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Weak and strong convergence theorems for nonlinear mappings and system of generalized mixed equilibrium problems in Hilbert spaces

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Abstract

In this paper, we construct two iteration schemes for approximating a common element of the set of solutions of equilibrium problems (GMEP and GEP) and the set of common fixed points of a finite family of k-strictly asymptotically pseudo-contractions in Hilbert spaces. Fixed point theorems are established in Hilbert spaces. Numerical examples and applications are provided. The main results of this paper modify and improve many important recent results in the literature.

Keywords: Equilibrium problem, Modified Ishikawa's iteration, hybrid algorithm, Hilbert space, weak and strong convergence.
2010 MSC: 47H05, 47H09, 47H10.

1. Introduction

Let H be an infinite dimensional real Hilbert space. Let C be a nonempty subset of H and let $T : C \to C$ be a mapping. Recall that T is said to be nonexpansive if and only if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C, \quad \forall n \geqslant 1.$$

T is said to be asymptotically nonexpansive if and only if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|, \quad \forall x, y \in C, \quad \forall n \geqslant 1.$$

A mapping T is called a k-strictly pseudocontractive mapping if and only if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2, \quad \forall x, y \in C.$$

A mapping T is called a k-strictly asymptotically pseudocontractive mapping if and only if there exist a constant $k \in [0, 1)$ and a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\|^2 \leqslant k_n^2 \|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C.$$

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Recall that T is called uniformly L-Lipschitzian if and only if there exists some $L > 0$ such that

$$\|T^n x - T^n y\| \leqslant L\|x-y\|, \quad \forall x, y \in C, \quad \forall n \geqslant 1.$$

It is obvious to observe that

(i) T is a k-strictly pseudocontractive mapping when $k_n \equiv 1$;

(ii) T is an asymptotically nonexpansive mapping when $k = 0$;

(iii) T is a nonexpansive mapping when $k_n \equiv 1$ and $k = 0$;

(iv) if $\{T_i\}_{1\leqslant i \leqslant N}$ is a finite family of $\{s_i\}_{1\leqslant i\leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence $\{s_i\} \subseteq [0, 1)$ and $\{k_{n,i}\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$, then we have

$$\|T_i^n x - T_i^n y\|^2 \leqslant k_n^2 \|x-y\|^2 + s\|(I-T_i^n)x - (I-T_i^n)y\|^2, \quad \forall x, y \in C,$$

for all $1 \leqslant i \leqslant N$, where constant $s = \max\{s_i : 1 \leqslant i \leqslant N\}$ and sequence $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ with $\lim_{n\to\infty} k_n = 1$.

Recently, the class of k-strictly asymptotically pseudocontractive mappings has been extensively investigated by many authors as an important extension of asymptotically nonexpansive mappings; see [13, 16, 22, 23] and the references therein. To study computational fixed points of nonlinear mappings, various iterative methods, such as mean valued iteration methods, projection iterative methods, splitting iterative methods, regularization iterative methods and so on, have been introduced and studied based on different analysis techniques; see [5, 7, 12, 18, 21] and the references.

Recently, Qin et al. [16] constructed a new iterative algorithm for approximating common fixed points of a finite family of k-strictly asymptotically pseudocontractive mappings in real Hilbert spaces by generating the sequence $\{x_n\}$ as follows:

$$x_n = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)} x_{n-1}, \quad n \geqslant 1. \tag{1.1}$$

Specifically,

$$\begin{cases} x_1 = \alpha_0 x_0 + (1-\alpha_0)T_1 x_0, \\ x_2 = \alpha_1 x_1 + (1-\alpha_1)T_2 x_1, \\ \vdots \\ x_N = \alpha_{N-1}x_{N-1} + (1-\alpha_{N-1})T_N x_{N-1}, \\ x_{N+1} = \alpha_N x_N + (1-\alpha_N)T_1^2 x_N, \\ \vdots \\ x_{2N} = \alpha_{2N-1}x_{N-1} + (1-\alpha_{2N-1})T_N^2 x_{2N-1}, \\ x_{2N+1} = \alpha_{2N}x_{2N} + (1-\alpha_{2N})T_1^3 x_{2N}, \\ \vdots \end{cases}$$

where we can write that $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, ...N\}, h = h(n) \geqslant 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. When sequence $\{\alpha_n\}$ satisfies certain conditions, the sequence $\{x_n\}$ generated by algorithm (1.1) converges weakly to a point in $\bigcap_{i=1}^N F(T_i)$.

Let $B : C \to H$ be a monotone mapping, that is,

$$\langle Bx - By, x - y\rangle \geqslant 0, \quad \forall x, y \in C.$$

Let $\varphi : C \to R$ be a semicontinuous and convex functional and $\Theta : C \times C \to R$ be a real-valued bifunction. Assume that Θ satisfies the following conditions:

(A1) $\Theta(x,x) = 0, \ \forall x \in C$;
(A2) Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \leqslant 0, \ \forall x,y \in C$;
(A3) $\limsup_{t \to 0} \Theta(x + t(z-x), y) \leqslant \Theta(x,y), \ \forall x,y \in C$;
(A4) the function $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous.

Recall the so-called the system of generalized mixed equilibrium problems (GMEP) is to find $x \in C$ such that

$$\Theta(x,y) + \langle Bx, y-x \rangle + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C. \tag{1.2}$$

We use $\mathrm{GMEP}(\Theta, B, \varphi)$ to denote the set of solutions to (1.2), i.e.,

$$\mathrm{GMEP}(\Theta, B, \varphi) = \{x \in C : \Theta(x,y) + \langle Bx, y-x \rangle + \varphi(y) - \varphi(x) \geqslant 0, \ \forall y \in C\}.$$

If $\varphi \equiv 0$, problem (1.2) turns into the mixed equilibrium problem for Θ, B, denoted by $\mathrm{GEP}(\Theta, B)$ which is to find $x \in C$ such that

$$\Theta(x,y) + \langle Bx, y-x \rangle \geqslant 0, \quad \forall y \in C.$$

If $B \equiv 0$ and $\varphi \equiv 0$, problem (1.2) turns into the equilibrium problem for Θ, denoted by $\mathrm{EP}(\Theta)$ which is to find $x \in C$ such that

$$\Theta(x,y) \geqslant 0, \quad \forall y \in C.$$

The generalized mixed equilibrium problem, which includes many important problems, for instance, complementarity problems, variational inequality problems, optimization problems, and fixed point problems as special cases, has been extensively investigated by many authors; see [2, 6, 9, 11] and the references therein. There are numerous problems in physics, optimization and economics which can be reduced to find a solution of generalized equilibrium problem. For exploring its solutions, various iterative methods have been proposed, see [15, 19, 24] and the references therein.

Due to extensive applications of equilibrium problems and k-strictly asymptotically pseudo-contractions, the topic of approximating common element of the set of solutions of the equilibrium problem and the set of the fixed points of k-strictly asymptotically pseudo-contractions attract more attention recently. For solving these problems, Liu [13] proposed the following iterative method:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_{n-1} = \alpha_{n-1} x_{n-1} + (1-\alpha_{n-1}) T_{i(n)}^{h(n)} x_{n-1}, \\ x_n \in C, \ \text{ such that } \ \Theta(x_n, y) + \langle B y_{n-1}, y - x_n \rangle + \frac{1}{\lambda_{n-1}} \langle y - x_n, x_n - y_{n-1} \rangle \geqslant 0, \ \ \forall y \in C, \end{cases} \tag{1.3}$$

where $\{T_i\}_{1 \leqslant i \leqslant N}$ is a finite family of strictly asymptotically pseudo-contractive mappings, $\varphi : C \to R$ is a proper lower semi-continuous and convex functional, $B : C \to H$ is a continuous and monotone mapping and $\Theta : C \times C \to R$ satisfies (A1)–(A4). Under appropriate conditions imposed on sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfied, they obtained weak and strong convergence theorems.

