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Fixed point results for generalized contractive mappings involving altering distance functions on complete quasi-metric spaces and applications

Yanbin Sang

School of Science, North University of China, Taiyuan, 030051, China.

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Abstract

In this paper, we introduce α-ψ-ϕ-Jachymski contractive mappings with generalized altering distance functions in the setting of quasi-metric spaces. Some theorems on the existence and uniqueness of fixed points for such mappings via admissible mappings are established. Utilizing above abstract results, we derive common fixed point theorem for two operators and multidimensional fixed point results for nonlinear mappings satisfying different kinds of contractive conditions on partially ordered metric spaces. Moreover, we present some examples and applications in a Fredholm integral equation and an initial value problem for partial differential equation of parabolic type.

Keywords: Admissible mapping, altering distance, multidimensional, w-distance, partial order.
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1. Introduction

One of the generalizations of metric spaces is the so-called quasi-metric spaces in which the commutativity condition does not hold in general [1, 2, 11, 17, 19]. Park [19] extended the notion of w-distance to quasi-metric spaces and obtained far-reaching generalized forms of Ekeland's principle and its six equivalents. Al-Homidan et al. [2] introduced the concept of Q-function defined on a quasi-metric space as a generalization of w-distances, and then established a Caristi-Kirk-type fixed point theorem, a Takahashi minimization theorem, and versions of Ekeland's principle in the setting of quasi-metric space with a Q-function, generalizing the main results of [19]. This approach has been continued by Marín et al. [17]. We would like to mention the result of Alegre [1]. In [1], Alegre et al. obtained a fixed point theorem for generalized contractions on complete quasi-metric spaces, which involves w-distances and functions of Meir-Keeler and Jachymski type. They established the following result.

Theorem 1.1 ([1])**.** *Let F be a self-map of a complete quasi-metric space (X, d). If there exist a w-distance p on (X, d) and a Jachymski function $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(t) < t$ for all $t > 0$, and*

$$p(Fx, Fy) \leqslant \phi(p(x, y)) \tag{1.1}$$

for all $x,\ y \in X$, then F has a unique fixed point $x \in X$. Moreover, $p(x, x) = 0$.

Email address: syb6662004@163.com (Yanbin Sang)

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In recent years, the method of α-admissible mappings has been effectively used for proving the existence and uniqueness of fixed points for contractive operators on complete metric spaces. In 2012, Samet et al. [24] introduced the concepts of α-admissible mappings as follows:

Definition 1.2 ([24]). Let $F : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be two given mappings. Then F is called an α-admissible mapping if

$$x,\ y \in X,\quad \alpha(x, y) \geqslant 1 \Longrightarrow \alpha(Fx, Fy) \geqslant 1.$$

Samet et al. [24] established various fixed point theorems for α-ψ-contractive mappings satisfying α-admissibility condition in complete metric spaces. Furthermore, Alsulami et al. [4] gave the definition of a class of α-admissible contraction via altering distance function. The results were reconsidered in the context of partially ordered metric spaces and applied to boundary value problems for differential equations with periodic boundary conditions. Very recently, Lakzian et al. [13] introduced the new concept of generalized α-ψ-contractive mappings in the setting of w-distances and proved some new fixed point results for such mappings which generalize fixed point theorems by Samet et al. [24].

On the other hand, there was much attention focused on Meir-Keeler contractive mappings with α-admissible conditions. Karapinar et al. [12] discussed an α-ψ-Meir Keeler contractive mapping in the setting of complete metric spaces via a triangular α-admissible mapping.

Definition 1.3 ([12]). A mapping $F : X \to X$ is called triangular α-admissible if it is α-admissible and satisfies

$$\alpha(x, y) \geqslant 1 \text{ and } \alpha(y, z) \geqslant 1 \text{ implies } \alpha(x, z) \geqslant 1,$$

where $x,\ y,\ z \in X$ and $\alpha : X \times X \to [0, +\infty)$ is a given function.

Subsequently, Alsulami et al. [3] investigated the existence of fixed points of Meir-Keeler type contractions defined on quasi-metric spaces and applied their results to G-metric spaces. They studied α-admissible Meir-Keeler contractions which can be regarded as generalizations of the Meir-Keeler contractions defined in [18]. In essence, the authors of [3, 12] inserted α-admissibility into the definition of the original Meir-Keeler contraction.

Definition 1.4 ([3]). Let (X, d) be a quasi-metric space, and $T : X \to X$ be a triangular α-admissible mappings. Suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leqslant d(x, y) < \epsilon + \delta \text{ implies } \alpha(x, y)d(Tx, Ty) < \epsilon \tag{1.2}$$

for all $x,\ y \in X$. Then T is called an α-admissible-Meir-Keeler contractive mapping.

It is known that common fixed point (and coincidence point) theorems are generalizations of fixed point theorems. However, Haghi et al. [8] proved that some generalizations in fixed point theory are not real generalizations. Moreover, Rosa and Vetro [14] established some common fixed point theorems for a large class of α-ψ-φ-contractions in generalized metric spaces. The purpose of this paper is to prove some fixed point theorems with respect to w-distances in quasi-metric spaces employing generalized altering distance functions and the notation of a function involving Jachymski type. It should be noted that coupled fixed point theorems of mappings satisfying certain α-admissible conditions can be ascribed to the corresponding fixed point results of dimension one. It is natural to apply our theorems to multidimensional fixed points results and common fixed point for multiple operators [5, 6, 21–23].

2. Preliminaries

Before presenting our results, we collect relevant definitions and results which will be needed in the proof of our main results.

Denote with $\mathbb{N}$ the set of positive integers. Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ for all $x,\ y \in X$, is also a quasi-metric on X, and the function $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x,\ y \in X$, is a metric on X.

Definition 2.1 ([14]). Let T, $S: X \to X$ and $\alpha : X \times X \to \mathbb{R}$. The mapping T is S-α-admissible if, for all x, $y \in X$ such that $\alpha(Sx, Sy) \geqslant 1$, we have $\alpha(Tx, Ty) \geqslant 1$. If S is the identity mapping, then T is called α-admissible.

Lemma 2.2 ([1]). *If q is a w-distance on a quasi-metric space (X, d), then for each $\epsilon > 0$ there exists $\delta > 0$ such that $q(x, y) \leqslant \delta$ and $q(x, z) \leqslant \delta$ imply $d^s(y, z) \leqslant \epsilon$.*

Definition 2.3 ([1]). A function $p : X \times X \to [0, +\infty)$ is said to be a w-distance on a quasi-metric space (X, d) if it satisfies the following conditions:

(W_1) $p(x, z) \leqslant p(x, y) + p(y, z)$ for any x, y, $z \in X$;

(W_2) $p(x, \cdot) : X \to [0, +\infty)$ is lower semi-continuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;

(W_3) for each $\epsilon > 0$ there exists $\delta > 0$ such that $p(x, y) \leqslant \delta$ and $p(x, z) \leqslant \delta$ imply $d(y, z) \leqslant \epsilon$.

Definition 2.4 ([20]). A self-mapping T on a metric (X, d) is said to be asymptotically regular if

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0 \quad \text{for each } x \in X.$$

Definition 2.5. A function $\phi(t) : [0, +\infty) \to [0, +\infty)$ is said to be a generalized altering distance function if it satisfied the following conditions:

(a) ϕ is non-decreasing;
(b) $\phi = 0$ if and only if $t = 0$.

Definition 2.6 ([1, 9]). A Jachymski function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfies:

(a) $\psi(0) = 0$;
(b) for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $t \in [0, +\infty)$,

$$\epsilon < t < \epsilon + \delta \quad \text{implies} \quad \psi(t) \leqslant \epsilon.$$

Lemma 2.7 ([12]). *Let T be triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then*

$$\alpha(x_m, x_n) \geqslant 1 \ \textit{for all } m,\ n \in \mathbb{N} \ \textit{with } m < n.$$

Definition 2.8 ([10]). Let (X, d) be a metric space, and let S, $T : X \to X$ be maps. Then S and T are called weakly comparable if $STx = TSx$, whenever $Sx = Tx$.

Lemma 2.9 ([8]). *Let X be a nonempty set and $T : X \to X$ be a function. Then there exists a subset $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \to X$ is one-to-one.*

Definition 2.10 ([21, 23]). An ordered metric space $(X, d, \preceq)$ is said to have the sequential monotone property if it satisfies:

(i) if $\{x_m\}$ is a non-increasing sequence and $\{x_m\} \to x$, then $x_m \succeq x$ for all m;
(ii) if $\{y_m\}$ is a non-decreasing sequence and $\{y_m\} \to y$, then $y_m \preceq y$ for all m.

Now, we denote

$$\Omega_{A,B} = \{\rho :\ \Lambda_n \to \Lambda_n :\ \rho(A) \subseteq A \text{ and } \rho(B) \subseteq B\},$$

and

$$\Omega'_{A,B} = \{\rho :\ \Lambda_n \to \Lambda_n :\ \rho(A) \subseteq B \text{ and } \rho(B) \subseteq A\},$$

where $\Lambda_n = \{1,\ 2,\ \cdots, n\}$, A and B are non-empty sets with $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$.

If $(X, \preceq)$ is a partially ordered space, $x,\ y \in X$, and $i \in \Lambda_n$, we will use the following notation:

$$x \preceq_i y \iff \begin{cases} x \preceq y, & \text{if } i \in A, \\ x \succeq y, & \text{if } i \in B. \end{cases}$$

The product space X^n is endowed with the following partial order:

for $X = (x_1, x_2, \cdots, x_n)$, $Y = (y_1, y_2, \cdots, y_n) \in X^n$,

$$X \sqsubseteq Y \iff x_i \preceq_i y_i, \text{ for all } i \in \Lambda_n.$$

Definition 2.11 ([21, 23])**.** Let $(X, \preceq)$ be a partially ordered space. We say $F : X^n \to X$ has the mixed monotone property with respect to the partition $\{A, B\}$ if F is non-decreasing in arguments of A and non-increasing in arguments of B, i.e., for all $x_1,\ x_2,\ \cdots, x_n,\ y,\ z \in X$ and all $i \in \Lambda_n$,

$$y \preceq z \implies F(x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n) \preceq_i F(x_1, \cdots, x_{i-1}, z, x_{i+1}, \cdots, x_n).$$

Let $\sigma_1,\ \sigma_2,\ \cdots, \sigma_n : \ \Lambda_n \to \Lambda_n$ be n mappings from Λ_n into itself and let Υ be the n-tuple $(\sigma_1,\ \sigma_2,\ \cdots, \sigma_n)$.

Definition 2.12 ([21, 23])**.** A point $(x_1, x_2, \cdots, x_n) \in X^n$ is called a Υ-fixed point of the mapping F if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}) = x_i \text{ for all } i \in \Lambda_n.$$

3. Main results

Theorem 3.1. *Let T be a triangular α-admissible and asymptotically regular mapping of a complete quasi-metric space (X, d). Suppose that there exists a w-distance p on (X, d) such that*

$$\alpha(x, y)\psi(p(Tx, Ty)) \leqslant \phi(D(x, y)) \tag{3.1}$$

for all $x,\ y \in X$, where

$$D(x, y) = p(x, y) + \gamma p(x, Tx),\ \gamma \geqslant 0, \tag{3.2}$$

ψ is a generalized altering distance function, and $\phi : [0, +\infty) \to [0, +\infty)$ is a Jachymski function such that $\phi(t) < \psi(t)$ for all $t > 0$. Furthermore, the following condition is satisfied

(H) *If $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \geqslant 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which $\alpha(x_{n_k}, x) \geqslant 1,\ k \geqslant 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, then T has a fixed point.

Proof. Let $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$. Since $\alpha(x_0, Tx_0) \geqslant 1$, it follows from Definition 1.3 that $\alpha(T^{n-1}x_0, T^n x_0) \geqslant 1$. By (3.1), we deduce that

$$\psi(p(x_{n+1}, x_{n+2})) \leqslant \alpha(x_n, x_{n+1})\psi(D(Tx_n, Tx_{n+1})) \leqslant \phi(D(x_n, x_{n+1})). \tag{3.3}$$

First, we shall prove that $\{x_n\}$ is a Cauchy sequence in (X, d^s).

