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Carathéodory's approximate solution to stochastic differential delay equation

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Abstract

In this paper, we show the difference between an approximate solution and an accurate solution for a stochastic differential delay equation, where the approximate solution, which is called by Carathéodory, is constructed by successive approximation. Furthermore, we study the p-th moment continuity of the approximate solution for this delay equation.

Keywords: Hölder's inequality, moment inequality, Carathéodory approximation procedure, stochastic differential delay equation.

2010 MSC: 60H05, 60H10, 60H20.

1. Introduction

An approximate solution is one of the fundamental concepts in stochastic differential systems. In the study of the stochastic differential delay equations, if there dose not exist an explicit solution, then "how can we obtain the approximate solution ?" is a very important problem.

In 2015, Kim [4] considered the following stochastic differential delay equation:

$$dx(t) = F(x(t), x(t-\tau), t)dt + G(x(t), x(t-\tau), t)dB(t), \quad t_0 \leqslant t \leqslant T, \tag{1.1}$$

and defined the Carathéodory approximation for a solution of the delay equation (1.1) as follows:

For each $n \geqslant 1$, define a function $x_n(t)$ on $[-\tau, T]$ by

$$x_n(t_0+\theta) = \xi(\theta), \quad -\tau \leqslant \theta \leqslant 0,$$

and

$$x_n(t) = \xi(0) + \int_{t_0}^{t} I_{D_n^c}(s)F(x_n(s-\frac{1}{n}), x_n(s-\tau), s)ds + \int_{t_0}^{t} I_{D_n}(s)F(x_n(s-1/n), x_n(s-\tau-\frac{1}{n}), s)ds$$

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$$+\int_{t_0}^{t} I_{D_n^c}(s)G(x_n(s-1/n), x_n(s-\tau), s)dB(s)$$
$$+\int_{t_0}^{t} I_{D_n}(s)G(x_n(s-1/n), x_n(s-\tau-\frac{1}{n}), s)dB(s)$$

for $t_0 \leqslant t \leqslant T$, where

$$D_n = \{t \in [t_0, T] : \tau < \frac{1}{n}\}$$

for $D_n^c = [t_0, T] - D_n$.

In [4], by employing the non-Lipschitz condition and the nonlinear growth condition, Kim established the following results for the second moment to stochastic differential delay equation:

Theorem 1.1. *Assume that there exists a constant* K *and a concave function* $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

(1) (*The non-Lipschitz condition*): *for all* $t \in [t_0, T]$ *and* $x, y, \overline{x}, \overline{y} \in \mathbb{R}$,

$$|F(x, y, t) - F(\overline{x}, \overline{y}, t)|^2 \vee |G(x, y, t) - G(\overline{x}, \overline{y}, t)|^2 \leqslant \kappa(|x-\overline{x}|^2 + |y-\overline{y}|^2),$$

where the concave function κ *is a nondecreasing function such that* $\kappa(0) = 0$, $\kappa(u) > 0$ *for any* $u > 0$ *and* $\int_{0+} \frac{1}{\kappa(u)} du = \infty$;

(2) (*The non-linear growth condition*): *there exists* $K > 0$ *such that, for all* $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times [t_0, T]$,

$$|F(0, 0, t)|^2 \vee |G(0, 0, t)|^2 \leqslant K.$$

Then we have

$$E\Big(\sup_{t_0 \leqslant t \leqslant T} |x(t) - x_n(t)|^2\Big) \leqslant \Big(\alpha\gamma(T-t_0) + \widehat{K}_1 + \widehat{K}_2\Big) e^{5\alpha\gamma(T-t_0)},$$

where $\gamma = 4(T - t_0 + 4)$,

$$\widehat{K}_1 = 8\alpha\gamma[C + 2T(\alpha(1+2C) + K)]\frac{1}{n}$$
$$\widehat{K}_2 = 8\alpha\gamma\Big([C + 4T(\alpha(1+2C) + K)]\frac{1}{n} + C\mu\Big),$$
$$C = \left(\frac{1}{2} + 4E\|\xi\|^2 + 6K(T-t_0+4)(T-t_0)\right) e^{12\alpha(T-t_0+4)(t-t_0)},$$

and $\mu = \{t \in [t_0, t_0 + 1 + \frac{1}{n}] : 0 < \tau < \frac{1}{n}\}$ *stands for the Lebesgue measure on* $\mathbb{R}$.

For some more details on stochastic differential equations, refer to [1–3, 5–11] and references therein. By using the nonlinear growth condition and nonlinear growth condition, in 2015, Kim [4] studied the difference between the approximate solution and the accurate solution to the stochastic differential delay equation (shortly, SDEs).

In this paper, motivated by the results mentioned above, we establish some exponential estimates for the p-th moment and show the difference between the approximate solution and the unique solution to the stochastic differential delay equation, which can be obtained from non-Lipschitz condition and special linear growth condition. For our main results in this paper, we use the Carathéodory approximation procedure.