Motivated by the above mentioned results and the on-going research, we construct several new iteration schemes for approximating a common element of the set of solutions of equilibrium problems (GMEP and GEP) and the set of common fixed points of a finite family of k-strictly asymptotically pseudo-contractions in Hilbert spaces.

The main contributions of this paper are the following:

(i) The Modified Mann iteration method in algorithm (1.3) is replaced by a new iteration in our paper. Moreover, we also consider approximating the common element of the set of solutions of generalized mixed equilibrium problems (GMEP), not only equilibrium problem (GEP), and the set of common fixed points of a finite family of k-strictly asymptotically pseudo-contractions. And we obtain two different weak convergence theorems.

(ii) By means of hybrid algorithms, we obtain two strong convergence theorems corresponding to weak convergence theorems.
(iii) We apply the results to approximate the common element of the set of solutions of equilibrium problems (GMEP and GEP) and the set of common fixed points of a finite family of asymptotically nonexpansive mappings under suitable conditions.
(iv) We apply our results to (mixed) equilibrium problem (EP and MEP), (mixed) variational inequality, convex minimization problem and convex feasibility problem.

The rest of the paper is organized as follows: Section 2 describes several definitions and lemmas which we will use in proving our main results. Also, we give an example of a k-strictly asymptotically pseudocontractive mapping with nonempty set of fixed points to support our results. Section 3 presents our main results which include two weak convergence theorems and two strong convergence theorems. Section 4 introduces several interesting applications of our results. Finally, we conclude our paper in Section 5.

2. Preliminaries

Throughout the paper, $\to$ and $\rightharpoonup$ denote the strong convergence and weak convergence, respectively. In addition, $F(T)$ and $\omega_w(x_n)$ denote the fixed point set of T and the weak ω-limit set of x_n, respectively, that is, $F(T) = \{x \in C : Tx = x\}$ and $\omega_w(x_n) = \{u : \exists x_{n_j} \rightharpoonup u\}$.

In an infinite dimensional real Hilbert space H, for all $x, y \in H$, the following properties hold:

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle,$$
$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall \lambda \in [0,1]. \tag{2.1}$$

Let C be a nonempty closed and convex subset of H, for each $x \in H$, there exists a nearest point from x to C. We denote the nearest point by $P_C x$, i.e., $\|x - P_C x\| = \inf\{\|x-y\| : y \in C\}$, where P_C is called metric projection from H onto C, and we have that

$$\langle x - P_C x, P_C x - y\rangle \geqslant 0, \quad \forall y \in C. \tag{2.2}$$

Let B be a mapping of C into H. Recall that B is an α-inverse-strongly monotone mapping if and only if there exists $\alpha > 0$ such that

$$\langle Bx - By, x-y\rangle \geqslant \alpha\|Bx - By\|^2, \forall x, y \in C.$$

It is clear that if B is an α-inverse-strongly monotone mapping, then it must be a $\frac{1}{\alpha}$-Lipschitz operator. Moreover, for all $x, y \in C$ and $r > 0$, we can observe that

$$\begin{aligned} \|(I-rB)x - (I-rB)y\|^2 &= \|(x-y) - r(Bx-By)\|^2 \\ &= \|x-y\|^2 - 2r\langle x-y, Bx-By\rangle + r^2\|Bx-By\|^2 \\ &\leqslant \|x-y\|^2 + r(r-2\alpha)\|Bx-By\|^2. \end{aligned} \tag{2.3}$$

From the last inequality, we can see that $I - rB : C \to H$ is nonexpansive when $r \leqslant 2\alpha$. The class of inverse-strongly monotone mappings has recently extensively investigated by many authors in different framework of spaces; see [3, 8, 17] and the references therein.

To obtain the main results of this paper, we also need the following lemmas:

Lemma 2.1 ([4, 10])**.** *Let C be a nonempty closed convex subset of H. Let Θ be a bifunction from $C \times C \to R$ satisfies (A1)-(A4), and let $\varphi : C \to R$ be a proper lower semicontinuous and convex function. Let $B : C \to H$ be a continuous monotone mapping. Then for $r > 0$ and $x \in H$, there exists $u \in C$ such that*

$$\Theta(u,y) + \langle Bu, y-u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y-u, u-x\rangle \geqslant 0, \quad \forall y \in C.$$

Define a mapping $K_r : H \to C$ *as follows:*

$$K_r x := \{u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geqslant 0,\ \forall y \in C\}$$

for all $x \in H$ *and* $r > 0$. *Then, the following hold:*

(i) *For each* $x \in H$, $K_r(x) \neq \emptyset$;
(ii) K_r *is single-valued;*
(iii) K_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|K_r x - K_r y\|^2 \leqslant \langle K_r x - K_r y, x - y\rangle;$$

(iv) $F(K_r) = GMEP(\Theta, \varphi, B)$;
(v) $GMEP(\Theta, \varphi, B)$ *is closed and convex.*

And notice that $\Omega(x, y) = \Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x)$ satisfies conditions (A1)–(A4) (see [10]). Additionally, if $B \equiv 0$ and $\varphi \equiv 0$, $GMEP(\Theta, \varphi, B)$ reduces to $EP(\Theta)$, that is,

$$\Theta(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geqslant 0, \quad \forall y \in C.$$

Then, define a mapping $T_r : H \to C$ and

$$T_r x := \{u \in C : \Theta(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geqslant 0,\ \forall y \in C\}$$

for all $x \in H$ and $r > 0$. It is obvious that the above conclusions of Lemma 2.1 are also suitable for $EP(\Theta)$.

Lemma 2.2 ([20])**.** *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1 + b_n)a_n + c_n, \quad \forall n \geqslant 0,$$

if $\sum_{n=0}^{\infty} b_n < \infty$ *and* $\sum_{n=0}^{\infty} c_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.3 ([16])**.** *Let* C *be a closed convex subset of a real Hilbert space* H *and* T *be an asymptotically* k*-strictly pseudocontractive.*

(i) T *is uniformly* L*-Lipschitzian.*
(ii) *If* $F(T)$ *is nonempty, then* $(I - T)$ *is demiclosed at zero, that is,*

$$x_n \rightharpoonup u \text{ and } (I - T)x_n \to 0 \Rightarrow (I - T)u = 0.$$

(iii) $F(T)$ *is closed and convex so that the projection* $P_{F(T)}$ *is well-defined.*

Lemma 2.4 ([11])**.** *Let* C *be a closed convex subset of a real Hilbert space* H *and* $x, y, z \in H$. *The set*

$$\{v \in C : \|y - v\|^2 \leqslant \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed), where a *is a real number.*

Lemma 2.5 ([1])**.** *Let* C *be a closed convex subset of a real Hilbert space* H *and sequence* $\{x_n\}$ *be bounded in* H. *If*

(i) $\omega_w(x_n) \subset C$;
(ii) $\lim_{n\to\infty} \|x_n - p\|$ *exists,* $\forall p \in C$.

Then sequence $\{x_n\}$ *converges weakly to a point in* C.

Lemma 2.6 ([14]). *Let C be a closed convex subset of a real Hilbert space H and sequence $\{x_n\}$ be bounded in H. Let $q = P_C x, x \in H$. Assume*

(i) $\omega_w(x_n) \subset C$;
(ii) $\|x_n - x\| \leqslant \|x - q\|$, $\forall n \in \mathbb{N}$.

Then sequence $\{x_n\}$ converges strongly to q.

Remark 2.7. We now give an example of a k-strictly asymptotically pseudocontractive mapping with nonempty set of fixed points.

