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Derivative polynomials of a function related to the Apostol-Euler and Frobenius-Euler numbers

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Abstract

In the paper, the authors find a simple and significant expression in terms of the Stirling numbers for derivative polynomials of a function with a parameter related to the higher order Apostol-Euler numbers and to the higher order Frobenius-Euler numbers. Moreover, the authors also present a common solution to a sequence of nonlinear ordinary differential equations.

Keywords: Derivative polynomial, Stirling number, nonlinear ordinary differential equation, solution.
2010 MSC: 11B68, 11B73, 34A34.

1. Introduction

In [4], Theorem 2.1 states that the differential euqation

$$\frac{d^n F(t,\lambda)}{d\,t^n} = \sum_{k=0}^{n} (-1)^{n-k} b_k(n) \lambda^k F^{k+1}(t,\lambda) \tag{1.1}$$

for $n \in \mathbb{N}$ has a solution

$$F(t,\lambda) = \frac{1}{e^t + \lambda}, \tag{1.2}$$

where

$$b_0(n) = 1, \quad b_n(n) = n!, \tag{1.3}$$

and

$$b_k(n) = k! \sum_{i_k=0}^{n-k} \sum_{i_{k-1}=0}^{n-i_k-k} \cdots \sum_{i_2=0}^{n-i_k-\cdots i_3-k} \sum_{i_1=0}^{n-i_k-\cdots i_2-k} 2^{i_1} 3^{i_2} \cdots (k+1)^{i_k} \tag{1.4}$$

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for $1 \leqslant k \leqslant n$. Theorems 2 and 3 in [4] read that

$$E_{n+m,1/\lambda} = \sum_{k=1}^{m} (-1)^{m-k} \frac{1}{2^k} E_{n,1/\lambda}^{(k+1)} b_k(m) + (-1)^m E_{n,1/\lambda} \tag{1.5}$$

and

$$H_{n+m}(-\lambda) = \sum_{k=1}^{m} (-1)^{m-k} \left(\frac{\lambda}{1+\lambda}\right)^k H_n^{(k+1)}(-\lambda) \frac{b_k(m)}{k!} + (-1)^m H_n(-\lambda) \tag{1.6}$$

for $n \geqslant 0$, $m \in \mathbb{N}$, and $\lambda \in \mathbb{C}$ with $\lambda \neq 0, -1$, where the quantities $E_{n,\lambda}$, $E_{n,\lambda}^{(k)}$, $H_n(u)$, and $H_n^{(k)}(u)$ for $k \in \mathbb{N}$, $n \geqslant 0$, $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, and $u \in \mathbb{C}$ with $u \neq 1$ are generated by

$$\left(\frac{2}{\lambda e^t + 1}\right)^k = \sum_{n=0}^{\infty} E_{n,\lambda}^{(k)} \frac{t^n}{n!}, \quad \left(\frac{1-u}{e^t - u}\right)^k = \sum_{n=0}^{\infty} H_n^{(k)}(u) \frac{t^n}{n!},$$

and

$$E_{n,\lambda} = E_{n,\lambda}^{(1)}, \quad H_n(u) = H_n^{(1)}(u).$$

In the literature, the quantities $E_{n,\lambda}$ and $H_n(u)$ are respectively called the Apostol-Euler numbers and the Frobenius-Euler numbers, and the quantities $E_{n,\lambda}^{(k)}$ and $H_n^{(k)}(u)$ are respectively called the higher order Apostol-Euler numbers and the higher order Frobenius-Euler numbers.

The quantities in (1.3) and (1.4) were derived in [4] by an elementary, inductive, and recurrent method. The method and idea in [4] have been employed in some papers, such as [5–8] and the closely related references therein, for many years. This means that the method used in [4–8] and other similar papers is effectual and applicable. On the other hand, a defect of the method is also apparent: those results obtained in the above mentioned papers are too complicated to be easily computed and understood in mathematics. In recent years, the defect has been discussed, avoided, and corrected in some papers such as [10–17] and the closely related references therein.

Alternatively, we can regard the differential equation (1.1) as derivative polynomials

$$P_n(x) = \sum_{k=0}^{n} (-1)^{n-k} b_k(n) \lambda^k x^{k+1} \tag{1.7}$$

with respect to the variable t of the function $F(t,\lambda)$ defined by (1.2). As usual, a sequence of polynomials P_n are called the derivative polynomials of a function $f(x)$ if and only if $f^{(n)}(x) = P_n(f(x))$ for $n \geqslant 0$. For more detailed information on the concept of derivative polynomials, please refer to [3, 9, 14, 15] and the closely related references therein.

In our eyes, the expression (1.4) is also not simple, not significant, not meaningful, and not easy to be computed by hands and computer softwares.

In this paper, we will simply find derivative polynomials $P_n(x)$ for the function $F(t,\lambda)$, which is simpler, more significant, and more meaningful than the one in (1.7). In other words, we will simply derive a simpler form for the quantities $b_k(n)$.

Our main results can be stated as the following theorems.

Theorem 1.1. *For $\lambda \neq 0$ and $n \in \mathbb{N}$, the derivative polynomials of the function $F(t,\lambda) = \frac{1}{e^t+\lambda}$ can be computed by*

$$P_n(x) = \sum_{k=0}^{n} (-1)^{n-k} k! S(n+1,k+1) \lambda^k x^{k+1}, \tag{1.8}$$

where $S(n,k)$, which can be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}, \quad k \in \mathbb{N},$$

stands for the Stirling numbers of the second kind.