2. Preliminary

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geqslant t_0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets). Let $|\cdot|$ denote the Euclidean norm in R^n if A is a vector or a matrix, its transpose is denoted by A^T; if$A$ is a matrix, its trace norm is represented by $|A| = \sqrt{\text{trace}(A^T A)}$. Assume that $B(t)$ is an m-dimensional

Brownian motion defined on complete probability space, that is,

$$B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^T.$$

Let $BC((-\infty, 0]; \mathbb{R}^d)$ denote the family of bounded continuous R^d-valued functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\| = \sup_{-\infty<\theta\leqslant 0} |\varphi|$. Let $\mathcal{M}^2((-\infty, 0]; \mathbb{R}^d)$ denote the family of $\mathcal{F}_{t_0}$-measurable and R^d-valued process $\varphi(t) = \varphi(t, \omega)$ for all $t \in (-\infty, 0]$ such that

$$E \int_{-\infty}^{0} |\varphi(t)|^2 dt < \infty.$$

In [7], Ren et al. considered the following d-dimensional stochastic functional differential equations:

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t), \quad t_0 \leqslant t \leqslant T, \tag{2.1}$$

where $x_t = \{x(t + \theta) : -\infty < \theta \leqslant 0\}$ can be regarded as a $BC((-\infty, 0]; \mathbb{R}^d)$-valued stochastic process, where

$$f : BC((-\infty, 0]; \mathbb{R}^d) \times [t_0, T] \to R^d, \quad g : BC((-\infty, 0]; \mathbb{R}^d) \times [t_0, T] \to \mathbb{R}^{d\times m}$$

are Borel measurable functions. Moreover, the initial value is as follows:

$$\begin{cases} x_{t_0} = \xi = \{\xi(\theta) : -\infty \leqslant \theta \leqslant 0\} \text{ is an } \mathcal{F}_{t_0}\text{-measurable;} \\ BC((-\infty, 0]; \mathbb{R}^d)\text{-valued random variable such that } \xi \in \mathcal{M}^2((-\infty, 0]; \mathbb{R}^d). \end{cases}$$

A special, but important class of stochastic functional differential equations is the *stochastic differential delay equation*.

Now, we consider the following stochastic differential delay equation:

$$dx(t) = F(x(t), x(t - \tau), t)dt + G(x(t), x(t - \tau), t)dB(t), \quad t_0 \leqslant t \leqslant T, \tag{2.2}$$

where $F : \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T] \to R^d$ and $G : R^d \times R^d \times [t_0, T] \to \mathbb{R}^{d\times m}$ are Borel measurable. Moreover, the initial value is as follows:

$$\begin{cases} x_{t_0} = \xi = \{\xi(\theta) : -\tau \leqslant \theta \leqslant 0\} \text{ is an } \mathcal{F}_{t_0}\text{-measurable;} \\ BC([-\tau, 0]; \mathbb{R}^d)\text{-valued random variable such that } \xi \in \mathcal{M}^2([-\tau, 0]; \mathbb{R}^d). \end{cases} \tag{2.3}$$

If we define

$$f(\varphi, t) = F(\varphi(0), \varphi(-\tau), t), \qquad g(\varphi, t) = G(\varphi(0), \varphi(-\tau), t)$$

for all $(\varphi, t) \in \mathcal{M}^2([-\tau, 0]; \mathbb{R}^d) \times [t_0, T]$, then equation (2.2) can be written as the equation (2.1) and so one can apply the existence and uniqueness theorem established in [7] to the delay equation (2.2).

Now, we need the following lemmas in order to show the main results:

Lemma 2.1 (Moment inequality, [5])**.** *If* $p \geqslant 2$ *and* $g \in \mathcal{M}^2([0, T]; \mathbb{R}^{d\times m})$ *such that*

$$E \int_0^T |g(s)|^p \, ds < \infty,$$

then

$$E\left|\int_0^T g(s)\, dB(s)\right|^p \leqslant \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E \int_0^T |g(s)|^p \, ds.$$

In particular, $E|\int_0^T g(s)\, dB(s)|^2 = E\int_0^T |g(s)|^2 \, ds$ *when* $p = 2$.

Lemma 2.2 (Moment inequality [5])**.** *Under the same assumptions as Lemma 2.1, we have*

$$E\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^t g(s)\, dB(s)\right|^p\right) \leqslant \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E \int_0^T |g(s)|^p \, ds.$$

3. Approximate solutions

First, we discuss the Carathéodory approximation procedure. Consider the stochastic differential delay equation (2.2) with initial data (2.3).

Now, define the Carathéodory approximation as follows: for each integer $n \geqslant 1$, define $x_n(t)$ on $[t_0 - \tau, T]$ by

$$x_n(t_0 + \theta) = \xi(\theta), \quad -\tau < \theta \leqslant 0,$$

and

$$x_n(t) = \xi(0) + \int_{t_0}^{t} F(x_n(s - \frac{1}{n}), x_n(s - \tau - \frac{1}{n}), s)ds + \int_{t_0}^{t} G(x_n(s - \frac{1}{n}), x_n(s - \tau - \frac{1}{n}), s)dB(s) \tag{3.1}$$

for $t_0 \leqslant t \leqslant T$. Note that, for $t_0 \leqslant t \leqslant t_0 + \frac{1}{n}$, $x_n(t)$ can be computed by