Suppose that $H := \mathbb{R}$ and $C := [-1, 1] \subset \mathbb{R}$. Let $T : C \to C$ be defined by

$$Tx := \begin{cases} x, & x \in [-1, 0); \\ \frac{1}{2}x, & x \in [0, 1]. \end{cases}$$

Then we observe that $F(T) = [-1, 0]$, and hence the set of the fixed points is nonempty.

Now, we show that T is a k-strictly asymptotically pseudocontractive mapping. Suppose that $C_1 = [-1, 0)$ and $C_2 = [0, 1]$.

Case 1. If $x, y \in C_1$, then, $T^n x = x, T^n y = y$, we have that

$$|T^n x - T^n y| = |x - y|. \tag{2.4}$$

Case 2. If $x, y \in C_2$, then, $T^n x = \frac{1}{2^n}x$, $T^n y = \frac{1}{2^n}y$, we have that

$$|T^n x - T^n y| = \frac{1}{2^n}|x - y|. \tag{2.5}$$

Case 3. If $x \in C_1, y \in C_2$, then, $T^n x = x, T^n y = \frac{1}{2^n}y$, we have that

$$|T^n x - T^n y| = |x - \frac{1}{2^n}y| \leqslant |x - y|. \tag{2.6}$$

Case 4. If $x \in C_2, y \in C_1$, then, $T^n x = \frac{1}{2^n}x$, $T^n y = y$, we have that

$$|T^n x - T^n y| = |\frac{1}{2^n}x - y| \leqslant |x - y|. \tag{2.7}$$

Therefore, from (2.4), (2.5), (2.6), and (2.7), it is obvious that T is an asymptotically nonexpansive mapping with $k_n \equiv 1$, then T is a k-strictly asymptotically pseudocontractive mapping for any $k \in [0, 1)$.

3. Main results

In this section, we first prove two weak convergence theorems via two kinds of iteration schemes for finding a common element of the set of solutions of equilibrium problems (GMEP and GEP) and the set of common fixed points of a finite family of k-strictly asymptotically pseudo-contractions in Hilbert spaces. Two strong convergence theorems are also established based on the hybrid algorithm.

3.1. Weak convergence theorems

Theorem 3.1. *Let C be a nonempty closed convex subset of an infinite dimensional real Hilbert space H and let $\{T_i\}_{1 \leqslant i \leqslant N}$ be a finite family of $\{s_i\}_{1 \leqslant i \leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence $\{s_i\} \subseteq [0, 1)$ and $\{k_{n,i}\} \subseteq [1, \infty)$ such that $\lim_{n \to \infty} k_{n,i} = 1$ and $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $\varphi : C \to \mathbb{R}$ be a proper lower semi-continuous and convex functional and let $B : C \to H$ be a continuous and monotone mapping. Assume that $\Theta : C \times C \to \mathbb{R}$ satisfies (A1)–(A4), $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ and $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap GMEP(\Theta, B, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ x_n \in C \ \textit{such that} \\ \qquad \Theta(x_n, y) + \langle Bx_n, y-x_n\rangle + \varphi(y) - \varphi(x_n) + \frac{1}{r_{n-1}}\langle y-x_n, x_n - z_{n-1}\rangle \geqslant 0, \ \ \forall y \in C, \ \ n \geqslant 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *and* $\{r_n\}$ *satisfying the following conditions:*

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;

(ii) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;

(iii) $\liminf_{n\to\infty} g_{n-1} > 0$, $\forall n \geqslant 1$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1}-s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$.

Then sequence $\{x_n\}$ *converges weakly to a point in* Γ.

Proof. Our proof is divided into the following three steps.

Step 1. We prove that $\lim_{n\to\infty}\|x_n - p\|$ exists, for all $p \in \Gamma$.

From (3.1) and (2.1), we have that

$$\begin{aligned} \|z_{n-1}-p\|^2 &= \|\alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1} - p\|^2 \\ &= \alpha_{n-1}\|x_{n-1}-p\|^2 + (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}y_{n-1}-p\|^2 \\ &\quad - \alpha_{n-1}(1-\alpha_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}y_{n-1}\|^2, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|y_{n-1}-p\|^2 &= \|\beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1} - p\|^2 \\ &= \beta_{n-1}\|x_{n-1}-p\|^2 + (1-\beta_{n-1})\|T_{i(n)}^{h(n)}x_{n-1}-p\|^2 \\ &\quad - \beta_{n-1}(1-\beta_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2. \end{aligned} \tag{3.3}$$

Since that T_i is a s_i-strictly asymptotically pseudo-contractive mapping, where $i \in \{1, 2, ..., N\}$, one has

$$\|T_{i(n)}^{h(n)}y_{n-1}-p\|^2 \leqslant k_{h(n)}^2\|y_{n-1}-p\|^2 + s\|y_{n-1} - T_{i(n)}^{h(n)}y_{n-1}\|^2, \tag{3.4}$$

and

$$\|T_{i(n)}^{h(n)}x_{n-1}-p\|^2 \leqslant k_{h(n)}^2\|x_{n-1}-p\|^2 + s\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2. \tag{3.5}$$

Substituting (3.5) into (3.3), we obtain that

$$\begin{aligned} \|y_{n-1}-p\|^2 &\leqslant \beta_{n-1}\|x_{n-1}-p\|^2 + (1-\beta_{n-1})(k_{h(n)}^2\|x_{n-1}-p\|^2 \\ &\quad + s\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2) - \beta_{n-1}(1-\beta_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &\leqslant k_{h(n)}^2\|x_{n-1}-p\|^2 - (\beta_{n-1}-s)(1-\beta_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2. \end{aligned} \tag{3.6}$$

Observe that

$$\begin{aligned} \|y_{n-1} - T_{i(n)}^{h(n)}y_{n-1}\|^2 &= \|\beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1} - T_{i(n)}^{h(n)}y_{n-1}\|^2 \\ &= \beta_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}y_{n-1}\|^2 + (1-\beta_{n-1})\|T_{i(n)}^{h(n)}x_{n-1} \\ &\quad - T_{i(n)}^{h(n)}y_{n-1}\|^2 - \beta_{n-1}(1-\beta_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &\leqslant \beta_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}y_{n-1}\|^2 + (1-\beta_n)L^2\|x_{n-1} - y_{n-1}\|^2 \\ &\quad - \beta_{n-1}(1-\beta_{n-1})|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &\leqslant \beta_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}y_{n-1}\|^2 + (1-\beta_n)^3L^2\|x_{n-1} \\ &\quad - T_{i(n)}^{h(n)}x_{n-1}\|^2 - \beta_{n-1}(1-\beta_{n-1})\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2. \end{aligned} \tag{3.7}$$

Substituting (3.6), (3.7) into (3.4), we get that

$$\begin{aligned}\|T_{i(n)}^{h(n)}y_{n-1}-p\|^2 \leqslant\ & k_{h(n)}^2[k_{h(n)}^2\|x_{n-1}-p\|^2-(\beta_{n-1}-s)(1-\beta_{n-1})\|x_{n-1}\\ &-T_{i(n)}^{h(n)}x_{n-1}\|^2]+s[\beta_{n-1}\|x_{n-1}-T_{i(n)}^{h(n)}y_{n-1}\|^2\\ &+(1-\beta_n)^3L^2\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|^2\\ &-\beta_{n-1}(1-\beta_{n-1})\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|^2]\\ =\ & k_{h(n)}^4\|x_{n-1}-p\|^2+s\beta_{n-1}\|x_{n-1}-T_{i(n)}^{h(n)}y_{n-1}\|^2\\ &-(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1}-s)-(1-\beta_{n-1})^2L^2s\\ &+s\beta_{n-1}]\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|^2.\end{aligned}\tag{3.8}$$