To this end put $r_n = p(x_n, x_{n+1}),\ n = 0, 1, 2, \cdots$. Since T is asymptotically regular, then $r_n \to 0$ $(n \to \infty)$.

If there is $n_0 \in \mathbb{N}$ such that $r_{n_0} = 0$, then $r_n = 0$ for all $n \geqslant n_0$, by (3.3), Definition 2.5, and our assumption that $\phi(0) = 0$. Therefore $p(x_i, x_j) = 0$ whenever $j > i \geqslant n_0$ by condition (W_1), and consequently, $d^s(x_i, x_j) = 0$ by Lemma 2.2. Otherwise, noting the increasing property of ψ, we have that the sequence $\{r_n\}$ is non-increasing and, consequently, there exists $r \geqslant 0$ such that $r_n \to r^+$ as $n \to \infty$. If $r > 0$, there exists $\delta = \delta(r)$ such that $\psi(r) < t < \psi(r) + \delta$ implies $\phi(t) \leqslant \psi(r)$. Take $N_\delta \in \mathbb{N}$ such that $r_n < \psi(r) + \delta$ for all $n \geqslant N_\delta$. Therefore $\phi(r_n) \leqslant \psi(r)$, so by (3.3), we derive that

$$\psi(r_{n+1}) \leqslant \phi(r_n) \leqslant \psi(r).$$

Since ψ is increasing, we have that $r_{n+1} \leqslant r$ for all $n \geqslant N_\delta$, a contradiction. Consequently, $r = 0$.

Now choose an arbitrary $\epsilon > 0$. There exists $\delta = \delta(\epsilon)$ with $\delta \in (0, \epsilon)$, for which condition (W_3) and Definition 2.6 hold. For $\delta_1 = \delta_1(\epsilon)(\delta_1 \in (0, \frac{\delta}{2}))$ there exists $\mu = \mu(\delta_1)$ with $\mu \in (0, \delta_1)$ such that $p(x, y) \leqslant \mu$ and $p(x, z) \leqslant \mu$ imply $d(y, z) \leqslant \delta_1$, and for each $t > 0$, $\delta_1 < t < \delta_1 + \mu$ implies $\phi(t) \leqslant \psi(\delta_1)$.

Since $r_n \to 0$ $(n \to \infty)$, there exists $N_0 \in \mathbb{N}$ such that $r_n < \frac{\mu}{1+\gamma}$ for all $n \geqslant N_0$.

In the following, we shall show that for each $n \in \mathbb{N}$, we have

$$p(x_k, x_{n+k}) < \delta_1 + \frac{\mu}{1+\gamma}. \tag{3.4}$$

Indeed, fix $k \geqslant N_0$. Since $p(x_k, x_{k+1}) < \frac{\mu}{1+\gamma}$, (3.4) follows for $n = 1$.

Assume that (3.4) holds for some $n \in \mathbb{N}$. We shall distinguish two cases.

Case 1. $D(x_k, x_{n+k}) > \delta_1$.

By Lemma 2.7, we know that $\alpha(x_k, x_{n+k}) \geqslant 1$, $k = 0, 1, 2, \cdots$, $n = 1, 2, \cdots$. Note that

$$D(x_k, x_{n+k}) = p(x_k, x_{n+k}) + \gamma p(x_k, x_{k+1}) < \delta_1 + \frac{\mu}{1+\gamma} + \frac{\gamma\mu}{1+\gamma} = \delta_1 + \mu.$$

Then we deduce from (3.1) that

$$\begin{aligned} \psi(p(x_{k+1}, x_{n+k+1})) = \psi(p(Tx_k, Tx_{n+k})) \leqslant \alpha(x_k, x_{n+k})\psi(p(Tx_k, Tx_{n+k})) &\leqslant \phi(D(x_k, x_{n+k})) \\ &\leqslant \psi(\delta_1). \end{aligned} \tag{3.5}$$

Since ψ is non-decreasing, it follows that $p(x_{k+1}, x_{n+k+1}) \leqslant \delta_1$. Therefore

$$p(x_k, x_{n+1+k}) \leqslant p(x_k, x_{k+1}) + p(x_{k+1}, x_{n+k+1}) < \mu + \delta_1 < \delta.$$

Case 2. $D(x_k, x_{n+k}) \leqslant \delta_1$.

If $D(x_k, x_{n+k}) = 0$, we derive that $p(x_{k+1}, x_{n+k+1}) = 0$ by (3.5). So, by (W_1),

$$p(x_k, x_{n+1+k}) \leqslant p(x_k, x_{n+k}) \leqslant \delta_1 < \delta_1 + \mu < \delta.$$

If $D(x_k, x_{n+k}) > 0$, we get that

$$p(x_k, x_{n+k+1}) \leqslant p(x_k, x_{k+1}) + p(x_{k+1}, x_{n+k+1}) \leqslant p(x_k, x_{k+1}) + D(x_k, x_{n+k}) < \mu + \delta_1 < \delta.$$

Now, take $i, j \in \mathbb{N}$ with $i, j > k$. Then $i = n_1 + k$ and $j = n_2 + k$ for some $n_1, n_2 \in \mathbb{N}$. Hence, by (3.4),

$$p(x_k, x_i) = p(x_k, x_{n_1+k}) < \delta_1 + \mu < \delta \quad \text{and} \quad p(x_k, x_j) = p(x_k, x_{n_2+k}) < \delta_1 + \mu < \delta.$$

It follows from Lemma 2.2 that $d^s(x_i, x_j) \leqslant \epsilon$ whenever $i, j > k$. We conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Since (X, d) is complete, there exists $a \in X$ such that $d(x_n, a) \to 0$ $(n \to \infty)$.

Next we show that $p(x_n, a) \to 0 (n \to \infty)$. In fact, choose an arbitrary $\epsilon > 0$. We have proved that there is $N_0 \in \mathbb{N}$ such that $p(x_k, x_{n+k}) < \epsilon$ for all $k \geqslant N_0$ and $n \in \mathbb{N}$. Fix $k \geqslant N_0$. Since $d(x_n, a) \to 0$ it follows from condition (W_2) that, for n sufficiently large,

$$p(x_k, a) < p(x_k, x_{n+k}) + \epsilon.$$

Hence $p(x_k, a) < 2\epsilon$ for all $k \geqslant N_0$. We deduce that $p(x_n, a) \to 0$ $(n \to \infty)$.

By condition (H) and (3.1), we obtain that

$$\psi(p(x_{n_k+1}, Ta)) = \psi(p(Tx_{n_k}, Ta)) \leqslant \alpha(x_{n_k}, a)\psi(p(Tx_{n_k}, Ta)) \leqslant \phi(D(x_{n_k}, a)) \to 0 \ (k \to \infty).$$

Thus, $p(x_{n_k+1}, Ta) \to 0$ $(k \to \infty)$. So $d^s(a, Ta) = 0$.

□

Theorem 3.2. *Let T be a triangular α-admissible mapping of a complete quasi-metric space (X, d). Suppose that there exists a w-distance p on (X, d) such that*

$$\alpha(x, y)\psi(p(Tx, Ty)) \leqslant \phi(p(x, y)) \tag{3.6}$$

for all $x,\ y \in X$, where ψ is a generalized altering distance function and $\phi : [0, +\infty) \to [0, +\infty)$ is a Jachymski function such that $\phi(t) < \psi(t)$ for all $t > 0$. Furthermore, the condition (H) *is satisfied. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, then T has a fixed point.*

Remark 3.3. T is said to be an α-ψ-ϕ–Jachymski contractive mapping if T satisfies the contractive condition (3.6). Since each Meir-Keeler function is a Jachymski function, we can know that our new concept generalizes and extends Definition 1.4.

Proof of Theorem 3.2. From the proof procedure of Theorem 3.1, we only need to show that $r_n \to 0$ $(n \to \infty)$.

If there is $n_0 \in \mathbb{N}$ such that $r_{n_0} = 0$, then $r_n = 0$ for all $n \geqslant n_0$, by (3.6), Definition 2.5, and our assumption that $\phi(0) = 0$. Therefore $p(x_i, x_j) = 0$ whenever $j > i \geqslant n_0$ by condition (W_1), and consequently, $d^s(x_i, x_j) = 0$ by Lemma 2.2. Otherwise, noting the increasing property of ψ, we have that the sequence $\{r_n\}$ is non-increasing and, consequently, there exists $r \geqslant 0$ such that $r_n \to r^+$ as $n \to \infty$. If $r > 0$ there exists $\delta = \delta(r)$ such that $\psi(r) < t < \psi(r) + \delta$ implies $\phi(t) \leqslant \psi(r)$. Take $N_\delta \in \mathbb{N}$ such that $r_n < \psi(r) + \delta$ for all $n \geqslant N_\delta$. Therefore $\phi(r_n) \leqslant \psi(r)$, so by (3.6), we derive that

$$\psi(r_{n+1}) \leqslant \phi(r_n) \leqslant \psi(r).$$

Since ψ is increasing, we have that $r_{n+1} \leqslant r$ for all $n \geqslant N_\delta$, a contradiction. Consequently, $r = 0$. □

We next discuss the condition for the uniqueness of the fixed point. A sufficient condition for the uniqueness of the fixed point in Theorem 3.2 can be stated as follows:

(U) For $u, v \in X$, there exists $w \in X$ such that $\alpha(w, u) \geqslant 1$ and $\alpha(w, v) \geqslant 1$.

Theorem 3.4. *If condition* (U) *is added to the hypotheses of Theorem* 3.2*, then the fixed point of T is unique.*

Proof. By Theorem 3.1, we have known that T has a fixed point z_1. Suppose that T has another fixed point z_2. In the following we shall show that $z_1 = z_2$.

From condition (U), there exists $w \in X$ such that $\alpha(w, z_1) \geqslant 1$ and $\alpha(w, z_2) \geqslant 1$. Then, since T is α-admissible, we obtain that

$$\alpha(T^n w, z_1) \geqslant 1, \quad \alpha(T^n w, z_2) \geqslant 1, \ \forall\, n \in \mathbb{N}.$$

Thus,

$$\psi(p(T^{n+1}w, z_1)) \leqslant \alpha(T^n w, z_1)\psi(p(T(T^n w), Tz_1)) \leqslant \phi(p(T^n w, z_1)).$$

Hence, the sequence $\{p(T^{n+1}w, z_1)\}$ is non-increasing and, consequently, there exists $L \geqslant 0$ such that $p(T^{n+1}w, z_1) \to L^+$, as $n \to \infty$. Similarly to the proof of $r = 0$, we can show that $p(T^{n+1}w, z_1) \to 0$ $(n \to \infty)$. In the same arguments, we have that $p(T^{n+1}w, z_2) \to 0$ $(n \to \infty)$. By Lemma 2.2, we deduce that $d^s(z_1, z_2) = 0$, i.e., $z_1 = z_2$. □

Theorem 3.5. *Let T be a triangular α-admissible mapping of a complete quasi-metric space (X, d) such that*

$$\alpha(x, y)\psi(d(Tx, Ty)) < \psi(M(x, y)) \ \textit{for any } x, y \in X, \tag{3.7}$$

$$\alpha(x, y)\psi(d(Tx, Ty)) \leqslant \phi(M(x, y)) \tag{3.8}$$

for all $x,\ y \in X$, where

$$M(x, y) := \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$

ψ is a generalized altering distance function, and $\phi : [0, +\infty) \to [0, +\infty)$ is a Jachymski function. Furthermore, the following condition is satisfied

(H′) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x \in X$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geqslant 1$ *and* $\alpha(x_{n+1}, x_n) \geqslant 1$ *for all* $n \geqslant 1$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *for which* $\alpha(x_{n_k}, x) \geqslant 1,\ k \geqslant 1$ *and* $\alpha(x, x_{n_k}) \geqslant 1,\ k \geqslant 1$.