$$x_n(t) = \xi(0) + \int_{t_0}^{t} F(\xi(0), \xi(0), s)ds + \int_{t_0}^{t} G(\xi(0), \xi(0), s)dB(s);$$

then, for $t_0 + \frac{1}{n} \leqslant t \leqslant t_0 + \frac{2}{n}$,

$$x_n(t) = x_n(t_0 + \frac{1}{n}) + \int_{t_0+\frac{1}{n}}^{t} F(x_n(s - \frac{1}{n}), x_n(s - \tau - \frac{1}{n}), s)ds + \int_{t_0+\frac{1}{n}}^{t} G(x_n(s - \frac{1}{n}), x_n(s - \tau - \frac{1}{n}), s)dB(s)$$

and so on. In other words, $x_n(t)$ can be computed step-by-step on the intervals $[t_0, t_0 + \frac{1}{n}]$, $(t_0 + \frac{1}{n}, t_0 + \frac{2}{n}]$, $\cdots$. Since our goal is to study exponential estimates on the difference between the approximate solutions and the uniqueness solution, we assume that there exists a unique solution $x(t)$ to the equation (2.2) under the non-Lipschitz condition and the special linear growth condition.

On the other hand, for our results, we impose the non-Lipschitz condition and the special linear growth condition, that is, for all $t \in [t_0, T]$, and $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$

$$|F(x, y, t) - F(\bar{x}, \bar{y}, t)|^2 \vee |G(x, y, t) - G(\bar{x}, \bar{y}, t)|^2 \leqslant \kappa(|x - \bar{x}|^2 + |y - \bar{y}|^2), \tag{3.2}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for all $u > 0$ and $\int_{0+} \frac{1}{\kappa(u)} du = \infty$, and there exists $K > 0$ such that, for all $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T]$,

$$|F(0, 0, t)|^2 \vee |G(0, 0, t)|^2 \leqslant K. \tag{3.3}$$

Now, we give an exponential estimate for p-th moment as follows:

Lemma 3.1. *Assume that the conditions (3.2) and (3.3) hold. Then we have*

$$E\Big(\sup_{t_0 - \tau < s \leqslant t} |x_n(s)|^p\Big) \leqslant C_k := C_2 \exp\big(12^{p-1} \alpha^{p/2} C_1 (t - t_0)\big) \tag{3.4}$$

for all $t \geqslant t_0$, *where*

$$C_1 = [(T - t_0)^{p-1} + (p^3/2(p-1))^{p/2} (T - t_0)^{(p-2)/2}]$$

and

$$C_2 = (1 + 3^{p-1}) E\|\xi\|^p + 6^{p-1}[K^{p/2} + 2^{(p-2)/2} \alpha^{p/2}](T - t_0) C_1.$$

Proof. Fix $n \geqslant 1$ arbitrarily. From the definition of $x_n(t)$ and the conditions (3.2) and (3.3), it is easy to see that $\{x_n(t)\}_{t_0\leqslant t\leqslant T} \in \mathcal{M}^2((t_0-\tau, T]; R^d)$. From (3.1), note that, for $t_0 \leqslant t \leqslant T$,

$$|x_n(s)|^p \leqslant 3^{p-1}|\xi(0)|^p + 3^{p-1}\Big|\int_{t_0}^t F(x_n(s-\frac{1}{n}), x_n(s-\tau-\frac{1}{n}), s)ds\Big|^p + 3^{p-1}\Big|\int_{t_0}^t G(x_n(s-\frac{1}{n}), x_n(s-\tau-\frac{1}{n}), s)dB(s)\Big|^p. \tag{3.5}$$

Using the Hölder inequality and Lemma 2.2, we can derive from (3.5) that, for $t_0 \leqslant t \leqslant T$,

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x_n(s)|^p\Big) \leqslant 3^{p-1}E|\xi(0)|^p + 3^{p-1}(t-t_0)^{p-1}E\int_{t_0}^t |F(x_n(s-\frac{1}{n}), x_n(s-\tau-\frac{1}{n}), s)|^p ds + 3^{p-1}\Big(\frac{p^3}{2(p-1)}\Big)^{p/2}(t-t_0)^{\frac{p-2}{2}}E\int_{t_0}^t |G(x_n(s-\frac{1}{n}), x_n(s-\tau-\frac{1}{n}), s)|^p ds.$$

By the conditions (3.2) and (3.3), we obtain

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x_n(s)|^p\Big) \leqslant 3^{p-1}E|\xi(0)|^p + 6^{p-1}(t-t_0)C_1K^{p/2} + 6^{p-1}C_1E\int_{t_0}^t [\kappa(|x_n(s-\frac{1}{n})|^2 + |x_n(s-\tau-\frac{1}{n})|^2)]^{p/2}ds,$$

where $C_1 = [(T-t_0)^{p-1} + (p^3/2(p-1))^{p/2}(T-t_0)^{(p-2)/2}]$. Since $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a positive constant α such that $\kappa(u) \leqslant \alpha(1+u)$ for all $u \geqslant 0$. Therefore, we have