Substituting (3.8) into (3.2), and combining with condition (i), we obtain that

$$\begin{aligned}\|z_{n-1}-p\|^2 =\ & \alpha_{n-1}\|x_{n-1}-p\|^2+(1-\alpha_{n-1})[k_{h(n)}^4\|x_{n-1}-p\|^2\\ &+s\beta_{n-1}\|x_{n-1}-T_{i(n)}^{h(n)}y_{n-1}\|^2-(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1}-s)\\ &-(1-\beta_{n-1})^2L^2s+s\beta_{n-1}]\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|^2]\\ &-\alpha_{n-1}(1-\alpha_{n-1})\|x_{n-1}-T_{i(n)}^{h(n)}y_{n-1}\|^2\\ \leqslant\ & k_{h(n)}^4\|x_{n-1}-p\|^2-(1-\alpha_{n-1})(\alpha_{n-1}-\beta_{n-1}s)\\ &\times\|x_{n-1}-T_{i(n)}^{h(n)}y_{n-1}\|^2-g_{n-1}\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|^2\\ \leqslant\ & k_{h(n)}^4\|x_{n-1}-p\|^2-g_{n-1}\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|^2.\end{aligned}\tag{3.9}$$

Since $x_n=K_{r_{n-1}}z_{n-1}$, it follows from Lemma 2.1 and condition (iii) that

$$\begin{aligned}\|x_n-p\|^2 &= \|K_{r_{n-1}}z_{n-1}-p\|^2\\ &\leqslant \|z_{n-1}-p\|^2\\ &\leqslant k_{h(n)}^4\|x_{n-1}-p\|^2\\ &= (1+k_{h(n)}^4-1)\|x_{n-1}-p\|^2.\end{aligned}\tag{3.10}$$

From condition $\sum_{n=0}^{\infty}(k_{n,i}-1)<\infty$, we have $\sum_{n=0}^{\infty}(k_n-1)<\infty$. Hence $\sum_{n=0}^{\infty}(k_{h(n)}^4-1)<\infty$. Again by Lemma 2.2, we have that $\lim_{n\to\infty}\|x_n-p\|$ exists. So, $\{\|x_n-p\|\}$ is bounded, this implies that $\{x_n\}$ is bounded.

Step 2. We prove that $\omega_w(x_n)\subseteq\Gamma$.

First, we prove that $\omega_w(x_n)\subseteq\bigcap_{l=1}^{N}F(T_l)$. In fact, we only prove that

$$\lim_{n\to\infty}\|x_n-T_lx_n\|\to 0,\quad \forall l\in\{1,2,...,N\}.$$

It follows from (3.9) that

$$g_{n-1}\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|^2\leqslant k_{h(n)}^4\|x_{n-1}-p\|^2-\|x_n-p\|^2.$$

From the fact that $\liminf_{n\to\infty}g_{n-1}>0$, $k_{h(n)}\to 1$ and $\lim_{n\to\infty}\|x_n-p\|$ exists, we obtain that

$$\|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|\to 0,\quad (n\to\infty).\tag{3.11}$$

From (3.1) and Lemma 2.3, we have that

$$\begin{aligned}\|z_{n-1}-x_{n-1}\| &= \|\alpha_{n-1}x_{n-1}+(1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}-x_{n-1}\|\\ &= (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}y_{n-1}-x_{n-1}\|\\ &\leqslant (1-\alpha_{n-1})[\|T_{i(n)}^{h(n)}y_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|+\|T_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\|]\\ &\leqslant (1-\alpha_{n-1})[L\|y_{n-1}-x_{n-1}\|+\|T_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\|]\\ &= (1-\alpha_{n-1})[(1-\beta_{n-1})L+1]\|T_{i(n)}^{h(n)}x_{n-1}-x_{n-1}\| \to 0.\end{aligned}$$

From Lemma 2.1, we get that

$$\begin{aligned}\|x_n-p\|^2 &= \|K_{r_{n-1}}z_{n-1}-p\|^2\\ &\leqslant \langle x_n-p, z_{n-1}-p\rangle\\ &= \frac{1}{2}(\|z_{n-1}-p\|^2+\|x_n-p\|^2-\|z_{n-1}-x_n\|^2).\end{aligned}$$

It follows that

$$\begin{aligned}\|z_{n-1}-x_n\|^2 &\leqslant \|z_{n-1}-p\|^2-\|x_n-p\|^2\\ &\leqslant k_{h(n)}^4\|x_{n-1}-p\|^2-\|x_n-p\|^2 \to 0.\end{aligned}$$

Hence, we have

$$\|x_n-x_{n-1}\| \leqslant \|x_n-z_{n-1}\|+\|z_{n-1}-x_{n-1}\| \to 0, \quad (n\to\infty).$$

It is obvious that

$$\lim_{n\to\infty}\|x_n-x_{n+j}\|=0, \quad \forall j\in\{1,2,...,N\}. \tag{3.12}$$

Notice that

$$\begin{aligned}\|x_n-T_{n+j}x_n\| &\leqslant \|x_n-x_{n+j}\|+\|x_{n+j}-T_{n+j}x_{n+j}\|+\|T_{n+j}x_{n+j}-T_{n+j}x_n\|\\ &\leqslant (1+L)\|x_n-x_{n+j}\|+\|x_{n+j}-T_{n+j}x_{n+j}\|,\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\|x_n-T_nx_n\| &\leqslant \|x_n-x_{n-1}\|+\|x_{n-1}-T_nx_{n-1}\|+\|T_nx_{n-1}-T_nx_n\|\\ &\leqslant (1+L)\|x_n-x_{n-1}\|+\|x_{n-1}-T_nx_{n-1}\|.\end{aligned} \tag{3.14}$$

For all $n>N$, we can write $n=(h(n)-1)N+i(n)$, where $i(n)\in\{1,2,...N\}$. Then,

$$n-N=(h(n)-1-1)N+i(n)=(h(n-N)-1)N+i(n-N),$$

that is, $h(n-N)=h(n)-1, i(n-N)=i(n)$. We can obtain that

$$\begin{aligned}\|x_{n-1}-T_nx_{n-1}\| &\leqslant \|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|+\|T_{i(n)}^{h(n)}x_{n-1}-T_nx_{n-1}\|\\ &\leqslant \|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|+\|T_{i(n)}^{h(n)}x_{n-1}-T_{i(n)}x_{n-1}\|\\ &\leqslant \|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|+L\|T_{i(n)}^{h(n)-1}x_{n-1}-x_{n-1}\|\\ &\leqslant \|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|+L(\|T_{i(n)}^{h(n)-1}x_{n-1}-T_{i(n-N)}^{h(n)-1}x_{n-N}\|\\ &\quad+\|T_{i(n-N)}^{h(n)-1}x_{n-N}-x_{(n-N)-1}\|+\|x_{(n-N)-1}-x_{n-1}\|),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\|T_{i(n)}^{h(n)-1}x_{n-1}-T_{i(n-N)}^{h(n)-1}x_{n-N}\| &= \|T_{i(n)}^{h(n)-1}x_{n-1}-T_{i(n)}^{h(n)-1}x_{n-N}\|\\ &\leqslant L\|x_{n-1}-x_{n-N}\|,\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\|T_{i(n-N)}^{h(n)-1}x_{n-N}-x_{(n-N)-1}\| &\leqslant \|T_{i(n-N)}^{h(n)-1}x_{n-N}-T_{i(n-N)}^{h(n-N)}x_{(n-N)-1}\| \\ &\quad+\|T_{i(n-N)}^{h(n-N)}x_{(n-N)-1}-x_{(n-N)-1}\| \\ &\leqslant \|T_{i(n-N)}^{h(n-N)}x_{(n-N)-1}-x_{(n-N)-1}\| \\ &\quad+L\|x_{n-N}-x_{(n-N)-1}\|.\end{aligned} \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), we have that

$$\begin{aligned}\|x_{n-1}-T_nx_{n-1}\| \leqslant \|x_{n-1}-T_{i(n)}^{h(n)}x_{n-1}\|+L(L\|x_{n-1}-x_{n-N}\| \\ +L\|x_{n-N}-x_{(n-N)-1}\|+\|T_{i(n-N)}^{h(n-N)}x_{(n-N)-1} \\ -x_{(n-N)-1}\|+\|x_{(n-N)-1}-x_{n-1}\|).\end{aligned}$$