If there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$ *and* $\alpha(Tx_0, x_0) \geqslant 1$*, then* T *has a fixed point.*

Proof. Take $x \in X$ and assume that $x \neq Tx$, we have that

$$\frac{1}{2}d(x, T^2x) \leqslant \frac{1}{2}[d(x, Tx) + d(Tx, T^2x)].$$

Hence, by (3.7), we may conclude that

$$\alpha(x, Tx)\psi(d(Tx, T^2x)) < \psi(M(x, Tx)) = \psi(d(x, Tx)). \tag{3.9}$$

Define $x_n := T^n x_0$ and $a_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$. We shall show that $a_n \to 0$ $(n \to \infty)$. We can restrict to the case that $a_n > 0$. Since $\alpha(T^n x_0, T^{n+1}x_0) \geqslant 1$, $n \in \mathbb{N}$, it follows from (3.9) and the increasing property of ψ that $a_{n+1} < a_n$, so $\{a_n\}$ converges to some a in $\mathbb{R}_+$. Assume that $a > 0$. There exists $\delta = \delta(a)$ such that $\psi(a) < t < \psi(a) + \delta$ implies $\phi(t) \leqslant \psi(a)$. Take $k \in \mathbb{N}$ such that $a_n < \psi(a) + \delta$ for all $n \geqslant k$. Therefore $\phi(a_n) \leqslant \psi(a)$, so by (3.8),

$$\psi(a_{n+1}) \leqslant \alpha(x_n, x_{n+1})\psi(a_{n+1}) \leqslant \phi(a_n) \leqslant \psi(a).$$

Note that ψ is increasing, we obtain that $a_{n+1} \leqslant a$ for all $n \geqslant k$, a contradiction. Consequently, $a = 0$. In the same arguments, we can prove that $\widetilde{a}_n = d(x_{n+1}, x_n) \to 0$ $(n \to \infty)$.

Now, fix an $\epsilon > 0$. There exists $\delta = \delta(\epsilon)$, with $0 < \delta < \epsilon$, for which ϕ is a Jachymski function, i.e., for each $t > 0$, $\epsilon < t < \delta + \epsilon$ implies $\phi(t) \leqslant \psi(\epsilon)$. Since $a_n \to 0$, there exists i in $\mathbb{N}$ such that $a_n < \frac{1}{2}\delta$ for $n \geqslant i \geqslant k$. We shall apply induction to show that, for any $n \in \mathbb{N}$,

$$d(x_i, x_{i+n}) < \epsilon + \frac{1}{2}\delta. \tag{3.10}$$

Obviously, (3.10) holds for $n = 1$. Assume that (3.10) holds for some n, we shall prove it for $n + 1$. Similar to the proof of Theorem 3.1, we only consider the case that $\epsilon < M(x_i, x_{i+n})$. It suffices to show that $M(x_i, x_{i+n}) < \epsilon + \delta$. By the definition of $M(x, y)$,

$$M(x_i, x_{i+n}) = \max\left\{d(x_i, x_{i+n}), a_i, a_{i+n}, \frac{1}{2}[d(x_i, x_{i+n+1}) + d(x_{i+1}, x_{i+n})]\right\}.$$

By the induction hypothesis and the definition of a_i, we get

$$d(x_i, x_{i+n}) < \epsilon + \frac{1}{2}\delta, \quad a_i < \frac{1}{2}\delta, \quad \widetilde{a}_i < \frac{1}{2}\delta, \quad a_{i+n} < \frac{1}{2}\delta.$$

Hence we have

$$\frac{1}{2}[d(x_i, x_{i+n+1}) + d(x_{i+1}, x_{i+n})] \leqslant \frac{1}{2}[d(x_i, x_{i+n}) + a_{i+n} + \widetilde{a}_i + d(x_i, x_{i+n})] < \epsilon + \delta.$$

Thus, $M(x_i, x_{i+n}) < \epsilon + \delta$. Since $\alpha(x_i, x_{i+n}) \geqslant 1$ $(n = 1, 2, 3, \cdots)$, we deduce that

$$\psi(d(x_{i+1}, x_{n+i+1})) = \psi(d(Tx_i, Tx_{n+i})) \leqslant \alpha(x_i, x_{n+i})\psi(d(Tx_i, Tx_{n+i})) \leqslant \phi(M(x_i, x_{n+i})) \leqslant \psi(\epsilon).$$

Since ψ is non-decreasing, it follows that $d(x_{i+1}, x_{n+i+1}) \leqslant \epsilon$. Therefore

$$d(x_i, x_{n+1+i}) \leqslant d(x_i, x_{i+1}) + d(x_{i+1}, x_{n+i+1}) < \frac{1}{2}\delta + \epsilon.$$

Obviously, (3.10) implies that $\{x_n\}$ is a Cauchy sequence. The remainder of proof is similar to that of Theorem 3.1, we omit the detail. □

Example 3.6. Let d be the quasi-metric on $\mathbb{R}^+$ given by $d(x,y) = \max\{y-x, 0\}$ for all $x,\ y \in \mathbb{R}^+$. Since d^s is the usual metric on $\mathbb{R}^+$ it immediately follows that $(\mathbb{R}^+, d)$ is complete.

Define $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as $p(x,y) = y$. It is clear that p is a w-distance on $(\mathbb{R}^+, d)$. Define T by

$$Tx = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{t}{3}, & \frac{1}{2} < x \leqslant 1, \\ 6x - \frac{1}{3}, & x > 1. \end{cases}$$

Choose

$$\phi(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{t}{3}, & \frac{1}{2} < x \leqslant 1, \\ 1, & x > 1, \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1, & x,\ y \in [0,1], \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \psi(x) = x.$$

If $x,\ y \in [0,1]$,

$$\alpha(x,y)p(Tx,Ty) = Ty = \phi(y) = \phi(p(x,y)).$$

If $x,\ y \notin [0,1]$,

$$\alpha(x,y)p(Tx,Ty) = 0 \leqslant \phi(p(x,y)).$$

In the following, we shall prove that ϕ is a Jachymski function. Now, given $\epsilon > 0$, we distinguish the following cases.

(1) if $0 < \epsilon < \frac{1}{2}$, we take $\delta = \frac{1}{2} - \epsilon$, and thus, from $\epsilon < t < \epsilon + \delta = \frac{1}{2}$, it follows $\phi(t) = 0 < \epsilon$;
(2) if $\epsilon = \frac{1}{2}$, we take $\delta = \frac{1}{3}$, and thus, from $\epsilon < t < \epsilon + \delta$, it follows $\phi(t) = \frac{t}{3} < \frac{1}{2} = \epsilon$;
(3) if $\frac{1}{2} < \epsilon < 1$, we take $\delta = 1 - \epsilon$, and thus, from $\epsilon < t < \epsilon + \delta = 1$, it follows $\phi(t) = \frac{t}{3} < \frac{1}{3} < \epsilon$;
(4) if $\epsilon \geqslant 1$, we take $\delta = \frac{1}{2}$, and thus, from $\epsilon < t < \epsilon + \frac{1}{2}$, it follows $\phi(t) = 1 \leqslant \epsilon$.

Therefore, all conditions of Theorem 3.2 are satisfied.

However, the contractive condition (1.1) is not satisfied. Indeed, if no, for any function $\phi : [0,\infty) \to [0,+\infty)$ with $\phi(t) < t$ for all $t > 0$, we have

$$p(T2, T3) = T3 = \frac{53}{3} \leqslant \phi(p(2,3)) < p(2,3) = 3,$$

which is a contradiction.

Example 3.7. Define a complete metric space (X, d) by $X = [0, +\infty)$ and $d(x,y) = x + y$ for $x,\ y \in X$ with $x \neq y$. Define

$$Tx = \begin{cases} 0, & 0 \leqslant x \leqslant 1, \\ 1, & 1 < x \leqslant 2, \\ 5x - 1, & x > 2, \end{cases} \qquad \psi(t) = \begin{cases} t, & t \leqslant 1, \\ \frac{3}{2} + t, & t > 1, \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1, & x,\ y \in [0,2], \\ 0, & \text{otherwise}, \end{cases} \qquad \phi(t) = \begin{cases} \frac{t}{2}, & t \leqslant 1, \\ \frac{3}{2} + \frac{t}{2}, & t > 1. \end{cases}$$

It is easy to check that

$$\alpha(x,y)\psi(d(Tx,Ty)) \leqslant \phi(d(x,y)), \quad x,\ y \in [0,2].$$

In fact, $\{d(Tx,Ty) : x,\ y \in [0,2]\} = \{0,1\}$, and hence $\{\psi(d(Tx,Ty)) : x,\ y \in [0,2]\} = \{0,1\}$. If $\psi(d(Tx,Ty)) = 1$ and $x > y$, then $x > 1$ and $0 \leqslant y \leqslant 1$, and hence $\phi(d(x,y)) = \phi(x+y) > \phi(x) > 2$. Therefore,

$$\alpha(x,y)\psi(d(Tx,Ty)) \leqslant 1 < \frac{3}{2} + \frac{x+y}{2}, \quad x,\ y \in [0,2].$$

Then all the assumptions of Theorem 3.2 are satisfied. From [10], we have known that T is not a Meir-Keeler contractive mapping. Furthermore, ψ is non-decreasing and

$$\psi(d(T2,T3)) = \psi(16) \leqslant \phi(d(2,3)) = \phi(5) < \psi(5),$$

which is a contradiction. Thus, the condition (C_4) of Theorem 2.7 in [26] is not satisfied. We can know that Theorem 2.7 in [26] cannot be applied to this example.

Example 3.8. Let $X = [0, +\infty)$. Let $d(x, y)$ be defined as follows:

$$d(x,y) := \max\{x,y\} \text{ for } x \neq y, \quad \text{and} \quad d(x,y) := 0 \text{ for } x = y.$$

Then (X, d) is a complete metric space. Define T by

$$Tx = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{n+2}, & \text{if } \frac{1}{n+1} < x \leqslant \frac{1}{n}, \\ 1, & \text{if } 1 < x \leqslant 2, \\ 3x^2 - \frac{1}{2}, & \text{if } x > 2. \end{cases}$$

First, we prove that all conditions of Theorem 3.5 are satisfied. Take

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,\ y \in [0,2], \\ 0, & \text{otherwise}, \end{cases}$$

$\psi(x) = x$, $\phi(x) = Tx$. For $x \in (2, +\infty)$, there exists $\delta = \sqrt{\frac{2\epsilon+1}{6}} - \epsilon > 0$ such that $\epsilon < x < \epsilon + \delta$ implies $3x^2 - \frac{1}{2} \leqslant \epsilon$. Thus, ϕ is a Jachymski function.

In the following, we shall show that T is not an α-admissible-Meir-Keeler contractive mapping.

For any $\delta > 0$ $(\delta < 1)$, there exists $\epsilon = 1$, such that

$$1 \leqslant \alpha(x,y) d\left(1, 1 + \frac{\delta}{2}\right) < 1 + \delta, \quad x,\ y \in [0,2]$$

implies

$$\alpha(x,y) d\left(T1, T(1 + \frac{\delta}{2})\right) = 1 = \epsilon, \quad x,\ y \in [0,2].$$

Therefore, the contractive condition (1.2) is not satisfied.

The next is an example where we can apply Theorem 3.1 for an appropriate w-distance p on a complete quasi-metric space (X, d) but not for d.