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x_n(s)|^p\Big) \leqslant 3^{p-1}E|\xi(0)|^p + 6^{p-1}(t-t_0)C_1K^{p/2} + 6^{p-1}2^{(p-2)/2}\alpha^{p/2}(t-t_0)C_1 + 6^{p-1}2^{p-1}\alpha^{p/2}C_1\int_{t_0}^t E\Big(\sup_{t_0-\tau<r\leqslant s}|x_n(r)|^p\Big)ds.$$

Note that

$$E\Big(\sup_{t_0-\tau\leqslant s\leqslant t}|x_n(s)|^p\Big) \leqslant C_2 + 12^{p-1}\alpha^{p/2}C_1\int_{t_0}^t E\Big(\sup_{t_0-\tau<r\leqslant s}|x_n(r)|^p\Big)ds,$$

where $C_2 = (1+3^{p-1})E\|\xi\|^p + 6^{p-1}[K^{p/2} + 2^{(p-2)/2}\alpha^{p/2}](T-t_0)C_1$. An application of the Gronwall inequality implies that

$$E\Big(\sup_{t_0-\tau\leqslant s\leqslant t}|x_n(s)|^p\Big) \leqslant C_2\exp(12^{p-1}\alpha^{p/2}C_1(t-t_0))$$

and so the desired inequality follows immediately. This completes the proof. □

In other words, the estimate for $E|x_n(t)|^p$ can be done via the estimate for the p-th moment. In view of Lemma 3.1, we know that the p-th moment of the solution satisfies

$$E\Big(\sup_{t_0-\tau<s\leqslant t}|x(s)|^p\Big) \leqslant C_j. \tag{3.6}$$

This means that the p-th moment grows at most exponentially with some exponent. Making use of the approximation (3.1) we can show the following theorem in the same way as Lemma 3.1.

Theorem 3.2. *Suppose that (3.2) and (3.3) hold. Then, for any* $t_0 \leqslant s < t \leqslant T$ *with* $t - s < 1$,

$$E\Big(|x_n(t) - x_n(s)|^p\Big) \leqslant \big[4^{p-1}C_3K^{p/2} + 4^{p-1}2^{(p-2)/2}\alpha^{p/2}C_3 + 8^{p-1}\alpha^{p/2}C_3C_k\big](t-s),$$

where C_1 *and* C_k *are defined in Lemma 3.1 and*

$$C_3 = [(T-t_0)^{p-1} + (p(p-1)/2)^{p/2}(T-t_0)^{(p-2)/2}].$$

Proof. Using the Hölder's inequality and Lemma 2.1, it follows from (3.1) that, for $t_0 \leqslant t \leqslant T$,

$$\begin{aligned} E\Big(|x_n(t) - x_n(s)|^p\Big) \leqslant\ & 2^{p-1}(t-s)^{p-1}E\int_s^t |F(x_n(r-\frac{1}{n}), x_n(r-\tau-\frac{1}{n}), r)|^p\,dr \\ & + 2^{p-1}\Big(\frac{p(p-1)}{2}\Big)^{p/2}(t-s)^{(p-2)/2}E\int_s^t |G(x_n(r-\frac{1}{n}), x_n(r-\tau-\frac{1}{n}), r)|^p\,dr. \end{aligned}$$

By the conditions (3.2) and (3.3), we obtain

$$E\Big(|x_n(t) - x_n(s)|^p\Big) \leqslant 4^{p-1}C_3K^{p/2}(t-s) + 4^{p-1}C_3E\int_s^t [\kappa(|x_n(r-\frac{1}{n})|^2 + |x_n(r-\tau-\frac{1}{n})|^2)]^{p/2}dr,$$

where $C_3 = [(T-t_0)^{p-1} + (p(p-1)/2)^{p/2}(T-t_0)^{(p-2)/2}]$. Since $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a positive constant α such that $\kappa(u) \leqslant \alpha(1+u)$ for all $u \geqslant 0$. Therefore, we have

$$E\Big(|x_n(t) - x_n(s)|^p\Big) \leqslant 4^{p-1}[K^{p/2} + 2^{(p-2)/2}\alpha^{p/2}]C_3(t-s) + 8^{p-1}\alpha^{p/2}C_3\int_s^t E\Big(\sup_{t_0-\tau<r\leqslant s}|x_n(r)|^p\Big)ds.$$

Hence, by Lemma 3.1, it follows that

$$E\Big(|x_n(t) - x_n(s)|^p\Big) \leqslant 4^{p-1}[K^{p/2} + 2^{(p-2)/2}\alpha^{p/2}]C_3(t-s) + 8^{p-1}\alpha^{p/2}C_3C_k(t-s)$$

and so the desired inequality follows immediately. This completes the proof. □

As another application of Lemma 3.1, we showed the continuity of the p-th moment of the sequence given in Theorem 3.2. In view of Theorem 3.2, we know that the p-th moment of the solution satisfies

$$E\Big(|x(t) - x(s)|^p\Big) \leqslant C_l(t-s). \tag{3.7}$$

This means that the p-th moment of the solution is continuous.

The following theorem shows that the Carathéodory sequence converges to the unique solution of the equation (2.2) and gives an estimate for the difference between the approximate solution $x_n(t)$ and the accurate solution $x(t)$.