From (3.11) and (3.12), we have that

$$\lim_{n\to\infty}\|x_{n-1}-T_nx_{n-1}\|=0. \tag{3.18}$$

Substituting (3.18) into (3.14), we can obtain

$$\lim_{n\to\infty}\|x_n-T_nx_n\|=0. \tag{3.19}$$

It follows from (3.12), (3.13) and (3.19) that

$$\lim_{n\to\infty}\|x_n-T_lx_n\|=0,\quad \forall l\in\{1,2,...,N\}.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}\rightharpoonup z$. Since $\|x_{n_j}-T_lx_{n_j}\|\to 0, (n\to\infty)$, from Lemma 2.3, we have $z\in F(T_l)$, for all $l\in\{1,2,...,N\}$. That is, $\omega_w(x_n)\subseteq\bigcap_{l=1}^N F(T_l)$.

Next, we show $\omega_w(x_n)\subseteq GMEP(\Theta,B,\varphi)$. From $\lim_{n\to\infty}\|z_{n-1}-x_n\|=0$ and condition (ii), we have that

$$\lim_{n\to\infty}\frac{\|z_{n-1}-x_n\|}{r_{n-1}}=0.$$

Since $x_n=K_{r_{n-1}}z_{n-1}$, we also have

$$\Omega(x_n,y)+\frac{1}{r_{n-1}}\langle y-x_n,x_n-z_{n-1}\rangle\geqslant 0,\quad \forall y\in C,$$

where

$$\Omega(x_n,y)=\Theta(x_n,y)+\langle Bx_n,y-x_n\rangle+\varphi(y)-\varphi(x_n).$$

It follows from (A2) that

$$\frac{1}{r_{n-1}}\langle y-x_n,x_n-z_{n-1}\rangle\geqslant-\Omega(x_n,y)\geqslant\Omega(y,x_n),\quad \forall y\in C.$$

Again from (A4) and $\lim_{n\to\infty}\frac{\|z_{n-1}-x_n\|}{r_{n-1}}=0$, we can obtain that

$$\Omega(y,z)\leqslant 0,\quad \forall y\in C,\quad \forall z\in\omega_w(x_n).$$

Put $y_t=ty+(1-t)z$, for all $t\in(0,1)$, $y\in C$ and $z\in\omega_w(x_n)$, then, $y_t\in C$. Therefore, $\Omega(y_t,z)\leqslant 0$. From (A1), we can obtain that $0=\Omega(y_t,y_t)\leqslant t\Omega(y_t,y)+(1-t)\Omega(y_t,z)\leqslant t\Omega(y_t,y)$. So, $\Omega(y_t,y)\geqslant 0$, for all $y\in C$. Taking $t\to 0$, we have $\Omega(z,y)\geqslant 0$, for all $y\in C$, then, $z\in GMEP(\Theta,B,\varphi)$. That is, $\omega_w(x_n)\subseteq GMEP(\Theta,B,\varphi)$, and $\omega_w(x_n)\subseteq\Gamma$.

Step 3. We prove that sequence $\{x_n\}$ converges weakly to a point in Γ.

Since $\omega_w(x_n)\subseteq\Gamma$ and $\lim_{n\to\infty}\|x_n-p\|$ exists, it follows from Lemma 2.5 that sequence $\{x_n\}$ converges weakly to a point in Γ. □

Theorem 3.2. *Let C be a nonempty closed convex subset of an infinite dimensional real Hilbert space H and let $\{T_i\}_{1\leqslant i\leqslant N}$ be a finite family of $\{s_i\}_{1\leqslant i\leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence $\{s_i\} \subseteq [0,1)$ and $\{k_{n,i}\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ and $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. Let $B : C \to H$ be an α-inverse-strongly monotone mapping. Assume that $\Theta : C \times C \to R$ satisfies (A1)–(A4), $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ and $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap GEP(\Theta, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ x_n \in C \ \textit{such that} \\ \qquad \Theta(x_n, y) + \langle Bz_{n-1}, y - x_n\rangle + \frac{1}{r_{n-1}}\langle y - x_n, x_n - z_{n-1}\rangle \geqslant 0, \quad \forall y \in C, \ \ n \geqslant 1, \end{cases} \tag{3.20}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and $\{r_n\}$ satisfying the following conditions:

(i) $\frac{\alpha_n}{\beta_n} \geqslant s, \forall n \geqslant 0$;

(ii) $r_n \in [a,b]$ *for some* $0 < a < b < 2\alpha, \forall n \geqslant 0$;

(iii) $\liminf_{n\to\infty} g_{n-1} > 0$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1}-s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$, *for all* $n \geqslant 1$.

Then sequence $\{x_n\}$ converges weakly to a point in Γ.

Proof. Our proof is divided into the following steps.

Step 1. We prove that $\lim_{n\to\infty}\|x_n - p\|$ exists, for all $p \in \Gamma$.

From algorithm (3.20), we can obtain that $x_n = T_{r_{n-1}}(z_{n-1} - r_{n-1}Bz_{n-1})$. Again from Lemma 2.1 and (2.3), we have that

$$\begin{aligned} \|x_n - p\|^2 &= \|T_{r_{n-1}}(z_{n-1} - r_{n-1}Bz_{n-1}) - T_{r_{n-1}}(p - r_{n-1}Bp)\|^2 \\ &\leqslant \|z_{n-1} - r_{n-1}Bz_{n-1} - (p - r_{n-1}Bp)\|^2 \\ &\leqslant \|z_{n-1} - p\|^2 + r_{n-1}(r_{n-1} - 2\alpha)\|Bz_{n-1} - Bp\|^2. \end{aligned}$$

Similar to the first step of Theorem 3.1, we find from conditions (ii) and (iii) that

$$\begin{aligned} \|x_n - p\|^2 &\leqslant \|z_{n-1} - p\|^2 + r_{n-1}(r_{n-1} - 2\alpha)\|Bz_{n-1} - Bp\|^2 \\ &\leqslant k_{h(n)}^4\|x_{n-1} - p\|^2 - g_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &\quad + r_{n-1}(r_{n-1} - 2\alpha)\|Bz_{n-1} - Bp\|^2 \\ &\leqslant (1 + k_{h(n)}^4 - 1)\|x_{n-1} - p\|^2. \end{aligned} \tag{3.21}$$

Since $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$, we have $\sum_{n=0}^{\infty}(k_n - 1) < \infty$. Hence $\sum_{n=0}^{\infty}(k_{h_n}^4 - 1) < \infty$. Again by Lemma 2.2, we have that $\lim_{n\to\infty}\|x_n - p\|$ exists. So, $\{\|x_n - p\|\}$ is bounded, this implies that $\{x_n\}$ is bounded.