Example 3.9. Let $X = \mathbb{N}$ and let d be the quasi-metric on X defined as

$$d(x,y) = \begin{cases} 0, & x = y, \text{ and } x,\ y \in X, \\ \frac{1}{x}, & x \in X\backslash\{0\}, \text{ and } y = 0, \\ 1, & x = 0, \text{ and } y \in X\backslash\{0\}, \\ |\frac{1}{x} - \frac{1}{y}|, & x,\ y \in X\backslash\{0\}. \end{cases}$$

Clearly (X, d) is complete (observe that $\{n\}$ is a Cauchy sequence in (X, d^s) with $d(n, 0) \to 0$). Let p be the w-distance on (X, d) given by $p(x, y) = y$ for all $x,\ y \in X$. Now define $T : X \to X$ as

$$Tx = \begin{cases} 0, & x = 0, \\ x - 1, & x \in \{1,\ 2,\ 3, \cdots, 2016\}, \\ x^3, & x \in X\backslash\{0,\ 1,\ 2,\ 3, \cdots,\ 2016\}, \end{cases} \qquad \psi(x) = \frac{1008x}{10075}, \quad \phi(x) = \frac{x}{10},$$

and

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,\ y \in \{0,\ 1,\ 2,\ \cdots, 2016\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is routine to check that ϕ is a Jachymski function satisfying $\phi(t) < \psi(t)$ for all $t > 0$. Since $p(Tx, T0) = 0$ for all $x \in X$, and for each $x,\ y \in X$ with $y \neq 0$, we have

$$\alpha(x,y)\psi(p(fx,fy)) = \frac{1008}{10075}(y-1) \leqslant \frac{1}{10}y = \phi(p(x,y)), \quad x,\ y \in \{1,\ 2,\ \cdots,\ 2016\},$$

it follows that all conditions of Theorem 3.1 are satisfied. In fact $x = 0$ is the unique fixed point of T.

However, the contractive condition 3.1 is not satisfied for d. Indeed, for any $y \in \{2,\ 3, \cdots, 2016\}$, we have

$$\alpha(0,n)\psi(d(T0,Ty)) = \psi(d(0,y-1)) = \frac{1008}{10075} > \frac{1}{10} = \phi(d(0,y)).$$

Therefore, Theorem 2.16 in [15] cannot be applied to operator T.

In the following, we utilize Theorem 3.2 to obtain common fixed points results for two mappings with α-admissible conditions.

Theorem 3.10. *Let T be a triangular S-α-admissible mapping of a complete quasi-metric space (X,d) such that $T(X) \subset S(X)$ and $\alpha(Sx,Sy) \leqslant \alpha(x,y)$ for all $x,\ y \in X$. Suppose that the following conditions are satisfied:*

(i) *there exists a w-distance p on (X,d) such that*

$$\alpha(x,y)\psi(p(Tx,Ty)) \leqslant \phi(p(Sx,Sy)) \tag{3.11}$$

for all $x,\ y \in X$, where ψ is a generalized altering distance function and $\phi : [0,+\infty) \to [0,+\infty)$ is a Jachymski function such that $\phi(t) < \psi(t)$ for all $t > 0$;

(ii) *there exists $x_0 \in X$ such that $\alpha(Sx_0, Tx_0) \geqslant 1$;*

(iii) *if every sequence $\{x_n\} \subseteq X$ such that $\alpha(Sx_n, Sx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $Sx_n \to Sx \in SX$ as $n \to \infty$, then there exists a subsequence $\{Sx_{n_k}\}$ of $\{Sx_n\}$ such that for all $k \in \mathbb{N}$, $\alpha(Sx_{n_k}, Sx) \geqslant 1$.*

If $S(X)$ is a complete subspace of X, then S and T have a coincidence point. Moreover, suppose that

(iv) *for all coincidence points z of S and T, $\alpha(Tz,z) \geqslant 1$, and S and T are weakly compatible,*

then S and T have a common fixed point in X.

Proof. By Lemma 2.9, there exists a subset Y of X such that $S(Y) = S(X)$ and $S : Y \to X$ is one-to-one.

Define a map $A : S(Y) \to S(Y)$ by $A(Sx) = Tx$. Then A is well-defined, because A is one-to-one.

From (3.11) we have

$$\alpha(Sx,Sy)\psi(p(A(Sx),A(Sy))) \leqslant \alpha(x,y)\psi(p(Tx,Ty)) \leqslant \phi(p(Sx,Sy))$$

for all $Sx,\ Sy \in S(Y)$. Hence, A is a α-ψ-ϕ-contractive type mapping on $S(X)$.

In addition, $\alpha(Sx_0, Tx_0) \geqslant 1$ implies that $\alpha(Sx_0, A(Sx_0)) \geqslant 1$, and condition (iv) implies that condition (H) holds. By Theorem 3.2, A has a fixed point in $S(X)$. That is, there exists $\tilde{x} \in X$ such that $A(S\tilde{x}) = S(\tilde{x})$. By definition of A, $S(\tilde{x}) = T(\tilde{x})$. Thus, $\tilde{x}$ is a coincidence point of S and T.

We now show the existence of common fixed points of S and T with their weak compatibility. Assume that condition (iv) holds. Let $z = S(\tilde{x}) = T(\tilde{x})$. Then $Sz = Tz$. Since $\alpha(z,\tilde{x}) = \alpha(T\tilde{x},\tilde{x}) \geqslant 1$, from (3.11) we have

$$\psi(p(Tz,z)) = \psi(p(Tz,T\tilde{x})) \leqslant \alpha(z,\tilde{x})\psi(p(Tz,T\tilde{x})) \leqslant \phi(p(Sz,S\tilde{x})) = \phi(p(Tz,z)).$$

If $p(Tz,z) > 0$, it follows that

$$\psi(p((Tz,z)) \leqslant \phi(p(Tz,z)) < \psi(p(Tz,z)),$$

which is a contradiction. Hence, $p(Tz,z) = 0$. Thus $Tz = z = Sz$, and z is a common fixed point. □

4. Multidimensional fixed point theorems

Consider the following conditions:

(i) for all $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in X^n$, we have

$$\alpha(\mathbb{X}, \mathbb{Y}) \geqslant 1 \Longrightarrow \alpha(T\mathbb{X}, T\mathbb{Y}) \geqslant 1,$$

and

$$\alpha(\mathbb{X}, \mathbb{Y}) \geqslant 1 \text{ and } \alpha(\mathbb{Y}, \mathbb{Z}) \geqslant 1 \text{ implies } \alpha(\mathbb{X}, \mathbb{Z}) \geqslant 1,$$

where

$$\mathbb{X} = (x_1, x_2, \cdots, x_n), \quad \mathbb{Y} = (y_1, y_2, \cdots, y_n), \quad \mathbb{Z} = (z_1, z_2, \cdots, z_n),$$

and T is defined by

$$\begin{aligned} T(x_1, x_2, \cdots, x_n) = (&F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \cdots, x_{\sigma_1(n)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \cdots, x_{\sigma_2(n)}), \cdots \\ &, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \cdots, x_{\sigma_n(n)})); \end{aligned}$$

(ii) there exists $(z_1, z_2, \cdots, z_n) \in X^n$ such that

$$\begin{aligned} \alpha\Big(&(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \cdots, z_{\sigma_i(n)}), \big(F_{\sigma_i(1)}(z_{\sigma_1(1)}, z_{\sigma_1(2)}, \cdots, z_{\sigma_1(n)}), F_{\sigma_i(2)}(z_{\sigma_2(1)}, z_{\sigma_2(2)}, \cdots, z_{\sigma_2(n)}), \\ &\cdots, F_{\sigma_i(n)}(z_{\sigma_n(1)}, z_{\sigma_n(2)}, \cdots, z_{\sigma_n(n)})\big)\Big) \geqslant 1, \quad i = 1,\ 2,\ 3,\ \cdots, n; \end{aligned}$$

(iii) if $\{(w_1)_m\}$, $\{(w_2)_m\}$, $\cdots$, $\{(w_n)_m\}$ are sequences in X such that

$$\begin{aligned} \alpha(&((w_{\sigma_i(1)})_m, (w_{\sigma_i(2)})_m, (w_{\sigma_i(3)})_m, \cdots, (w_{\sigma_i(n)})_m), \\ &((w_{\sigma_i(1)})_{m+1}, (w_{\sigma_i(2)})_{m+1}, (w_{\sigma_i(3)})_{m+1}, \cdots, (w_{\sigma_i(n)})_{m+1})) \geqslant 1, \quad i = 1,\ 2,\ 3,\ \cdots, n, \end{aligned}$$

$(w_1)_m \to w_1 \in X$, $(w_2)_m \to w_2 \in X$, $\cdots$, $(w_n)_m \to w_n \in X$ ($n \to \infty$), then

$$\begin{aligned} \alpha\big(&((w_{\sigma_i(1)})_m, (w_{\sigma_i(2)})_m, (w_{\sigma_i(3)})_m, \cdots, (w_{\sigma_n(n)})_m,), \\ &(w_{\sigma_i(1)}, w_{\sigma_i(2)}, w_{\sigma_i(3)}, \cdots, w_{\sigma_i(n)})\big) \geqslant 1, \quad i = 1,\ 2,\ 3, \cdots n \end{aligned}$$

for all $m \in \mathbb{N}$.

Theorem 4.1. *Let (X, d) be a complete metric space and $F : X^n \to X$. Suppose that there exists a function $\alpha : X^n \times X^n \to [0, +\infty)$ such that*

$$\alpha(\mathbb{X}, \mathbb{Y})\psi\left(\frac{\sum\limits_{i=1}^{n} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \cdots, y_{\sigma_i(n)}))}{n}\right) < \psi\left(\frac{\sum\limits_{i=1}^{n} d(x_i, y_i)}{n}\right) \tag{4.1}$$

for $x_i \neq y_i$, $i = 1, 2, \cdots, n$,

$$\alpha(\mathbb{X}, \mathbb{Y})\psi\left(\frac{\sum\limits_{i=1}^{n} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \cdots, y_{\sigma_i(n)}))}{n}\right) \leqslant \phi\left(\frac{\sum\limits_{i=1}^{n} d(x_i, y_i)}{n}\right) \tag{4.2}$$

for all $\mathbb{X}, \mathbb{Y} \in X^n$, where ψ is a generalized altering distance function and $\phi : [0, +\infty) \to [0, +\infty)$ is a Jachymski function. Furthermore, conditions (i)-(iii) *are satisfied. Then F has a Υ-fixed point.*

Proof. The idea consists in transporting the problem to the complete metric space (Y,δ), where $Y = X^n$, and $\delta(\mathbb{X},\mathbb{Y}) = \frac{1}{n}\sum_{i=1}^{n} d(x_i,y_i)$ for all $\mathbb{X},\mathbb{Y} \in X^n$. By (4.2), we have

$$\begin{aligned}
\alpha((x_{\sigma_1(1)},x_{\sigma_1(2)},\cdots,x_{\sigma_1(n)}),(y_{\sigma_1(1)},y_{\sigma_1(2)},\cdots,y_{\sigma_1(n)}))\psi(\delta(T\mathbb{X},T\mathbb{Y})) &\leqslant \phi(\delta(\mathbb{X},\mathbb{Y})),\\
\alpha((x_{\sigma_2(1)},x_{\sigma_2(2)},\cdots,x_{\sigma_2(n)}),(y_{\sigma_2(1)},y_{\sigma_2(2)},\cdots,y_{\sigma_2(n)}))\psi(\delta(T\mathbb{X},T\mathbb{Y})) &\leqslant \phi(\delta(\mathbb{X},\mathbb{Y})),\\
&\vdots\\
\alpha((x_{\sigma_n(1)},x_{\sigma_n(2)},\cdots,x_{\sigma_n(n)}),(y_{\sigma_n(1)},y_{\sigma_n(2)},\cdots,y_{\sigma_n(n)}))\psi(\delta(T\mathbb{X},T\mathbb{Y})) &\leqslant \phi(\delta(\mathbb{X},\mathbb{Y})).
\end{aligned}$$