Theorem 3.3. *Suppose that (3.2) and (3.3) hold. Then we have*

$$E\Big(\sup_{t_0\leqslant s\leqslant T}|x(s) - x_n(s)|^p\Big) \leqslant 2^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1\Big[(T-t_0) + C_4\Big]\exp\Big(2^{2p-1}3^{(p/2)-1}\alpha^{p/2}C_1(t-t_0)\Big), \tag{3.8}$$

where C_1 *is defined in Lemma 3.1 and*

$$C_4 = 2^{p-1}[2^pC_j + (T-(t_0+\frac{1}{n}))C_l + 2^pC_j + (T-(t_0+\tau+\frac{1}{n}))C_l][1/n].$$

Proof. From (2.2) and (3.1), it follows that, for $t_0 \leqslant t \leqslant T$,

$$|x(s)-x_n(s)|^p \leqslant 2^{p-1}\Big|\int_{t_0}^{t}[F(x(s),x(s-\tau),s)-F(x_n(s-\frac{1}{n}),x_n(s-\tau-\frac{1}{n}),s)]ds\Big|^p + 2^{p-1}\Big|\int_{t_0}^{t}[G(x(s),x(s-\tau),s)-G(x_n(s-\frac{1}{n}),x_n(s-\tau-\frac{1}{n}),s)]dB(s)\Big|^p.$$

By the Hölder inequality, Lemma 2.2, and the condition (3.4), it follows that

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 2^{p-1}C_1 E\int_{t_0}^{t}[\kappa(|x(s)-x_n(s-\frac{1}{n})|^2+|x(s-\tau)-x_n(s-\tau-\frac{1}{n})|^2)]^{p/2}ds.$$

By the conditions (3.2), (3.3), and the definition of $\kappa(\cdot)$, we obtain

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 2^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1\Big[(t-t_0)+\int_{t_0}^{t}\Big(E|x(s)-x_n(s-\frac{1}{n})|^p+E|x(s-\tau)-x_n(s-\tau-\frac{1}{n})|^p\Big)ds\Big].$$

Therefore, we have

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 2^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1\Big[(t-t_0)+2^{p-1}\int_{t_0}^{t}\Big(E|x(s)-x(s-1/n)|^p+E|x(s-\tau)-x(s-\tau-1/n)|^p\Big)ds + 2^p\int_{t_0}^{t}\Big(E\sup_{t_0\leqslant r\leqslant s}|x(r)-x_n(r)|^p\Big)ds\Big].$$

But, using the inequalities (3.6) and (3.7), we can estimate

$$\int_{t_0}^{t}E|x(s)-x(s-\frac{1}{n})|^p ds = \Big[2^pC_j+(T-(t_0+\frac{1}{n}))C_l\Big]\frac{1}{n}$$

and

$$\int_{t_0}^{t}E|x(s-\tau)-x(s-\tau-\frac{1}{n})|^p ds = \Big[2^pC_j+(T-(t_0+\tau+\frac{1}{n}))C_l\Big]\frac{1}{n}.$$

Hence an application of the Gronwall inequality implies that

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 2^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1\Big[(T-t_0)+C_4\Big]\exp\Big(2^{2p-1}3^{(p/2)-1}\alpha^{p/2}C_1(t-t_0)\Big),$$

where

$$C_4 = 2^{p-1}[2^pC_j+(T-(t_0+\frac{1}{n}))C_l+2^pC_j+(T-(t_0+\tau+\frac{1}{n}))C_l][1/n]$$

and the required result (3.8) follows. This completes the proof. □

Let us continue with the discussion of the following stochastic differential delay equation:

$$dx(t) = F(x(t),x(t-\delta(t)),t)dt + G(x(t),x(t-\delta(t)),t)dB(t) \tag{3.9}$$

on $t\in[t_0,T]$ with initial data (2.3), where $\delta:[t_0,T]\to[0,\tau]$, $F:\mathbb{R}^d\times\mathbb{R}^d\times[t_0,T]\to\mathbb{R}^d$ and $G:\mathbb{R}^d\times\mathbb{R}^d\times[t_0,T]\to\mathbb{R}^{d\times m}$ are Borel measurable functions.

If we define

$$f(\varphi,t)=F(\varphi(0),\varphi(-\delta(t)),t),\quad g(\varphi,t)=G(\varphi(0),\varphi(-\delta(t)),t)$$

for all $(\varphi,t)\in\mathcal{M}^2([-\tau,0];\mathbb{R}^d)\times[t_0,T]$, then the equation (3.9) can be written as the equation (2.1) and so one can apply the existence and uniqueness theorem established in [7] to the delay equation (3.9).