Step 2. We prove that $\omega_w(x_n) \subseteq \Gamma$. The difference between Theorem 3.1 and Theorem 3.2 is the proof of $\lim_{n\to\infty}\|x_n - T_l x_n\| \to 0$, for all $l \in \{1, 2, ..., N\}$ and $\omega_w(x_n) \subseteq GEP(\Theta, B)$. From (3.21), we have that

$$g_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \leqslant k_{h(n)}^4\|x_{n-1} - p\|^2 - \|x_n - p\|^2,$$

and

$$r_{n-1}(2\alpha - r_{n-1})\|Bz_{n-1} - Bp\|^2 \leqslant k_{h(n)}^4\|x_{n-1} - p\|^2 - \|x_n - p\|^2.$$

Again from conditions (ii) and (iii), we have that

$$\lim_{n\to\infty}\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\| = 0,$$

and

$$\lim_{n\to\infty} \|Bz_{n-1} - Bp\| = 0.$$

Observe that

$$\begin{aligned}
\|z_{n-1} - x_{n-1}\| &= \|\alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1} - x_{n-1}\| \\
&= (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}y_{n-1} - x_{n-1}\| \\
&\leqslant (1-\alpha_{n-1})\|T_{i(n)}^{h(n)}y_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\| \\
&\quad + \|T_{i(n)}^{h(n)}x_{n-1} - x_{n-1}\| \\
&\leqslant (1-\alpha_{n-1})[L\|y_{n-1} - x_{n-1}\| \\
&\quad + \|T_{i(n)}^{h(n)}x_{n-1} - x_{n-1}\|] \\
&= (1-\alpha_{n-1})[(1-\beta_{n-1})L + 1]\|T_{i(n)}^{h(n)}x_{n-1} - x_{n-1}\| \\
&\to 0.
\end{aligned} \tag{3.22}$$

From (3.20) and Lemma 2.1, we have that

$$\begin{aligned}
\|x_n - p\|^2 &= \|T_{r_{n-1}}(z_{n-1} - r_{n-1}Bz_{n-1}) - T_{r_{n-1}}(p - r_{n-1}Bp)\|^2 \\
&\leqslant \langle z_{n-1} - r_{n-1}Bz_{n-1} - (p - r_{n-1}Bp), x_n - p\rangle \\
&= \frac{1}{2}(\|(z_{n-1} - r_{n-1}Bz_{n-1}) - (p - r_{n-1}Bp)\|^2) + \|x_n - p\|^2 \\
&\quad - \|(z_{n-1} - r_{n-1}Bz_{n-1}) - (p - r_{n-1}Bp) - (x_n - p)\|^2 \\
&\leqslant \frac{1}{2}(\|z_{n-1} - p\|^2 + \|x_n - p\|^2 - \|(z_{n-1} - x_n) - r_{n-1}(Bz_{n-1} - Bp)\|^2) \\
&= \frac{1}{2}(\|z_{n-1} - p\|^2 + \|x_n - p\|^2 - \|z_{n-1} - x_n\|^2 - r_{n-1}^2\|Bz_{n-1} - Bp\|^2 \\
&\quad + 2r_{n-1}\langle z_{n-1} - x_n, Bz_{n-1} - Bp\rangle).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_n - p\|^2 &\leqslant \|z_{n-1} - p\|^2 - \|z_{n-1} - x_n\|^2 - r_{n-1}^2\|Bz_{n-1} - Bp\|^2 \\
&\quad + 2r_{n-1}\langle z_{n-1} - x_n, Bz_{n-1} - Bp\rangle \\
&\leqslant k_{h(n)}^4\|x_{n-1} - p\|^2 - \|z_{n-1} - x_n\|^2 - r_{n-1}^2\|Bz_{n-1} - Bp\|^2 \\
&\quad + 2r_{n-1}\langle z_{n-1} - x_n, Bz_{n-1} - Bp\rangle.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|z_{n-1} - x_n\|^2 &\leqslant \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + (k_{h(n)}^4 - 1)\|x_{n-1} - p\|^2 \\
&\quad - r_{n-1}^2\|Bz_{n-1} - Bp\|^2 + 2r_{n-1}\langle z_{n-1} - x_n, Bz_{n-1} - Bp\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \|Bz_{n-1} - Bp\| = 0$ and $\lim_{n\to\infty} \|x_n - p\|$ exists, we obtain that

$$\lim_{n\to\infty} \|z_{n-1} - x_n\| = 0.$$

Therefore,

$$\|x_n - x_{n-1}\| \leqslant \|x_n - z_{n-1}\| + \|z_{n-1} - x_{n-1}\| \to 0.$$

The remaining is the same to the proof of $\lim_{n\to\infty} \|x_n - T_l x_n\| \to 0$, for all $l \in \{1, 2, ..., N\}$.

Next, we prove that $\omega_w(x_n) \subseteq GEP(\Theta, B)$. Since $x_n = T_{r_{n-1}}(z_{n-1} - r_{n-1}Az_{n-1})$, we have

$$\Theta(x_n, y) + \langle Bz_{n-1}, y - x_n\rangle + \frac{1}{r_{n-1}}\langle y - x_n, x_n - z_{n-1}\rangle \geqslant 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\langle Bz_{n-1}, y - x_n\rangle + \frac{1}{r_{n-1}}\langle y - x_n, x_n - z_{n-1}\rangle \geqslant \Theta(y, x_n). \tag{3.23}$$

Let $y_t = ty + (1-t)z$, where $y \in C$, $z \in \omega_w(x_n)$ and $t \in (0,1)$, it is easy to see that $y_t \in C$. Combining with (3.23), we can obtain that

$$\begin{aligned}\langle By_t, y_t - x_n\rangle &\geqslant \langle By_t, y_t - x_n\rangle - \langle Bz_{n-1}, y_t - x_n\rangle \\ &\quad - \frac{1}{r_{n-1}}\langle y_t - x_n, x_n - z_{n-1}\rangle + \Theta(y_t, x_n) \\ &\geqslant \langle By_t - Bx_n, y_t - x_n\rangle + \langle Bx_n - Bz_{n-1}, y_t - x_n\rangle \\ &\quad - \langle y_t - x_n, \frac{x_n - z_{n-1}}{r_{n-1}}\rangle + \Theta(y_t, x_n).\end{aligned}$$

It follows from $\lim_{n\to\infty}\|z_{n-1} - x_n\| = 0$ that $\lim_{n\to\infty}\frac{\|z_{n-1}-x_n\|}{r_{n-1}} = 0$. On the other hand, we have $\|Bx_n - Bz_{n-1}\| \to 0$. Again by monotonicity of B, we obtain that $\langle By_t - Bx_n, y_t - x_n\rangle \geqslant 0$. Then, replacing n by n_k and letting $k \to \infty$, from (A4), we can easily observe that

$$\langle By_t, y_t - z\rangle \geqslant \Theta(y_t, z).$$

Again combining with (A1) and (A4), we have

$$\begin{aligned}0 &= \Theta(y_t, y_t) \\ &\leqslant t\Theta(y_t, y) + (1-t)\Theta(y_t, z) \\ &\leqslant t\Theta(y_t, y) + (1-t)t\langle By_t, y - z\rangle.\end{aligned}$$

It follows that

$$\Theta(y_t, y) + (1-t)\langle By_t, y - z\rangle \geqslant 0.$$

Taking $t \to 0$, we obtain that

$$\Theta(z, y) + \langle Bz, y - z\rangle \geqslant 0, \quad \forall y \in C.$$

Then $z \in GEP(\Theta, B)$, that is, $\omega_w(x_n) \subseteq GEP(\Theta, B)$. The remaining is the same to the proof of Theorem 3.1. □

Remark 3.3. Since asymptotically nonexpansive mappings are 0-strict asymptotically pseudo-contractions, therefore, we can obtain two kinds of weak convergence theorems for a finite family of asymptotically nonexpansive mappings by putting $s_i = 0$, for all $i \in \{1, 2, ..., N\}$ in Theorem 3.1 with the condition replaced of the following:

(i) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;
(ii) $\{\alpha_n\} \subset [0,1)$, $\{\beta_n\} \subset (0,1)$;
(iii) $\limsup_{n\to\infty}\alpha_n < 1$, $0 < \liminf_{n\to\infty}\beta_n \leqslant \limsup_{n\to\infty}\beta_n < 1$,

and in Theorem 3.2 with the condition replaced of the following:

(i) $r_n \in [a, b]$ for some $0 < a < b < 2\alpha$, $\forall n \geqslant 0$;
(ii) $\{\alpha_n\} \subset [0,1)$, $\{\beta_n\} \subset (0,1)$;
(iii) $\limsup_{n\to\infty}\alpha_n < 1$, $0 < \liminf_{n\to\infty}\beta_n \leqslant \limsup_{n\to\infty}\beta_n < 1$.

