} \right)^r, t \right).$$

By (1.2) and (2.11), we get

$$\left\langle \frac{[2]_q^r}{(\lambda e_q(t)+1)^r} e_q(yt)|x^n \right\rangle = E_{n,q}^{(r)}(y\mid\lambda) = \sum_{l=0}^{n} \binom{n}{l}_q E_{n-l,q}^{(r)}(\lambda)\, y^l. \tag{2.12}$$

We thus find that

$$\left\langle \left(\frac{[2]_q}{\lambda e_q(t)+1}\right)^r \mid x^n \right\rangle = \left\langle \frac{[2]_q}{\lambda e_q(t)+1} \cdots \frac{[2]_q}{\lambda e_q(t)+1} \mid x^n \right\rangle$$

$$= \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1,\cdots,i_r}_q E_{i_1,q}(\lambda)\cdots E_{i_r,q}(\lambda), \tag{2.13}$$

where

$$\binom{n}{i_1,\cdots,i_r}_q = \frac{[n]_q!}{[i_1]_q!\cdots[i_r]_q!}.$$

By using (2.12), we have

$$\left\langle \left(\frac{[2]_q}{\lambda e_q(t)+1}\right)^r |x^n\rangle = E_{n,q}^{(r)}(\lambda). \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.4. *Let* $n \geqq 0$*. Then*

$$E_{n,q}^{(r)}(\lambda) = \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1,\cdots,i_r}_q E_{i_1,q}(\lambda)\cdots E_{i_r,q}(\lambda).$$

Let us take

$$q(x) = E_{n,q}^{(r)}(x\mid\lambda) \in \mathbb{P}_n.$$

Then, by Theorem 2.2, we write

$$E_{n,q}^{(r)}(x\mid\lambda) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x\mid\lambda), \tag{2.15}$$

where the coefficient $b_{k,q}$ is given by

$$b_{k,q} = \frac{1}{[2]_q\,[k]_q!}\left\langle(\lambda e_q(t)+1)\,t^k\mid q(x)\right\rangle = \frac{1}{[2]_q\,[k]_q!}\left\langle(\lambda e_q(t)+1)\,t^k\mid q(x)\right\rangle$$

$$= \frac{\binom{n}{k}_q}{[2]_q}\left\langle(\lambda e_q(t)+1)\mid E_{n-k,q}^{(r)}(x\mid\lambda)\right\rangle = \frac{\binom{n}{k}_q}{[2]_q}\left(\lambda E_{n-k,q}^{(r)}(1\mid\lambda)+E_{n-k,q}^{(r)}(\lambda)\right).$$

From (2.11), we have

$$\sum_{k=0}^{\infty}\left(\lambda E_{n,q}^{(r)}(1\mid\lambda)+E_{n,q}^{(r)}(\lambda)\right)\frac{t^n}{[n]_q!} = \left(\frac{[2]_q}{\lambda e_q(t)+1}\right)^r(\lambda e_q(t)+1)$$

$$= [2]_q\left(\frac{[2]_q}{\lambda e_q(t)+1}\right)^{r-1}$$

$$= \sum_{n=0}^{\infty}[2]_q\, E_{n,q}^{(r-1)}(\lambda)\frac{t^n}{[n]_q!}.$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we get

$$\lambda E_{n,q}^{(r)}(1\mid\lambda)+E_{n,q}^{(r)}(\lambda) = [2]_q\, E_{n,q}^{(r-1)}(\lambda). \tag{2.16}$$

Thus, clearly, the following theorem is derivable by applying (2.15) and (2.16).

Theorem 2.5. *Let* $n \in \mathbb{Z}_{\geq 0}$ *and* $r \in \mathbb{Z}_{>0}$*. Then*

$$E_{n,q}^{(r)}(x \mid \lambda) = \sum_{k=0}^{n} \binom{n}{k}_q E_{n-k,q}^{(r-1)}(\lambda) E_{k,q}(x \mid \lambda).$$

Let us assume that

$$q(x) = \sum_{k=0}^{n} b_{k,q}^r E_{k,q}^{(r)}(x \mid \lambda) \in \mathbb{P}_n.$$

By a similar method, we find the coefficient $b_{k,q}^r$ as follows:

$$b_{k,q}^r = \frac{1}{[2]_q^r [k]_q!} \sum_{l=0}^{r} \binom{r}{l} \lambda^l \sum_{m \geq 0} \sum_{i_1 + \cdots + i_l = m} \binom{m}{i_1, \cdots, i_l}_q \frac{1}{[m]_q!} q^{(m+k)}(0).$$

Therefore, we obtain the following theorem.

Theorem 2.6. *For* $n \geq 0$*, let*

$$q(x) = \sum_{k=0}^{n} b_{k,q}^r E_{k,q}^{(r)}(x \mid \lambda) \in P_n.$$

Then

$$b_{k,q}^r = \frac{1}{[2]_q^r [k]_q!} \left\langle (\lambda e_q(t) + 1)^r t^k \mid q(x) \right\rangle$$

$$= \frac{1}{[2]_q^r [k]_q!} \sum_{m \geq 0} \sum_{l=0}^{r} \binom{r}{l} \lambda^l \sum_{i_1 + \cdots + i_l = m} \binom{m}{i_1, \cdots, i_l}_q \frac{1}{[m]_q!} q^{(m+k)}(0),$$

where

$$q^{(k)}(x) = D_q^k q(x).$$

Let us consider $q(x) = E_{n,q}(x \mid \lambda) \in \mathbb{P}_n$. Then, by Theorem 2.6, we write

$$E_{n,q}(x \mid \lambda) = \sum_{k=0}^{n} b_{k,q}^r E_{k,q}^{(r)}(x \mid \lambda). \tag{2.17}$$

By applying Theorem 2.6 and (2.17), we are led to the following result.

Theorem 2.7. *For* $n, r \geq 0$*, the following assertion holds true*:

$$E_{n,q}(x \mid \lambda) = \frac{1}{[2]_q^r} \sum_{k=0}^{n} \left(\sum_{m=0}^{n-k} \sum_{l=0}^{r} \sum_{i_1 + \cdots + i_l = m} \lambda^l \binom{r}{l} \binom{m}{i_1, \cdots, i_l}_q \binom{m+k}{m}_q \binom{n}{m+k}_q \right.$$
$$\left. \cdot E_{n-m-k,q}(\lambda) \right) E_{k,q}^{(r)}(x \mid \lambda).$$

3. Concluding remarks and observations

We have investigated various properties of a new q-generalization of the Apostol-Euler polynomials which we introduced by using the usual q-exponential function $e_q(x)$. These properties and other related identities are shown to arise from the q-umbral calculus. Two of the main results presented in our investigation (Theorem 2.2 and Theorem 2.6) seem to be sufficiently deep and general for obtaining not only new, but also interesting, identities related to some special polynomials in terms of other new q-generalizations of the Euler polynomials of the Apostol type.

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