Denote $\beta : Y \times Y \to [0,+\infty)$ by

$$\begin{aligned}
\beta(\mathbb{X},\mathbb{Y}) = \min\{&\alpha((x_{\sigma_1(1)},x_{\sigma_1(2)},\cdots,x_{\sigma_1(n)}),(y_{\sigma_1(1)},y_{\sigma_1(2)},\cdots,y_{\sigma_1(n)}))\\
&,\alpha((x_{\sigma_2(1)},x_{\sigma_2(2)},\cdots,x_{\sigma_2(n)}),(y_{\sigma_2(1)},y_{\sigma_2(2)},\cdots,y_{\sigma_2(n)})),\cdots\\
&,\alpha((x_{\sigma_n(1)},x_{\sigma_n(2)},\cdots,x_{\sigma_n(n)}),(y_{\sigma_n(1)},y_{\sigma_n(2)},\cdots,y_{\sigma_n(n)}))\}.
\end{aligned} \tag{4.3}$$

Subsequently, we deduce that

$$\beta(\mathbb{X},\mathbb{Y})\psi(\delta(T\mathbb{X},T\mathbb{Y})) \leqslant \phi(\delta(\mathbb{X},\mathbb{Y})).$$

Then T is a β-ψ-ϕ-Jachymski contractive mapping. Using condition (i), we know that T is β-admissible. Moreover, from condition (ii), we obtain that there exists $(z_1,z_2,\cdots,z_n) \in Y$ such that

$$\beta((z_1,z_2,\cdots,z_n),T(z_1,z_2,\cdots,z_n)) \geqslant 1.$$

Let $\{(w_1)_m,(w_2)_m,\cdots,(w_n)_m\}$ be a sequence in Y such that

$$\beta\left(((w_1)_m,(w_2)_m,\cdots,(w_n)_m),((w_1)_{m+1},(w_2)_{m+1},\cdots,(w_n)_{m+1})\right) \geqslant 1$$

and

$$((w_1)_m,(w_2)_m,\cdots,(w_n)_m) \to (w_1,w_2,\cdots,w_n)\ (m\to\infty).$$

It follows from condition (iii) that

$$\beta\left(((w_1)_m,(w_2)_m,\cdots,(w_n)_m),(w_1,w_2,\cdots,w_n)\right) \geqslant 1.$$

All the hypotheses of Theorem 3.5 are satisfied, and so we derive the existence of a fixed point of T that gives us the existence of Υ-fixed point of F. ☐

Theorem 4.2. *Let (X,d) be a complete metric space and $F : X^n \to X$. Suppose that there exists a function $\alpha : X^n \times X^n \to [0,+\infty)$ such that*

$$\alpha(\mathbb{X},\mathbb{Y})\psi(d(F\mathbb{X},F\mathbb{Y})) < \psi\left(\max_{1\leqslant i\leqslant n} d(x_i,y_i)\right) \tag{4.4}$$

for $x_i \neq y_i$, $i = 1,2,\cdots,n$,

$$\alpha(\mathbb{X},\mathbb{Y})\psi(d(F\mathbb{X},F\mathbb{Y})) \leqslant \phi\left(\max_{1\leqslant i\leqslant n} d(x_i,y_i)\right)$$

for all $\mathbb{X},\mathbb{Y} \in X^n$, where ψ is a generalized altering distance function and $\phi : [0,+\infty) \to [0,+\infty)$ is a Jachymski function. Furthermore, conditions (i)-(iii) *are satisfied. Then F has a Υ-fixed point.*

Proof. From (4.3) and (4.4), for all $\mathbb{X},\mathbb{Y} \in X^n$, we have

$$\alpha((x_{\sigma_1(1)},x_{\sigma_1(2)},\cdots,x_{\sigma_1(n)}),(y_{\sigma_1(1)},y_{\sigma_1(2)},\cdots,y_{\sigma_1(n)})),$$

$$
\begin{aligned}
&\cdot\psi\left(d(F(x_{\sigma_1(1)},x_{\sigma_1(2)},\cdots,x_{\sigma_1(n)}),F(y_{\sigma_1(1)},y_{\sigma_1(2)},\cdots,y_{\sigma_1(n)}))\right)\leqslant\varphi\left(\max_{1\leqslant i\leqslant n}d(x_i,y_i)\right),\\
&\alpha((x_{\sigma_2(1)},x_{\sigma_2(2)},\cdots,x_{\sigma_2(n)}),(y_{\sigma_2(1)},y_{\sigma_2(2)},\cdots,y_{\sigma_2(n)})),\\
&\cdot\psi\left(d(F(x_{\sigma_2(1)},x_{\sigma_2(2)},\cdots,x_{\sigma_2(n)}),F(y_{\sigma_2(1)},y_{\sigma_2(2)},\cdots,y_{\sigma_2(n)}))\right)\leqslant\varphi\left(\max_{1\leqslant i\leqslant n}d(x_i,y_i)\right),\\
&\vdots\\
&\alpha((x_{\sigma_n(1)},x_{\sigma_n(2)},\cdots,x_{\sigma_n(n)}),(y_{\sigma_n(1)},y_{\sigma_n(2)},\cdots,y_{\sigma_n(n)})),\\
&\cdot\psi\left(d(F(x_{\sigma_n(1)},x_{\sigma_n(2)},\cdots,x_{\sigma_n(n)}),F(y_{\sigma_n(1)},y_{\sigma_n(2)},\cdots,y_{\sigma_n(n)}))\right)\leqslant\varphi\left(\max_{1\leqslant i\leqslant n}d(x_i,y_i)\right).
\end{aligned}
$$

This implies (since ψ is nondecreasing) that for all $\mathbb{X},\mathbb{Y}\in X^n$,

$$
\begin{aligned}
\beta(\mathbb{X},\mathbb{Y})\psi\big(\max\{&d(F(x_{\sigma_1(1)},x_{\sigma_1(2)},\cdots,x_{\sigma_1(n)}),F(y_{\sigma_1(1)},y_{\sigma_1(2)},\cdots,y_{\sigma_1(n)})),\\
&d(F(x_{\sigma_2(1)},x_{\sigma_2(2)},\cdots,x_{\sigma_2(n)}),F(y_{\sigma_2(1)},y_{\sigma_2(2)},\cdots,y_{\sigma_2(n)})),\\
&\vdots\\
&d(F(x_{\sigma_n(1)},x_{\sigma_n(2)},\cdots,x_{\sigma_n(n)}),F(y_{\sigma_n(1)},y_{\sigma_n(2)},\cdots,y_{\sigma_n(n)}))\}\big)\leqslant\varphi\left(\max_{1\leqslant i\leqslant n}d(x_i,y_i)\right),
\end{aligned}
$$

where β is defined in (4.3). Denote $\eta : Y^n\to[0,+\infty)$ as

$$\eta(\mathbb{X},\mathbb{Y})=\max_{1\leqslant i\leqslant n}d(x_i,y_i).$$

Thus,

$$\beta(\mathbb{X},\mathbb{Y})\psi(\eta(T\mathbb{X},T\mathbb{Y}))\leqslant\varphi(\eta(\mathbb{X},\mathbb{Y})).$$

Therefore, we prove that the mapping T satisfies the conditions of Theorem 3.5. The rest of the proof is similar to the above proof. □

Theorem 4.3. *Assume that* (H) *is satisfied. Let* $T : X\to X$ *be a nondecreasing and asymptotically regular mapping of a partially ordered complete quasi-metric space* $(X,d,\preceq)$. *Suppose that there exists a w-distance* p *on* (X,d) *such that*

$$\psi(p(Tx,Ty))\leqslant\varphi(D(x,y))$$

for all $x,\ y\in X$ *with* $x\preceq y$, *where* $D(x,y)$ *is defined in* (3.2), ψ *is a generalized altering distance function, and* $\varphi : [0,+\infty)\to[0,+\infty)$ *is a Jachymski function such that* $\varphi(t)<\psi(t)$ *for all* $t>0$. *Suppose that there exists* $x_0\preceq Tx_0$ *and the following condition holds:*

(R) *for every nondecreasing sequence* $\{x_n\}\subset X$ *which converges to* $x\in X$, *there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *satisfying* $x_{n_k}\preceq x$.

Then T *has a fixed point.*

Proof. Define the mapping $\alpha : X\times X\to[0,+\infty)$ as

$$\alpha(x,y)=\begin{cases}1, & \text{if } x\preceq y \text{ or } y\preceq x,\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that T satisfies

$$\alpha(x,y)\psi(p(Tx,Ty))\leqslant\varphi(D(x,y)),\quad\forall\ x,\ y\in X.$$

Let $x_0\in X$ satisfy $x_0\preceq Tx_0$. Then, $\alpha(x_0,Tx_0)\geqslant 1$. On the other hand, since T is nondecreasing, then T is α-admissible. Indeed, $\alpha(x,y)\geqslant 1\Longrightarrow x\preceq y$ or $y\preceq x\Longrightarrow Tx\preceq Ty$ or $Ty\preceq Tx\Longrightarrow\alpha(Tx,Ty)\geqslant 1$. Note also that if $x \preceq Tx$ then $Tx \preceq T^2x$, and hence $x \preceq T^2x$; that is, if $\alpha(x, Tx) \geqslant 1$ then $\alpha(Tx, T^2x) \geqslant 1$ and $\alpha(x, T^2x) \geqslant 1$. Similar conclusion can be done if $x \succeq Tx$. Therefore, T is weak triangular α-admissible. By condition (R), every nondecreasing sequence $\{x_n\}$ which converges to x has a subsequence $\{x_{n_k}\}$ for which $x_{n_k} \preceq x$ holds for all $k \in \mathbb{N}$. Hence, $\alpha(x_n, x_{n+1}) \geqslant 1$ implies $\alpha(x_{n_k}, x) \geqslant 1$ for all $k \in \mathbb{N}$. In other words, the set X satisfies condition (H). By Theorem 3.1, the mapping T has a fixed point. □

The uniqueness of a fixed point on partially ordered metric spaces requires an additional assumption on the set X. This assumption reads as follows.

(C) For all x, $y \in X$ there exists $z \in X$ which is comparable to both x and y.

Theorem 4.4. *Adding condition* (C) *to the hypotheses of Theorem 4.3 one obtains the uniqueness of the fixed point.*

Proof. The proof is trivial, here we omit the detail. The readers are referred to the proof of Theorem 20 in [4]. □

Theorem 4.5. *Let* (X, d) *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is completely metric space. Let* $F : X^n \to X$ *be a mapping having the mixed monotone property. Assume that the following contractive conditions are satisfied*

$$\psi\left(\frac{\sum_{i=1}^{n} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \cdots, y_{\sigma_i(n)}))}{n}\right) < \psi\left(\frac{\sum_{i=1}^{n} d(x_i, y_i)}{n}\right)$$

for $x_i \neq y_i$, $i = 1, 2, \cdots, n$ *with* $x_i \preceq_i y_i$ *for all* $i \in \Lambda_i$,

$$\psi\left(\frac{\sum_{i=1}^{n} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \cdots, y_{\sigma_i(n)}))}{n}\right) \leqslant \phi\left(\frac{\sum_{i=1}^{n} d(x_i, y_i)}{n}\right)$$

for which $x_i \preceq_i y_i$ *for all* $i \in \Lambda_n$, *where* ψ *is a generalized altering distance function and* $\phi : [0, +\infty) \to [0, +\infty)$ *is a Jachymski function. Suppose that* $(X, d, \preceq)$ *has the sequential monotone property. Suppose that there exist* x_0^1, x_0^2, $\cdots, x_0^n \in X$ *verifying*

$$x_0^i \preceq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \cdots, x_0^{\sigma_i(n)}) \tag{4.5}$$

for all $i \in \Lambda_n$. *Furthermore, if for all* $A, B \in X^n$ *there exists* $U \in X^n$ *such that* $A \sqsubseteq U$ *and* $B \sqsubseteq U$, *then* F *has a unique* Υ*-fixed point.*

Proof. Define the mapping $\alpha : X^n \times X^n \to [0, +\infty)$ as

$$\alpha(\mathbb{X}, \mathbb{Y}) = \begin{cases} 1, & \text{if } x_i \preceq y_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

It is easy to check that

$$\alpha(\mathbb{X}, \mathbb{Y})\psi\left(\frac{\sum_{i=1}^{n} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \cdots, y_{\sigma_i(n)}))}{n}\right) < \psi\left(\frac{\sum_{i=1}^{n} d(x_i, y_i)}{n}\right)$$

for $x_i \neq y_i$, $i = 1, 2, \cdots, n$ with $x_i \preceq_i y_i$ for all $i \in \Lambda_n$,

$$\alpha(\mathbb{X}, \mathbb{Y})\psi\left(\frac{\sum_{i=1}^{n} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \cdots, y_{\sigma_i(n)}))}{n}\right) \leqslant \phi\left(\frac{\sum_{i=1}^{n} d(x_i, y_i)}{n}\right)$$

for which $x_i \preceq_i y_i$ for all $i \in \Lambda_n$. It implies that F is an α-ψ-ϕ-Jachymski contractive mapping.