Now, we define the Carathéodory approximation as follows: for each integer $n\geqslant 1$, define $x_n(t)$ on $[t_0-\tau,T]$ by

$$x_n(t_0+\theta)=\xi(\theta),\quad -\tau<\theta\leqslant 0,$$

and

$$\begin{aligned} x_n(t)=\xi(0)&+\int_{t_0}^t I_{D_n^c}(s)F(x_n(s-\frac{1}{n}),x_n(s-\delta(s)),s)ds\\ &+\int_{t_0}^t I_{D_n}(s)F(x_n(s-\frac{1}{n}),x_n(s-\delta(s)-\frac{1}{n}),s)ds\\ &+\int_{t_0}^t I_{D_n^c}(s)G(x_n(s-\frac{1}{n}),x_n(s-\delta(s)),s)dB(s)\\ &+\int_{t_0}^t I_{D_n}(s)G(x_n(s-\frac{1}{n}),x_n(s-\delta(s)-\frac{1}{n}),s)dB(s) \end{aligned}\tag{3.10}$$

for $t_0\leqslant t\leqslant T$, where

$$D_n=\{t\in[t_0,T]:\delta(t)<\frac{1}{n}\}$$

for $D_n^c=[t_0,T]-D_n$.

In the sequel of this section, $x_n(t)$ always means the Carathéodory approximation (3.10) rather than the Carathéodory one (3.1).

The following lemma shows that the Carathéodory approximation sequence is bounded in p-th moment.

Lemma 3.4. *Suppose that (3.2) and (3.3) hold. Then, for all $n\geqslant 1$, we have*

$$E\Big(\sup_{t_0-\tau<s\leqslant t}|x_n(s)|^p\Big)\leqslant C_m:=C_5\exp\big(20^{p-1}(2\alpha)^{p/2}C_1(t-t_0)\big)$$

for all $t\geqslant t_0$, where C_1 is defined in Lemma 3.1 and

$$C_5=(1+5^{p-1})E\|\xi\|^p+5^{p-1}[2^pK^{p/2}+4^{p-1}\alpha^{p/2}]C_1(T-t_0).$$

Proof. By the Hölder inequality and Lemma 2.2, we can derive, from (3.10), that, for $t_0\leqslant t\leqslant T$,

$$\begin{aligned} E\Big(\sup_{t_0\leqslant s\leqslant t}|x_n(s)|^p\Big)\leqslant 5^{p-1}E|\xi(0)|^p&+5^{p-1}(t-t_0)^{p-1}E\Big(\int_{t_0}^t I_{D_n^c}(s)|F(x_n(s-\frac{1}{n}),x_n(s-\delta(s)),s)|^p ds\\ &+E\int_{t_0}^t I_{D_n}(s)|F(x_n(s-\frac{1}{n}),x_n(s-\delta(s)-\frac{1}{n}),s)|^p ds\Big)\\ &+5^{p-1}\Big(\frac{p^3}{2(p-1)}\Big)^{p/2}(t-t_0)^{\frac{p-2}{2}}E\Big(\int_{t_0}^t I_{D_n^c}(s)|G(x_n(s-\frac{1}{n}),x_n(s-\delta(s)),s)|^p ds\\ &+\int_{t_0}^t I_{D_n}(s)|G(x_n(s-\frac{1}{n}),x_n(s-\delta(s)-\frac{1}{n}),s)|^p ds\Big). \end{aligned}$$

By the conditions (3.2) and (3.3), we obtain

$$\begin{aligned}E\Big(\sup_{t_0\leqslant s\leqslant t}|x_n(s)|^p\Big) &\leqslant 5^{p-1}E|\xi(0)|^p + 2^p 5^{p-1}C_1K^{p/2}(t-t_0)\\ &\quad + 10^{p-1}(t-t_0)^{p-1}E\Big(\int_{t_0}^t [\kappa(|x_n(s-\frac{1}{n})|^2+|x_n(s-\delta(s))|^2)]^{p/2}ds\\ &\quad + \int_{t_0}^t [\kappa(|x_n(s-\frac{1}{n})|^2+|x_n(s-\delta(s)-\frac{1}{n})|^2)]^{p/2}ds\Big)\\ &\quad + 10^{p-1}\Big(\frac{p^3}{2(p-1)}\Big)^{p/2}(t-t_0)^{\frac{p-2}{2}}E\Big(\int_{t_0}^t [\kappa(|x_n(s-\frac{1}{n})|^2+|x_n(s-\delta(s))|^2)]^{p/2}ds\\ &\quad + \int_{t_0}^t [\kappa(|x_n(s-\frac{1}{n})|^2+|x_n(s-\delta(s)-\frac{1}{n})|^2)]^{p/2}ds\Big),\end{aligned}$$

where $C_1 = [(T-t_0)^{p-1} + (p^3/2(p-1))^{p/2}(T-t_0)^{(p-2)/2}]$. Since $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a positive constant α such that $\kappa(u) \leqslant \alpha(1+u)$ for all $u \geqslant 0$. Therefore, we have

$$\begin{aligned}E\Big(\sup_{t_0\leqslant s\leqslant t}|x_n(s)|^p\Big) &\leqslant 5^{p-1}E|\xi(0)|^p + 2^p5^{p-1}K^{p/2}C_1(t-t_0)\\ &\quad + 20^{p-1}\alpha^{p/2}C_1(t-t_0) + 20^{p-1}(2\alpha)^{p/2}C_1\int_{t_0}^t E\Big(\sup_{t_0-\tau<r\leqslant s}|x_n(r)|^p\Big)ds.\end{aligned}$$