3.2. Strong convergence theorems

Theorem 3.4. *Let C be a nonempty closed convex subset of an infinite dimensional real Hilbert space H and let $\{T_i\}_{1\leqslant i\leqslant N}$ be a finite family of $\{s_i\}_{1\leqslant i\leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence*

$\{s_i\} \subseteq [0,1)$ *and* $\{k_{n,i}\} \subseteq [1,\infty)$ *such that* $\lim_{n\to\infty} k_{n,i} = 1$ *and* $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. *Let* $\varphi : C \to R$ *be a proper lower semi-continuous and convex functional, and let* $B : C \to H$ *be a continuous and monotone mapping. Assume that* $\Theta : C \times C \to R$ *satisfies* (A1)–(A4), $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ *and* $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap GMEP(\Theta, B, \varphi) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ u_{n-1} \in C, \text{ such that for all } y \in C, \\ \qquad \Theta(u_{n-1}, y) + \langle Bu_{n-1}, y - u_{n-1}\rangle + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - z_{n-1}\rangle \geqslant 0, \\ C_{n-1} = \{v \in C : \|u_{n-1} - v\|^2 \leqslant \|x_{n-1} - v\|^2 + \theta_{n-1} - h_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2\}, \\ Q_{n-1} = \{v \in C : \langle x_0 - x_{n-1}, x_{n-1} - v\rangle \geqslant 0\}, \\ x_n = P_{C_{n-1}\bigcap Q_{n-1}}x_0, \quad \forall n \geqslant 1, \end{cases}$$

where $\theta_n = (k_{h(n)}^4 - 1)\rho_n$, $\rho_n = \sup\{\|x_n - p\| : p \in \Gamma\} < \infty$. *When* $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ *and* $\{r_n\}$ *satisfying the following conditions:*

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;
(ii) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;
(iii) $\liminf_{n\to\infty} g_{n-1} > 0$, $\forall n \geqslant 1$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1} - s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$.

Then sequence $\{x_n\}$ *converges strongly to* $P_\Gamma x_0$.

Proof. Our proof is divided into the following five steps.

Step 1. We show that C_{n-1} and Q_{n-1} are closed and convex for all $n \geqslant 1$.

From the definition of C_{n-1} and Q_{n-1}, it is obvious that C_{n-1} is closed and Q_{n-1} is closed and convex. From Lemma 2.4, we know that C_{n-1} is also convex.

Step 2. We show that $\Gamma \subseteq C_{n-1} \bigcap Q_{n-1}$, for all $n \geqslant 1$.

The proof of $\Gamma \subseteq C_{n-1}$ is similar to the first step of Theorem 3.1, we only replace x_n with u_{n-1} in (3.9) and (3.10). Then, we can obtain for all $p \in \Gamma$,

$$\begin{aligned} \|u_{n-1} - p\|^2 &\leqslant \|z_{n-1} - p\|^2 \\ &\leqslant k_{h(n)}^4\|x_{n-1} - p\|^2 - g_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2 \\ &\leqslant \|x_{n-1} - p\|^2 + \theta_{n-1} - g_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2, \end{aligned}$$

that is, $\Gamma \subseteq C_{n-1}$, for all $n \geqslant 1$.

We prove that $\Gamma \subseteq Q_{n-1}$, for all $n \geqslant 1$ by induction. We have $\Gamma \subseteq C = Q_0$. Suppose $\Gamma \subseteq Q_{n-1}$. Since $x_n = P_{C_{n-1}\bigcap Q_{n-1}}x_0$, we find from (2.2) that

$$\langle x_0 - x_n, x_n - u\rangle \geqslant 0, \quad \forall u \in C_{n-1} \bigcap Q_{n-1}.$$

Since $\Gamma \subseteq C_{n-1} \bigcap Q_{n-1}$, we have $\Gamma \subseteq Q_n$.

Step 3. We prove that $\{x_n\}$ is bounded.

Since the definition of Q_{n-1}, we know that $x_{n-1} = P_{Q_{n-1}}x_0$. Also, $x_n \in C_{n-1} \bigcap Q_{n-1} \subseteq Q_{n-1}$. Then

$$\|x_{n-1} - x_0\| \leqslant \|x_n - x_0\|.$$

Therefore, the sequence $\{\|x_n - x_0\|\}$ is nondecreasing. Again since $\Gamma \subseteq Q_{n-1}$, for all $n \geqslant 1$. It follows that

$$\|x_{n-1} - x_0\| \leqslant \|p - x_0\|, \quad \forall p \in \Gamma. \tag{3.24}$$

So, we have $\lim_{n\to\infty}\|x_n - x_0\|$ exists, that is, $\{x_n\}$ is bounded.

Step 4. We prove that $\omega_w(x_n) \subseteq \Gamma$.

The difference between Theorem 3.4 and Theorem 3.1 is the proof of

$$\lim_{n\to\infty}\|x_n - T_l x_n\| \to 0, \quad \forall l \in \{1, 2, ..., N\},$$

and $\omega_w(x_n) \subseteq GMEP(\Theta, B, \varphi)$.

First, we show that $\lim_{n\to\infty}\|x_n - T_l x_n\| \to 0$, for all $l \in \{1, 2, ..., N\}$. Obviously, we only need to prove that $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty}\|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| = 0$. Since the definition of Q_{n-1}, we know that $x_{n-1} = P_{Q_{n-1}} x_0$. Considering $x_n \in C_{n-1} \bigcap Q_{n-1} \subseteq Q_{n-1}$, we can obtain that

$$\langle x_n - x_{n-1}, x_{n-1} - x_0\rangle \geqslant 0.$$

From (2.2), we know that

$$\begin{aligned}\|x_n - x_{n-1}\|^2 &= \|(x_n - x_0) - (x_{n-1} - x_0)\|^2\\ &= \|x_n - x_0\|^2 - \|x_{n-1} - x_0\|^2\\ &\quad - 2\langle x_n - x_{n-1}, x_{n-1} - x_0\rangle\\ &\leqslant \|x_n - x_0\|^2 - \|x_{n-1} - x_0\|^2,\end{aligned}$$

which together with the existence of $\lim_{n\to\infty}\|x_n - x_0\|$, we get that

$$\|x_n - x_{n-1}\| \to 0, \quad (n \to \infty). \tag{3.25}$$

By the definition of C_{n-1}, we have

$$\|u_{n-1} - v\|^2 \leqslant \|x_{n-1} - v\|^2 + \theta_{n-1}, \quad \forall v \in C_{n-1}.$$

Again since $x_n = P_{C_{n-1}\bigcap Q_{n-1}} x_0 \in C_{n-1}$, we have

$$\|u_{n-1} - x_n\|^2 \leqslant \|x_{n-1} - x_n\|^2 + \theta_n.$$

Combining (3.25) and $\theta_n \to 0 (n \to \infty)$, it is obvious that $\lim_{n\to\infty}\|u_{n-1} - x_n\| = 0$.

$$\|u_{n-1} - x_{n-1}\| \leqslant \|u_{n-1} - x_n\| + \|x_n - x_{n-1}\| \to 0, \quad (n \to \infty). \tag{3.26}$$

From the definition of C_{n-1}, we have

$$g_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\|^2 \leqslant \|x_{n-1} - p\|^2 + \theta_n - \|u_{n-1} - p\|^2.$$

From (3.26), $\theta_n \to 0 (n \to \infty)$ and condition (iii), we can obtain

$$\|x_{n-1} - T_{i(n)}^{h(n)} x_{n-1}\| \to 0, \quad (n \to \infty).$$

The remaining is the same to Theorem 3.1.