Remark 1.2. Comparing (1.1) with (1.8) reveals that $b_k(n) = k!S(n+1,k+1)$ for $n \geqslant k \geqslant 0$ and that

$$S(n+1,k+1) = \sum_{i_k=0}^{n-k} \sum_{i_{k-1}=0}^{n-i_k-k} \cdots \sum_{i_2=0}^{n-i_k-\cdots i_3-k} \sum_{i_1=0}^{n-i_k-\cdots i_2-k} 2^{i_1}3^{i_2}\cdots(k+1)^{i_k}$$

for $n \geqslant k \geqslant 1$. Hence, the identities (1.5) and (1.6) can be significantly simplified as

$$E_{n+m,1/\lambda} = \sum_{k=1}^{m}(-1)^{m-k}S(m+1,k+1)\frac{k!}{2^k}E^{(k+1)}_{n,1/\lambda} + (-1)^m E_{n,1/\lambda}$$

and

$$H_{n+m}(\lambda) = (-1)^m \sum_{k=1}^{m} S(m+1,k+1)\left(\frac{\lambda}{1-\lambda}\right)^k H^{(k+1)}_n(\lambda) + (-1)^m H_n(\lambda)$$

for $n \geqslant 0$, $m \in \mathbb{N}$, and $\lambda \in \mathbb{C}$ with $\lambda \neq 0, 1$. These imply that the derivative polynomials $P_n(x)$ in (1.8) is simpler, more significant, more meaningful, and easier to be computed by hands and computer softwares than the above-mentioned result in [4, Theorem 2.1].

Theorem 1.3. *For $\lambda \neq 0$ and $n \in \mathbb{N}$, the nonlinear differential equation*

$$\sum_{k=0}^{n}(-1)^k s(n+1,k+1)\frac{d^k F(t,\lambda)}{d\,t^k} = (-1)^n n!\lambda^n F^{n+1}(t,\lambda)$$

has a common solution $F(t,\lambda) = \frac{1}{e^t+\lambda}$, where $s(n,k)$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \quad |x| < 1$$

for $k \in \mathbb{N}$, stands for the Stirling numbers of the first kind.

2. A lemma

In order to prove Theorems 1.1 and 1.3, we need the following lemma.

Lemma 2.1 ([1, Theorem 2.1] and [19, Theorems 3.1 and 3.2])**.** *Let $\alpha, \beta \neq 0$ be real constants and $k \in \mathbb{N}$. When $\beta > 0$ and $t \neq -\frac{\ln\beta}{\alpha}$ or when $\beta < 0$ and $t \in \mathbb{R}$, we have*

$$\frac{d^k}{d\,t^k}\left(\frac{1}{\beta e^{\alpha t}-1}\right) = (-1)^k\alpha^k \sum_{m=1}^{k+1}(m-1)!S(k+1,m)\left(\frac{1}{\beta e^{\alpha t}-1}\right)^m \tag{2.1}$$

and

$$\left(\frac{1}{\beta e^{\alpha t}-1}\right)^k = \frac{1}{(k-1)!}\sum_{m=1}^{k}\frac{(-1)^{m-1}}{\alpha^{m-1}}s(k,m)\frac{d^{m-1}}{d\,t^{m-1}}\left(\frac{1}{\beta e^{\alpha t}-1}\right). \tag{2.2}$$

Remark 2.2. For motivations and further developments about the papers [1, 19], please refer to [2, 18, 20] and the closely related references therein.

3. Proofs of Theorems 1.1 and 1.3

Now we are in a position to prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. Taking $\alpha = 1$ and $\beta = -\frac{1}{\lambda}$ for $\lambda \neq 0$ in (2.1) gives

$$\frac{d^k}{d\,t^k}\left(\frac{1}{-e^t/\lambda-1}\right) = (-1)^k \sum_{m=1}^{k+1}(m-1)!S(k+1,m)\left(\frac{1}{-e^t/\lambda-1}\right)^m$$

which can be rearranged as

$$\begin{aligned}\frac{d^k}{d\,t^k}\left(\frac{1}{e^t+\lambda}\right)&=(-1)^{k+1}\sum_{m=1}^{k+1}(-1)^m(m-1)!S(k+1,m)\lambda^{m-1}\left(\frac{1}{e^t+\lambda}\right)^m\\&=(-1)^{k+1}\sum_{m=0}^{k}(-1)^{m+1}m!S(k+1,m+1)\lambda^m\left(\frac{1}{e^t+\lambda}\right)^{m+1}.\end{aligned}$$

The proof of Theorem 1.1 is thus complete. □

Proof of Theorem 1.3. Letting $\alpha=1$ and $\beta=-\frac{1}{\lambda}$ for $\lambda\neq 0$ in (2.2) arrives at

$$\left(\frac{1}{-e^t/\lambda-1}\right)^k=\frac{1}{(k-1)!}\sum_{m=1}^{k}(-1)^{m-1}s(k,m)\frac{d^{m-1}}{d\,t^{m-1}}\left(\frac{1}{-e^t/\lambda-1}\right)$$

which can be rewritten as

$$(-1)^{k-1}\lambda^{k-1}\left(\frac{1}{e^t+\lambda}\right)^k=\frac{1}{(k-1)!}\sum_{m=1}^{k}(-1)^{m-1}s(k,m)\frac{d^{m-1}}{d\,t^{m-1}}\left(\frac{1}{e^t+\lambda}\right).$$

The proof of Theorem 1.3 is thus complete. □

Remark 3.1. This paper is a slightly revised version of the preprint [21].

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