Note that

$$E\Big(\sup_{t_0-\tau\leqslant s\leqslant t}|x_n(s)|^p\Big) \leqslant C_5 + 20^{p-1}(2\alpha)^{p/2}C_1\int_{t_0}^t E\Big(\sup_{t_0-\tau<r\leqslant s}|x_n(r)|^p\Big)ds,$$

where $C_5 = (1+5^{p-1})E\|\xi\|^p + 5^{p-1}[2^pK^{p/2} + 4^{p-1}\alpha^{p/2}]C_1(T-t_0)$. An application of the Gronwall inequality implies that

$$E\Big(\sup_{t_0-\tau\leqslant s\leqslant t}|x_n(s)|^p\Big) \leqslant C_5\exp(20^{p-1}(2\alpha)^{p/2}C_1(t-t_0))$$

and so the desired inequality follows immediately. This completes the proof. □

In other words, the estimate for $E|x_n(t)|^p$ can be done via the estimate for the p-th moment. This means that the p-th moment grows at most exponentially with some exponents.

By using the approximation (3.10), we can show the following theorem in the same way as Lemma 3.4.

Theorem 3.5. *Suppose that (3.2) and (3.3) hold. Then, for any $t_0 \leqslant s < t \leqslant T$ with $t-s<1$, we have*

$$E\Big(|x_n(t)-x_n(s)|^p\Big) \leqslant \big[2^p4^{p-1}K^{p/2}C_3 + 2^p4^{p-1}\alpha^{p/2}C_3(t-s) + 2^p4^{p-1}(2\alpha)^{p/2}C_3C_m\big](t-s),$$

where C_3 and C_m are defined in Theorem 3.2 and Lemma 3.4.

Proof. Using the Hölder inequality and Lemma 2.1, we can derive from (3.10) that, for $t_0 \leqslant t \leqslant T$,

$$\begin{aligned}E\Big(|x_n(t)-x_n(s)|^p\Big) &\leqslant 4^{p-1}(t-s)^{p-1}E\int_s^t |F(x_n(r-\frac{1}{n}), x_n(r-\delta(r)), r)|^p dr\\ &\quad + 4^{p-1}(t-s)^{p-1}E\int_s^t |F(x_n(r-\frac{1}{n}), x_n(r-\delta(r)-\frac{1}{n}), r)|^p dr\end{aligned}$$

$$+4^{p-1}\Big(\frac{p(p-1)}{2}\Big)^{p/2}(t-s)^{(p-2)/2}E\int_s^t |G(x_n(r-\frac{1}{n}),x_n(r-\delta(r)),r)|^p dr$$
$$+4^{p-1}\Big(\frac{p(p-1)}{2}\Big)^{p/2}(t-s)^{(p-2)/2}E\int_s^t |G(x_n(r-\frac{1}{n}),x_n(r-\delta(r)-\frac{1}{n}),r)|^p dr.$$

By the conditions (3.2) and (3.3), we obtain

$$E\Big(|x_n(t)-x_n(s)|^p\Big) \leqslant 2^p4^{p-1}K^{p/2}C_3(t-s)+8^{p-1}C_3E\int_s^t[\kappa(|x_n(r-\frac{1}{n})|^2+|x_n(r-\delta(r))|^2)]^{p/2}dr$$
$$+8^{p-1}C_3E\int_s^t[\kappa(|x_n(r-\frac{1}{n})|^2+|x_n(r-\delta(r)-\frac{1}{n})|^2)]^{p/2}dr,$$

where C_3 is defined in Theorem 3.2. Since $\kappa(\cdot)$ is concave, we can find a positive constant α such that $\kappa(u) \leqslant \alpha(1+u)$ for all $u \geqslant 0$. Therefore, we have

$$E\Big(|x_n(t)-x_n(s)|^p\Big) \leqslant 2^p4^{p-1}K^{p/2}C_3(t-s)+2^p4^{p-1}\alpha^{p/2}C_3(t-s)$$
$$+2^p4^{p-1}(2\alpha)^{p/2}C_3\int_s^t E\Big(\sup_{t_0-\tau<r\leqslant s}|x_n(r)|^p\Big)ds.$$

Hence, by Lemma 3.4, it follows that

$$E\Big(|x_n(t)-x_n(s)|^p\Big) \leqslant \big[2^p4^{p-1}K^{p/2}C_3+2^p4^{p-1}\alpha^{p/2}C_3(t-s)+2^p4^{p-1}(2\alpha)^{p/2}C_3C_m\big](t-s).$$

and so the desired inequality follows immediately. This completes the proof. □

As another application of Lemma 3.4, we showed the continuity of the p-th moment of the sequence given in Theorem 3.5. In view of Theorem 3.5, we know that the p-th moment of the solution is continuous.

The following theorem shows that the Carathéodory sequence (3.10) converges to the unique solution of the equation (3.9) and gives an estimate for difference between the approximate solution $x_n(t)$ and the accurate solution $x(t)$.