Let $\mathbb{X}$, $\mathbb{Y} \in X^n$ such that $\alpha(\mathbb{X}, \mathbb{Y}) \geqslant 1$. By the definition of α, this implies that $x_i \preceq_i y_i$ for all $i \in \Lambda_n$. Since F satisfies the mixed monotone property, we have that

$$\begin{aligned} &i \in A,\ x_i \preceq y_i \Longrightarrow F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}) \leqslant F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \cdots, y_{\sigma_i(n)}),\\ &i \in B,\ x_i \succeq y_i \Longrightarrow F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \cdots, x_{\sigma_i(n)}) \geqslant F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \cdots, y_{\sigma_i(n)}).\end{aligned}$$

By the definition of α, we deduce that $\alpha(T\mathbb{X}, T\mathbb{Y}) \geqslant 1$. It follows from (4.5) that

$$\begin{aligned} &\alpha\Big((x_0^{\sigma_1(1)}, x_0^{\sigma_1(2)}, \cdots, x_0^{\sigma_1(n)}), \Big(F_{\sigma_1(1)}(x_0^{\sigma_1(1)}, x_0^{\sigma_1(2)}, \cdots, x_0^{\sigma_1(n)}),\\ &F_{\sigma_1(2)}(x_0^{\sigma_2(1)}, x_0^{\sigma_2(2)}, \cdots, x_0^{\sigma_2(n)}), \cdots, F_{\sigma_1(n)}(x_0^{\sigma_n(1)}, x_0^{\sigma_n(2)}, \cdots, x_0^{\sigma_n(n)})\Big)\Big) \geqslant 1,\\ &\alpha\Big((x_0^{\sigma_2(1)}, x_0^{\sigma_2(2)}, \cdots, x_0^{\sigma_2(n)}), \Big(F_{\sigma_2(1)}(x_0^{\sigma_1(1)}, x_0^{\sigma_1(2)}, \cdots, x_0^{\sigma_1(n)}),\\ &F_{\sigma_2(2)}(x_0^{\sigma_2(1)}, x_0^{\sigma_2(2)}, \cdots, x_0^{\sigma_2(n)}), \cdots, F_{\sigma_2(n)}(x_0^{\sigma_n(1)}, x_0^{\sigma_n(2)}, \cdots, x_0^{\sigma_n(n)})\Big)\Big) \geqslant 1,\\ &\vdots\\ &\alpha\Big((x_0^{\sigma_n(1)}, x_0^{\sigma_n(2)}, \cdots, x_0^{\sigma_n(n)}), \Big(F_{\sigma_n(1)}(x_0^{\sigma_1(1)}, x_0^{\sigma_1(2)}, \cdots, x_0^{\sigma_1(n)}),\\ &F_{\sigma_n(2)}(x_0^{\sigma_2(1)}, x_0^{\sigma_2(2)}, \cdots, x_0^{\sigma_2(n)}), \cdots, F_{\sigma_n(n)}(x_0^{\sigma_n(1)}, x_0^{\sigma_n(2)}, \cdots, x_0^{\sigma_n(n)})\Big)\Big) \geqslant 1.\end{aligned}$$

That is, condition (ii) is satisfied.

In the following, we check that condition (iii) holds with α given by (4.6). Let $\{(c^i)_m\}$, $i \in \Lambda_n$ are n sequences in X such that $(c^1)_m \to c^1$, $(c^2)_m \to c^2$, $\cdots$, $(c^n)_m \to c^n$, $(m \to \infty)$. Suppose that

$$\begin{aligned} &\alpha\Big(\Big((c^{\sigma_1(1)})_m, (c^{\sigma_1(2)})_m, \cdots, (c^{\sigma_1(n)})_m\Big), \Big((c^{\sigma_1(1)})_{m+1}, (c^{\sigma_1(2)})_{m+1}, \cdots, (c^{\sigma_1(n)})_{m+1}\Big)\Big) \geqslant 1,\\ &\alpha\Big(\Big((c^{\sigma_2(1)})_m, (c^{\sigma_2(2)})_m, \cdots, (c^{\sigma_2(n)})_m\Big), \Big((c^{\sigma_2(1)})_{m+1}, (c^{\sigma_2(2)})_{m+1}, \cdots, (c^{\sigma_2(n)})_{m+1}\Big)\Big) \geqslant 1,\\ &\vdots\\ &\alpha\Big(\Big((c^{\sigma_n(1)})_m, (c^{\sigma_n(2)})_m, \cdots, (c^{\sigma_n(n)})_m\Big), \Big((c^{\sigma_n(1)})_{m+1}, (c^{\sigma_n(2)})_{m+1}, \cdots, (c^{\sigma_n(n)})_{m+1}\Big)\Big) \geqslant 1.\end{aligned}$$

By the definition of α, we deduce that

$$(c^{\sigma_i(n)})_m \preceq_i (c^{\sigma_i(n)})_{m+1} \text{ for all } i \in \Lambda_n.$$

That is, for $i \in A$, $\{(c^{\sigma_i(n)})_m\}$ is non-decreasing and $(c^{\sigma_i(n)})_m \to c^{\sigma_i(n)}(m \to \infty)$. Since $(X, d, \preceq)$ has sequential monotone property, we have that $(c^{\sigma_i(n)})_m \leqslant c^{\sigma_i(n)}$. Similarly, for $i \in B$, $\{(c^{\sigma_i(n)})_m\}$ is non-increasing and $(c^{\sigma_i(n)})_m \to c^{\sigma_i(n)}(m \to \infty)$. Thus $(c^{\sigma_i(n)})_m \geqslant c^{\sigma_i(n)}$. By the definition of α, we derive that

$$\alpha\Big(\Big((c^{\sigma_i(1)})_m, (c^{\sigma_i(2)})_m, \cdots, (c^{\sigma_i(n)})_m\Big), \Big(c^{\sigma_i(1)}, c^{\sigma_i(2)}, \cdots, c^{\sigma_i(n)}\Big)\Big) = 1.$$

□

Example 4.6. Let $X = [0, +\infty)$ endowed with the usual order $\preceq$ in $\mathbb{R}$ and the standard metric $d(x, y) = |x - y|$ for all x, $y \in X$. Define the continuous mapping $F : X \times X \to X$ by

$$F(x, y) = 5x - \frac{y}{7} \text{ for all } x,\ y \in X.$$

Now we prove that (3.13) of [25] is not satisfied. Indeed, assume that there exist a generalized altering distance function ψ and a Jachymski function ϕ with $\phi(t) < \psi(t)$ for all $t > 0$ such that

$$\psi\left(\frac{|5x - \frac{y}{7} - (5u - \frac{v}{7})| + |5y - \frac{x}{7} - (5v - \frac{u}{7})|}{2}\right) \leqslant \phi\left(\frac{|x - u| + |y - v|}{2}\right) \tag{4.7}$$

for all $x,\ y,\ u,\ v \in X$. Taking $x = 6,\ u = 5,\ y = 2,\ v = 1$ in (4.7), we have

$$\psi\left(\frac{25 - \frac{1}{7} + 5 - \frac{1}{7}}{2}\right) < \psi(1),$$

which is a contradiction. This implies that F does not satisfy (3.13) of [25].

Define the mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$ by

$$\alpha((x,y),(u,v)) = \begin{cases} \frac{1}{100}, & \text{if } (x,y,u,v) \neq (0,0,0,0), \\ 1, & \text{if } x = y = u = v = 0. \end{cases}$$

For all $x,\ y,\ u,\ v \in X$, we have

$$|5x - \frac{y}{7} - (5u - \frac{v}{7})| + |5y - \frac{x}{7} - (5v - \frac{u}{7})| \leqslant \frac{72}{7}\frac{|x-u| + |y-v|}{2}.$$

Taking $\psi(x) = \frac{1}{2}x,\ \phi(x) = \frac{1}{7}x$, for all $x \geqslant 0$, we get that (4.1) and (4.2) of our Theorem 4.1 are satisfied. Let $(x,y),(u,v) \in X \times X$ such that $\alpha((x,y),(u,v)) \geqslant 1$. This means that $x = y = u = v = 0$ and $F(x,y) = F(y,x) = F(u,v) = F(v,u) = 0$. It follows that $\alpha((F(x,y),F(y,x)),(F(u,v),F(v,u))) = 1$. Then conditions (i) and (iii) of Theorem 4.1 hold, and condition (ii) of the same theorem is also satisfied with $x_0 = y_0 = 0$. Now, all the hypotheses of Theorem 4.1 are true, and so we deduce the existence of a coupled fixed point of F.

5. Some applications

Motivated by the works in [16, 25], we study the existence of solutions for the integral equations in the following system

$$\begin{cases} x(t) = \int_a^b G_1(t,s)[f(s,x(s)) + g(s,y(s))]ds \\ \qquad + \int_a^b G_2(t,s)[f(s,y(s)) + g(s,x(s))]ds + h(t), \quad t \in I = [a,b], \\ y(t) = \int_a^b G_1(t,s)[f(s,y(s)) + g(s,x(s))]ds \\ \qquad + \int_a^b G_2(t,s)[f(s,x(s)) + g(s,y(s))]ds + h(t), \quad t \in I. \end{cases} \tag{5.1}$$

Let $C(I,\mathbb{R})$ be the space of all continuous functions defined on I. It is well-known that such a space with the metric given by

$$d(x,y) = \sup_{t \in I} |x(t) - y(t)|$$

is a complete metric space. We will use the following assumptions:

(i) $G_1,\ G_2 \in C(I \times I, \mathbb{R})$, $G_1(t,s) \geqslant 0$ and $G_2(t,s) \leqslant 0$;

(ii) $h(t) \in C(I,\mathbb{R})$;

(iii) there exists a function $\xi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and positive numbers $\mu,\ \nu$ such that for all $(x,y),\ (u,v) \in C^2(I,\mathbb{R})$ with $\xi((x,y),(u,v)) \geqslant 0$, we have

$$|f(t,x) - f(t,u)| \leqslant \mu\phi(|x-u|),$$

and

$$|g(t,y) - g(t,v)| \leqslant \nu\phi(|y-v|),$$

where $\phi \in \Phi$, and Φ is the family of all non-decreasing functions $\phi : [0,+\infty) \to [0,+\infty)$ satisfying the following condition:

(C) there exists a Jachymski function $\psi : [0,+\infty) \to [0,+\infty)$ with $\psi(t) < t$ for all $t > 0$ such that $\phi(t) = \psi(t/2)$.

Moreover,

$$\xi((x_1,y_1),(x_2,y_2)) \geqslant 0 \text{ and } \xi((x_2,y_2),(x_3,y_3)) \geqslant 0 \text{ imply } \xi((x_1,y_1),(x_3,y_3)) \geqslant 0$$

for all $(x_1,y_1),\ (x_2,y_2),\ (x_3,y_3) \in C^2(I,\mathbb{R})$;

(iv) there exist $p > 1$ and $q > 0$ with $1/p + 1/q = 1$ such that

$$(\mu+\nu)\sup_{t\in I}\left(\int_a^b (G_1(t,s)-G_2(t,s))^p\, ds\right)^{1/p}(b-a)^{1/q} \leqslant 1;$$

(v) there exists $(x_0,y_0) \in C^2(I,\mathbb{R})$ such that for all $t \in I$, we have

$$\begin{aligned}\xi\Bigg((x_0(t),y_0(t)),\Bigg(&\int_a^b G_1(t,s)[f(s,x_0(s))+g(s,y_0(s))]ds\\ &+\int_a^b G_2(t,s)[f(s,y_0(s))+g(s,x_0(s))]ds+h(t),\int_a^b G_1(t,s)[f(s,y_0(s))+g(s,x_0(s))]ds\\ &+\int_a^b G_2(t,s)[f(s,x_0(s))+g(s,y_0(s))]ds+h(t)\Bigg)\Bigg) \geqslant 0;\end{aligned}$$

(vi) for all $t \in I$, $(x,y),\ (u,v) \in C^2(I,\mathbb{R})$,

$$\begin{aligned}&\xi((x(t),y(t)),(u(t),v(t))) \Longrightarrow\\ &\xi\Bigg(\Bigg(\Bigg(\int_a^b G_1(t,s)[f(s,x(s))+g(s,y(s))]ds+\int_a^b G_2(t,s)[f(s,y(s))+g(s,x(s))]ds+h(t),\\ &\qquad\int_a^b G_1(t,s)[f(s,y(s))+g(s,x(s))]ds+\int_a^b G_2(t,s)[f(s,x(s))+g(s,y(s))]ds+h(t)\Bigg),\\ &\quad\Bigg(\int_a^b G_1(t,s)[f(s,u(s))+g(s,v(s))]ds+\int_a^b G_2(t,s)[f(s,v(s))+g(s,u(s))]ds+h(t),\\ &\qquad\int_a^b G_1(t,s)[f(s,v(s))+g(s,u(s))]ds+\int_a^b G_2(t,s)[f(s,u(s))+g(s,v(s))]ds+h(t)\Bigg)\Bigg) \geqslant 0;\end{aligned}$$

(vii) if $\{x_n\}$ and $\{y_n\}$ are sequences in $C(I,\mathbb{R})$ such that $\xi((x_n,y_n),(x_{n+1},y_{n+1}))\geqslant 0$ and $\xi((y_n,x_n),(y_{n+1},x_{n+1}))\geqslant 0$ for all n with $x_n \to x \in C(I,\mathbb{R})$ and $y_n \to y \in C(I,\mathbb{R})$ as $n\to\infty$, then $\xi((x_n,y_n),(x,y)) \geqslant 0$ and $\xi((y_n,x_n),(y,x)) \geqslant 0$ for all $n \in \mathbb{N}$.

Theorem 5.1. *Suppose that conditions* (i)-(vii) *hold. Then* (5.1) *has at least a solution* $(x,y) \in C^2(I,\mathbb{R})$.

Proof. Define the operator $F : C^2(I,\mathbb{R}) \to C(I,\mathbb{R})$ by

$$\begin{aligned}F(x,y)(t) = &\int_a^b G_1(t,s)[f(s,x(s))+g(s,y(s))]ds\\ &+\int_a^b G_2(t,s)[f(s,y(s))+g(s,x(s))]ds+h(t) \quad \text{for all } t\in I.\end{aligned}$$

Then problem (5.1) is equivalent to find $(x^*,y^*) \in C^2(I,\mathbb{R})$ that is a coupled fixed point of F.

Now, let $(x, y),\ (u, v) \in C^2(I, \mathbb{R})$ such that $\xi((x(t), y(t)), (u(t), v(t))) \geqslant 0$ for all $t \in I$. It follows from the monotonicity of ϕ, condition (iii) and $G_2(t, s) \leqslant 0$ that

$$\begin{aligned} d(F(x,y), F(u,v)) \leqslant \sup_{t\in I} \Bigg| &\int_a^b G_1(t,s)[\mu\phi(d(x,u)) + \nu\phi(d(y,v))]ds \\ &- \int_a^b G_2(t,s)[\mu\phi(d(v,y)) + \nu\phi(d(x,u))]ds \Bigg|. \end{aligned} \tag{5.2}$$

Similarly, we have

$$\begin{aligned} d(F(y,x), F(v,u)) \leqslant \sup_{t\in I} \Bigg| &\int_a^b G_1(t,s)[\nu\phi(d(x,u)) + \mu\phi(d(y,v))]ds \\ &- \int_a^b G_2(t,s)[\nu\phi(d(v,y)) + \mu\phi(d(x,u))]ds \Bigg|. \end{aligned} \tag{5.3}$$

By summing (5.2) and (5.3), and applying the condition (iv) and the definition of Φ, we deduce that

$$\begin{aligned} &\frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2} \\ &\leqslant \sup_{t\in I} \left(\int_a^b (G_1(t,s) - G_2(t,s))ds \right)^{1/p} (b-a)^{1/q} (\mu + \nu) \frac{\phi(d(x,u)) + \phi(d(v,y))}{2} \\ &\leqslant \frac{\phi(d(x,u)) + \phi(d(v,y))}{2} \leqslant \phi(d(x,u) + d(y,v)) = \psi\left(\frac{d(x,u) + d(y,v)}{2} \right). \end{aligned}$$

Define the function $\alpha : C^2(I, \mathbb{R}) \times C^2(I, \mathbb{R}) \to [0, +\infty)$ by

$$\alpha((x,y),(u,v)) = \begin{cases} 1, & \text{if } \xi((x(t),y(t)),(u(t),v(t))) \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

For all $(x, y),\ (u, v) \in C^2(I, \mathbb{R})$, we have

$$\alpha((x,y),(u,v)) \frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2} \leqslant \phi\left(\frac{d(x,u) + d(y,v)}{2} \right).$$

Then, T is an α-ψ-ϕ-contractive mapping. From condition (vi), for all $(x, y),\ (u, v) \in C^2(I, \mathbb{R})$,

$$\begin{aligned} \alpha((x,y),(u,v)) \geqslant 1 &\Longrightarrow \xi((x(t),y(t)),(u(t),v(t))) \geqslant 0 \\ &\Longrightarrow \xi((F(x(t),y(t)), F(y(t),x(t))), (F(u(t),v(t)), F(v(t),u(t)))) \geqslant 0 \\ &\Longrightarrow \alpha((F(x,y),F(y,x)),(F(u,v),F(v,u))) \geqslant 1. \end{aligned}$$

From (v), there exists $(x_0, y_0) \in C^2(I, \mathbb{R})$ such that

$$\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \geqslant 1.$$

Finally, from (vii), and using Theorem 4.1, we deduce the existence of $(x^*, y^*) \in C^2(I, \mathbb{R})$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$, that is (x^*, y^*) is a solution to (5.1).

Finally, we show the existence of solution for the following initial-value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + f(x,t,u,u_x) + g(x,t,v,v_x), & -\infty < x < \infty, \quad 0 < t \leqslant T, \\ u(x,0) = \varphi(x), & -\infty < x < \infty, \\ v_t(x,t) = v_{xx}(x,t) + f(x,t,v,v_x) + g(x,t,u,u_x), & -\infty < x < \infty, \quad 0 < t \leqslant T, \\ v(x,0) = \varphi(x), & -\infty < x < \infty, \end{cases} \tag{5.4}$$

where $f,\ g$ are continuous functions, φ is continuously differentiable, and φ and φ' are bounded. Define

$$\Omega = \{u(x,t):\ u,\ u_x \in C(\mathbb{R}\times[0,T]) \ \text{ and } \ \|u\| < \infty\},$$

where

$$\|u\| = \sup_{x\in\mathbb{R},\ t\in[0,T]} |u(x,t)| + \sup_{x\in\mathbb{R},\ t\in[0,T]} |u_x(x,t)|.$$

It is easy to check that such a space with the metric given by

$$d(u,v) = \sup_{x\in\mathbb{R},\ t\in[0,T]} |u(x,t)-v(x,t)| + \sup_{x\in\mathbb{R},\ t\in[0,T]} |u_x(x,t)-v_x(x,t)|$$

is complete metric space. □

Theorem 5.2. *Assume that the following conditions are satisfied*

(i) $f,\ g:\mathbb{R}\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are continuous. For any* $\epsilon>0$ *with* $|u|<\epsilon$ *and* $|v|<\epsilon$, *the functions* $f(x,t,u,v),\ g(x,t,u,v)$ *are uniformly Hölder continuous in* x *and* t *for each compact subset of* $\Omega\times[0,T]$;

(ii) *there exists a function* $\xi\in\Omega^2\times\Omega^2\to\mathbb{R}$ *and positive constants* c_1 *and* c_2 *such that for all* $(u_1,v_1),\ (u_2,v_2)\in\Omega^2$ *with* $\xi((u_1,v_1),(u_2,v_2))\geqslant 0$, *we have*

$$|f(x,t,u_1,(u_1)_x)-f(x,t,u_2,(u_2)_x)| \leqslant c_1\varphi\left(\frac{|u_1-u_2+(u_1)_x-(u_2)_x|}{2}\right),$$
$$|g(x,t,v_1,(v_1)_x)-g(x,t,v_2,(v_2)_x)| \leqslant c_2\varphi\left(\frac{|v_1-v_2+(v_1)_x-(v_2)_x|}{2}\right),$$

where $\varphi(t):[0,+\infty)\to[0,+\infty)$ *is a non-decreasing Jachymski function with* $\varphi(t)<t$ *for all* $t>0$. *In addition,* $\xi((u_1,v_1),(u_2,v_2))\geqslant 0$ *and* $\xi((u_2,v_2),(u_3,v_3))\geqslant 0$ *imply that* $\xi((u_1,v_1),(u_3,v_3))\geqslant 0$, *for all* $(u_i,v_i)\in\Omega^2$, $i=1,2,3$;

(iii) f,g *are bounded for bounded* u *and* v;

(iv) $c_1+c_2\leqslant(T+2\pi^{-1/2}T^{1/2})^{-1}$;

(v) *there exists* $(u_0,v_0)\in\Omega\times\Omega$ *such that for all* $x\in\mathbb{R}$ *and* $t\in[0,T]$, *we have*

$$\begin{aligned}\xi\Bigg((u_0(x,t),v_0(x,t)),\Bigg(&\int_{-\infty}^{\infty}k(x-\xi,t)\varphi(\xi)d\xi+\int_0^t\int_{-\infty}^{\infty}k(x-\xi,t-\tau)\\&\times[f(\xi,\tau,u_0(\xi,\tau),(u_0)_x(\xi,\tau))+g(\xi,\tau,v_0(\xi,\tau),(v_0)_x(\xi,\tau))]d\xi d\tau,\\&\int_{-\infty}^{\infty}k(x-\xi,t)\varphi(\xi)d\xi+\int_0^t\int_{-\infty}^{\infty}k(x-\xi,t-\tau)[f(\xi,\tau,v_0(\xi,\tau),(v_0)_x(\xi,\tau))\\&+g(\xi,\tau,u_0(\xi,\tau),(u_0)_x(\xi,\tau))]d\xi d\tau\Bigg)\Bigg)\geqslant 0;\end{aligned}$$

(vi) *for all* $x\in\mathbb{R}$ *and* $t\in[0,T]$, $(u_1,v_1),\ (u_2,v_2)\in\Omega\times\Omega$,

$$\begin{aligned}&\xi((u_1(x,t),v_1(x,t)),(u_2(x,t),v_2(x,t)))\Longrightarrow\\&\quad\xi\Bigg(\Bigg(\Bigg(\int_{-\infty}^{\infty}k(x-\xi,t)\varphi(\xi)d\xi+\int_0^t\int_{-\infty}^{\infty}k(x-\xi,t-\tau)[f(\xi,\tau,u_1(\xi,\tau),(u_1)_x(\xi,\tau))\\&\quad+g(\xi,\tau,v_1(\xi,\tau),(v_1)_x(\xi,\tau))]d\xi d\tau,\ \int_{-\infty}^{\infty}k(x-\xi,t)\varphi(\xi)d\xi\\&\quad+\int_0^t\int_{-\infty}^{\infty}k(x-\xi,t-\tau)[f(\xi,\tau,v_1(\xi,\tau),(v_1)_x(\xi,\tau))+g(\xi,\tau,u_1(\xi,\tau),(u_1)_x(\xi,\tau))]d\xi d\tau\Bigg),\end{aligned}$$

$$\Big(\int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi+\int_0^t\int_{-\infty}^{\infty} k(x-\xi,t-\tau)[f(\xi,\tau,u_2(\xi,\tau),(u_2)_x(\xi,\tau))$$
$$+g(\xi,\tau,v_2(\xi,\tau),(v_2)_x(\xi,\tau))]d\xi d\tau,\ \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi$$
$$+\int_0^t\int_{-\infty}^{\infty} k(x-\xi,t-\tau)[f(\xi,\tau,v_2(\xi,\tau),(v_2)_x(\xi,\tau))+g(\xi,\tau,u_2(\xi,\tau),(u_2)_x(\xi,\tau))]d\xi d\tau\Big)\Big);$$

(vii) *if* $\{u_n\}$ *and* $\{v_n\}$ *are sequences in* Ω *such that* $\xi((u_n,v_n),(u_{n+1},v_{n+1}))\geqslant 0$ *and* $\xi((v_n,u_n),(v_{n+1},u_{n+1}))\geqslant 0$ *for all* n *with* $u_n\to u\in\Omega$ *and* $v_n\to v\in\Omega$ *as* $n\to\infty$, *then* $\xi((u_n,v_n),(u,v))\geqslant 0$ *and* $\xi((v_n,u_n),(v,u))\geqslant 0$ *for all* $n\in\mathbb{N}$.