Theorem 3.6. *Suppose that* (3.2) *and* (3.3) *hold. Then we have*

$$E\Big(\sup_{t_0\leqslant s\leqslant T}|x(s)-x_n(s)|^p\Big) \leqslant 4^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1C_8\exp\Big(4^{p-1}3^{(p/2)-1}\alpha^{p/2}(1+2^p)C_1(t-t_0)\Big), \tag{3.11}$$

where C_1 *is defined in Lemma* 3.1,

$$C_6=[2^pC_j+(T-(t_0+\frac{1}{n}))C_l][1/n], C_7=2^pC_j(\frac{2}{n})+C_l(T-(t_0+\frac{2}{n}))(1/n),$$

and

$$C_8=(T-t_0)+2^{p-1}(2C_6+C_7).$$

Proof. By the Hölder inequality, Lemma 2.2, the equation (2.2), and the sequence (3.10), we can derive that

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 4^{p-1}(t-t_0)^{p-1}\int_{t_0}^t\Big(I_{D_n^c}(s)|F_1(x,x_n)|^p+I_{D_n}(s)|F_2(x,x_n)|^p\Big)ds$$
$$+4^{p-1}\Big(\frac{p^3}{2(p-1)}\Big)^{p/2}(t-t_0)^{p-1}\int_{t_0}^t\Big(I_{D_n^c}(s)|G_1(x,x_n)|^p+I_{D_n}(s)|G_2(x,x_n)|^p\Big)ds,$$

where

$$F_1(x,x_n)=F(x(s),x(s-\delta(s)),s)-F(x_n(s-\frac{1}{n}),x_n(s-\delta(s)),s),$$

$$F_2(x,x_n) = F(x(s),x(s-\delta(s)),s) - F(x_n(s-\frac{1}{n}),x_n(s-\delta(s)-\frac{1}{n}),s),$$
$$G_1(x,x_n) = G(x(s),x(s-\delta(s)),s) - G(x_n(s-\frac{1}{n}),x_n(s-\delta(s)),s),$$
$$G_2(x,x_n) = G(x(s),x(s-\delta(s)),s) - G(x_n(s-\frac{1}{n}),x_n(s-\delta(s)-\frac{1}{n}),s).$$

By the conditions (3.2) and (3.3), we obtain

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 4^{p-1}C_1E\int_{t_0}^t I_{D_n^c}(s)\Big[\kappa(|f_1(x,x_n)|^2+|f_2(x,x_n)|^2)\Big]^{p/2}ds$$
$$+4^{p-1}C_1E\int_{t_0}^t I_{D_n}(s)\Big[\kappa(|f_1(x,x_n)|^2+|f_3(x,x_n)|^2)\Big]^{p/2}ds,$$

where

$$f_1(x,x_n) = x(s)-x_n(s-\frac{1}{n}),\quad f_2(x,x_n) = x(s-\delta(s))-x_n(s-\delta(s)),$$
$$f_3(x,x_n) = x(s-\delta(s))-x_n(s-\delta(s)-\frac{1}{n}).$$

By the definition of $\kappa(\cdot)$, we obtain

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 4^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1(t-t_0)+8^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1E\int_{t_0}^t|x(s)-x(s-\frac{1}{n})|^p ds$$
$$+8^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1E\int_{t_0}^t I_{D_n}(s)|x(s-\delta(s))-x(s-\delta(s)-\frac{1}{n})|^p ds$$
$$+4^{p-1}3^{(p/2)-1}\alpha^{p/2}(1+2^p)C_1\int_{t_0}^t E\Big(\sup_{t_0\leqslant r\leqslant s}|x(r)-x_n(r)|^p\Big)ds.$$

But, using the inequalities (3.6) and (3.7), we can estimate

$$\int_{t_0}^t E|x(s)-x(s-\frac{1}{n})|^p ds = \Big[2^pC_j+(T-(t_0+\frac{1}{n}))C_l\Big]\frac{1}{n}.$$

Also, setting $D_0=\{t\in[t_0,T]:\delta(t)=0\}$, we have

$$\int_{t_0}^t E|x(s-\delta(s))-x(s-\delta(s)-\frac{1}{n})|^p ds \leqslant \Big[2^pC_j+(T-(t_0+\frac{1}{n}))C_l\Big]\frac{1}{n}+2^pC_j\frac{2}{n}+C_l(T-(t_0+\frac{2}{n}))\frac{1}{n}.$$

Therefore, we have

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 4^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1(t-t_0)$$
$$+8^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1C_6+8^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1(C_6+C_7)$$
$$+4^{p-1}3^{(p/2)-1}\alpha^{p/2}(1+2^p)C_1\int_{t_0}^t E\Big(\sup_{t_0\leqslant r\leqslant s}|x(r)-x_n(r)|^p\Big)ds,$$

where

$$C_6 = [2^pC_j+(T-(t_0+1/n))C_l][1/n],\quad C_7 = 2^pC_j(2/n)+C_l(T-(t_0+2/n))(1/n).$$

Hence an application of the Gronwall inequality implies that

$$E\Big(\sup_{t_0\leqslant s\leqslant t}|x(s)-x_n(s)|^p\Big) \leqslant 4^{p-1}3^{(p/2)-1}\alpha^{p/2}C_1C_8\exp\Big(4^{p-1}3^{(p/2)-1}\alpha^{p/2}(1+2^p)C_1(t-t_0)\Big),$$

where $C_8 = (T-t_0)+2^{p-1}(2C_6+C_7)$ and so the required result (3.11) follows. This completes the proof. □

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