Then initial-value problem (5.4) *has a solution.*

Proof. The problem (5.4) is equivalent to the integral equation

$$\begin{cases} u(x,t)=\int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi+\int_0^t\int_{-\infty}^{\infty} k(x-\xi,t-\tau)[f(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)) \\ \qquad\qquad +g(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau))]d\xi d\tau, \\ v(x,t)=\int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi+\int_0^t\int_{-\infty}^{\infty} k(x-\xi,t-\tau)[f(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau)) \\ \qquad\qquad +g(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))]d\xi d\tau \end{cases}$$

for all $x\in\mathbb{R}$ and $0<t\leqslant T$, where

$$k(x,t)=\frac{1}{\sqrt{4\pi t}}\exp\left\{\frac{-x^2}{4t}\right\} \tag{5.5}$$

for all $x\in\mathbb{R}$ and $t>0$. Define a mapping $F:\Omega\times\Omega\to\Omega$ by

$$F(u,v)(x,t)=\int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi+\int_0^t\int_{-\infty}^{\infty} k(x-\xi,t-\tau)[f(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))$$
$$+g(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau))]d\xi d\tau$$

for all $x\in\mathbb{R}$ and $t\in[0,T]$. We can see that (u,v) is a solution of the problem (5.4) if and only if $(u,v)\in\Omega\times\Omega$ is a coupled fixed point of F.

Now, let (u_1,u_2), $(v_1,v_2)\in\Omega\times\Omega$ such that

$$\xi((u_1(x,t),u_2(x,t)),(v_1(x,t),v_2(x,t))\geqslant 0$$

for all $x\in\mathbb{R}$ and $t\in[0,T]$. It follows from the increasing property of φ and (5.5) that

$$d(F(u_1,v_1),F(u_2,v_2))\leqslant\left(T+2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\right)\left[c_1\varphi\left(\frac{d(u_1,u_2)}{2}\right)+c_2\varphi\left(\frac{d(v_2,v_1)}{2}\right)\right]. \tag{5.6}$$

In similar arguments, we have

$$d(F(v_1,u_1),F(v_2,u_2))\leqslant\left(T+2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\right)\left[c_2\varphi\left(\frac{d(u_1,u_2)}{2}\right)+c_1\varphi\left(\frac{d(v_2,v_1)}{2}\right)\right]. \tag{5.7}$$

By summing (5.6) and (5.7), and combining the increasing property of φ and condition (iv), we get

$$\frac{d(F(u_1,v_1),F(u_2,v_2))+d(F(v_1,u_1),F(v_2,u_2))}{2}$$
$$\leqslant\left(T+2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\right)\left(\frac{c_1+c_2}{2}\right)\left[\varphi\left(\frac{d(u_1,u_2)}{2}\right)+\varphi\left(\frac{d(v_2,v_1)}{2}\right)\right]$$

$$\begin{aligned}
&\leqslant \left(T+2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\right)\left(\frac{c_1+c_2}{2}\right)2\varphi\left(\frac{d(u_1,u_2)+d(v_2,v_1)}{2}\right)\\
&\leqslant \varphi\left(\frac{d(u_1,u_2),d(v_2,v_1)}{2}\right).
\end{aligned}$$

Define the function $\alpha:\Omega^2\times\Omega^2\to[0,+\infty)$ by

$$\alpha((u_1,v_1),(u_2,v_2))=\begin{cases}1, & \text{if } \xi((u_1(x,t),v_1(x,t)),(u_2(x,t),v_2(x,t)))\geqslant 0,\\ 0, & \text{otherwise.}\end{cases}$$

For all $(u_1,v_1),\ (u_2,v_2)\in\Omega\times\Omega$, we have

$$\alpha((u_1,v_1),(u_2,v_2))\frac{d(F(u_1,v_1),F(u_2,v_2))+d(F(v_1,u_1),F(v_2,u_2))}{2}\leqslant\varphi\left(\frac{d(u_1,u_2)+d(v_1,v_2)}{2}\right).$$

Then, T is an α-ψ-φ-contractive mapping. From condition (vi), for all $(u_1,v_1),\ (u_2,v_2)\in\Omega\times\Omega$,

$$\begin{aligned}
&\alpha((u_1,v_1),(u_2,v_2))\geqslant 1\\
&\Longrightarrow \xi((u_1(x,t),v_1(x,t)),(u_2(x,t),v_2(x,t)))\geqslant 0\\
&\Longrightarrow \xi((F(u_1(x,t),v_1(x,t))),F(v_1(x,t),u_1(x,t))),(F(u_2(x,t),v_2(x,t)),F(v_2(x,t),u_2(x,t))))\geqslant 0\\
&\Longrightarrow \alpha((F(u_1,v_1),F(v_1,u_1)),(F(u_2,v_2),F(v_2,u_2)))\geqslant 1.
\end{aligned}$$

From (v), there exists $(x_0,y_0)\in\Omega\times\Omega$ such that

$$\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0)))\geqslant 1.$$

Finally, from (vii), and using Theorem 4.1, we deduce the existence of $(x^*,y^*)\in C^2(I,\mathbb{R})$ such that $x^*=F(x^*,y^*)$ and $y^*=F(y^*,x^*)$, that is (x^*,y^*) is a solution to (5.4). $\square$

Remark 5.3. Chaipunya et al. [7] considered the initial value problem (5.4) when $g(x,t,u,u_x)=g(x,t,v,v_x)\equiv 0$. Very recently, in [25], we discussed the corresponding initial-value problem of single parabolic equation for (5.4). Unlike to Theorem 4.2 of [25], it should be noted that we do not require f and g to have the mixed monotone properties.

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References

[1] C. Alegre, J. Marín, S. Romaguera, *A fixed point theorem for generalized contractions involving w-distances on complete quasi-metric spaces*, Fixed Point Theory Appl., **2014** (2014), 8 pages. 1, 1.1, 2.2, 2.3, 2.6

[2] S. Al-Homidan, Q. H. Ansari, J.-C. Yao, *Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory*, Nonlinear Anal., **69** (2008), 126–139. 1

[3] H. H. Alsulami, S. Gülyaz, İ. M. Erhan, *Fixed points of α-admissible Meir-Keeler contraction mappings on quasi-metric spaces*, J. Inequal. Appl., **2015** (2015), 15 pages. 1, 1.4

[4] H. H. Alsulami, S. Gülyaz, E. Karapınar, İ. M. Erhan, *Fixed point theorems for a class of α-admissible contractions and applications to boundary value problem*, Abstr. Appl. Anal., **2014** (2014), 10 pages. 1, 4

[5] V. Berinde, *Coupled fixed point theorems for ϕ-contractive mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **75** (2012), 3218–3228. 1

[6] M. Borcut, V. Berinde, *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput., **218** (2012), 5929–5936. 1

[7] P. Chaipunya, Y. J. Cho, P. Kumam, *Geraghty-type theorems in modular metric spaces with an application to partial differential equation*, Adv. Difference Equ., **2012** (2012), 12 pages. 5.3

[8] R. H. Haghi, S. Rezapour, N. Shahzad, *Some fixed point generalizations are not real generalizations*, Nonlinear Anal., **74** (2011), 1799–1803. 1, 2.9

[9] J. Jachymski, *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl., **194** (1995), 293–303. 2.6

[10] G. Jungck, B. E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., **29** (1998), 227–238. 2.8, 3.7

[11] E. Karapınar, İ. M. Erhan, A. Öztürk, *Fixed point theorems on quasi-partial metric spaces*, Math. Comput. Modelling, **57** (2013), 2442–2448. 1

[12] E. Karapınar, P. Kumam, P. Salimi, *On α-ψ-Meir-Keeler contractive mappings*, Fixed Point Theory Appl., **2013** (2013), 12 pages. 1, 1.3, 1, 2.7

[13] H. Lakzian, D. Gopal, W. Sintunavarat, *New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations*, J. Fixed Point Theory Appl., **18** (2016), 251–266. 1

[14] V. La Rosa, P. Vetro, *Common fixed points for α-ψ-ϕ-contractions in generalized metric spaces*, Nonlinear Anal. Model. Control, **19** (2014), 43–54. 1, 2.1

[15] X.-L. Liu, A. H. Ansari, S. Chandok, C.-K. Park, *Some new fixed point results in partial ordered metric spaces via admissible mappings and two new functions*, J. Nonlinear Sci. Appl., **9** (2016), 1564–1580. 3.9

[16] N. V. Luong, N. X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal., **74** (2011), 983–992. 5

[17] J. Marín, S. Romaguera, P. Tirado, *Generalized contractive set-valued maps on complete preordered quasi-metric spaces*, J. Funct. Spaces Appl., **2013** (2013), 6 pages. 1

[18] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969), 326–329. 1

[19] S. Park, *On generalizations of the Ekeland-type variational principles*, Nonlinear Anal., **39** (2000), 881–889. 1

[20] P. D. Proinov, *Fixed point theorems in metric spaces*, Nonlinear Anal., **64** (2006), 546–557. 2.4

[21] A. Roldán, J. Martínez-Moreno, C. Roldán, *Multidimensional fixed point theorems in partially ordered complete metric spaces*, J. Math. Anal. Appl., **396** (2012), 536–545. 1, 2.10, 2.11, 2.12

[22] A. Roldán, J. Martínez-Moreno, C. Roldán, Y. J. Cho, *Multidimensional fixed point theorems under (ψ, ϕ)-contractive conditions in partially ordered complete metric spaces*, J. Comput. Appl. Math., **273** (2015), 76–87.

[23] A. Roldán, J. Martínez-Moreno, C. Roldán, E. Karapınar, *Some remarks on multidimensional fixed point theorems*, Fixed Point Theory, **15** (2014), 545–558. 1, 2.10, 2.11, 2.12

[24] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1, 1.2, 1

[25] Y.-B. Sang, Q. Meng, *Fixed point theorems with generalized altering distance functions in partially ordered metric spaces via w-distances and applications*, Fixed Point Theory Appl., **2015** (2015), 25 pages. 4.6, 4.6, 5, 5.3

[26] T. Suzuki, *Meir-Keeler contractions of integral type are still Meir-Keeler contractions*, Int. J. Math. Math. Sci., **2007** (2007), 6 pages. 3.7