Next, we show that $\omega_w(x_n) \subseteq GMEP(\Theta, B, \varphi)$.

We only need to prove that $\lim_{n\to\infty}\|u_{n-1} - z_{n-1}\| = 0$, for all $n \geqslant 1$. From (3.22) and (3.26), we have

$$\|u_{n-1} - z_{n-1}\| \leqslant \|u_{n-1} - x_{n-1}\| + \|x_{n-1} - z_{n-1}\| \to 0. \tag{3.27}$$

The remaining is the same to Theorem 3.1. Therefore, $\omega_w(x_n) \subseteq \Gamma$.

Step 5. We prove that $x_n \to x^* = P_\Gamma x_0$.

Combining with (3.24) and lemma 2.6, we can obtain that sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma x_0$. □

Theorem 3.5. *Let C be a nonempty closed convex subset of an infinite dimensional real Hilbert space H and let $\{T_i\}_{1\leqslant i\leqslant N}$ be a finite family of $\{s_i\}_{1\leqslant i\leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence $\{s_i\} \subseteq [0,1)$ and $\{k_{n,i}\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ and $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. Let $B : C \to H$ be an α-inverse-strongly monotone mapping. Assume that $\Theta : C\times C \to R$ satisfies (A1)-(A4), $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ and $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap GEP(\Theta, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ u_{n-1} \in C, \ \textit{such that for all } y \in C, \\ \qquad \Theta(u_{n-1}, y) + \langle Bz_{n-1}, y-u_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y-u_{n-1}, u_{n-1}-z_{n-1}\rangle \geqslant 0, \quad \forall y \in C, \\ C_n = \{v \in C : \|u_{n-1}-v\|^2 \leqslant \|x_{n-1}-v\|^2 + \theta_{n-1} - h_{n-1}\|x_{n-1} - T_{i(n)}^{h(n)}x_{n-1}\|^2\}, \\ Q_n = \{v \in C : \langle x_0 - x_{n-1}, x_{n-1} - v\rangle \geqslant 0\}, \\ x_n = P_{C_{n-1}\bigcap Q_{n-1}}x_0, \quad \forall n \geqslant 1, \end{cases}$$

where $\theta_n = (k_{h(n)}^4 - 1)\rho_n$, $\rho_n = \sup\{\|x_n - p\| : p \in \Gamma\} < \infty$. When $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and $\{r_n\}$ satisfying the following conditions:

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;
(ii) $r_n \in [a,b]$ *for some* $0 < a < b < 2\alpha$, $\forall n \geqslant 0$;
(iii) $\liminf_{n\to\infty} g_{n-1} > 0$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h_n}^2(\beta_{n-1}-s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$, $\forall n \geqslant 1$.

Then sequence $\{x_n\}$ converges strongly to $P_\Gamma x_0$.

Proof. The process of proof is similar to the proof of Theorem 3.4 except $\omega_w(x_n) \subseteq GEP(\Theta, B)$. The proof of $\omega_w(x_n) \subseteq GEP(\Theta, B)$ is similar to the proof of Theorem 3.2, we only replace x_n by u_{n-1} and use (3.27). This completes the proof. □

Remark 3.6. Similarly, in Theorem 3.4 and Theorem 3.5, if $s_i = 0, 1 \leqslant i \leqslant N$ and the conditions which sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy in Theorem 3.4 and Theorem 3.5 replaced of the following:

(i) $\{\alpha_n\} \subset [0,1)$, $\{\beta_n\} \subset (0,1)$;
(ii) $\limsup_{n\to\infty} \alpha_n < 1$, $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$.

Then, two kinds of strong convergence theorems for a finite family of asymptotically nonexpansive mappings can be obtained.

4. Applications

In this section, we introduce the several applications of weak convergence theorem (Theorem 3.1). According to different situations, the corresponding strong convergence theorem (Theorem 3.4) has different results.

4.1. Application to (mixed) equilibrium problem (EP and MEP)

In (1.2), if $B = \varphi \equiv 0$, the generalized mixed equilibrium problem reduces the equilibrium problem (EP), that is, to find $x \in C$ such that

$$\Theta(x, y) \geqslant 0, \quad \forall y \in C.$$

If $B \equiv 0$, the generalized mixed equilibrium problem reduces the mixed equilibrium problem (MEP), that is, to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C.$$

Therefore, the result of Theorem 3.1 can be applied to (mixed) equilibrium problem (EP and MEP), then, we have the following results.

Theorem 4.1. *Let* C *be a nonempty closed convex subset of an infinite dimensional real Hilbert space* H *and let* $\{T_i\}_{1\leqslant i\leqslant N}$ *be a finite family of* $\{s_i\}_{1\leqslant i\leqslant N}$*-strictly asymptotically pseudo-contractive mappings with sequence* $\{s_i\} \subseteq [0,1)$ *and* $\{k_{n,i}\} \subseteq [1,\infty)$ *such that* $\lim_{n\to\infty} k_{n,i} = 1$ *and* $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. *Assume that* $\Theta : C \times C \to R$ *satisfies* (A1)–(A4), $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ *and* $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap EP(\Theta) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ x_n \in C, \quad \textit{such that} \quad \Theta(x_n, y) + \frac{1}{r_{n-1}}\langle y - x_n, x_n - z_{n-1}\rangle \geqslant 0, \quad \forall y \in C, \ \ n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *and* $\{r_n\}$ *satisfying the following conditions:*

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;
(ii) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;
(iii) $\liminf_{n\to\infty} g_{n-1} > 0$, $\forall n \geqslant 1$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1}-s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$.

Then sequence $\{x_n\}$ *converges weakly to a point in* Γ.

Theorem 4.2. *Let* C *be a nonempty closed convex subset of an infinite dimensional real Hilbert space* H *and let* $\{T_i\}_{1\leqslant i\leqslant N}$ *be a finite family of* $\{s_i\}_{1\leqslant i\leqslant N}$*-strictly asymptotically pseudo-contractive mappings with sequence* $\{s_i\} \subseteq [0,1)$ *and* $\{k_{n,i}\} \subseteq [1,\infty)$ *such that* $\lim_{n\to\infty} k_{n,i} = 1$ *and* $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. *Let* $\varphi : C \to R$ *be a proper lower semi-continuous and convex functional. Assume that* $\Theta : C \times C \to R$ *satisfies* (A1)-(A4), $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ *and* $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap MEP(\Theta,\varphi) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ x_n \in C, \quad \textit{such that} \quad \Theta(x_n, y) + \varphi(y) - \varphi(x_n) + \frac{1}{r_{n-1}}\langle y - x_n, x_n - z_{n-1}\rangle \geqslant 0, \quad \forall y \in C, \ \ n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *and* $\{r_n\}$ *satisfying the following conditions:*

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;
(ii) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;
(iii) $\liminf_{n\to\infty} g_{n-1} > 0$, $\forall n \geqslant 1$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1}-s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$.

Then sequence $\{x_n\}$ *converges weakly to a point in* Γ.

4.2. *Application to (mixed) variational inequality*

A variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geqslant 0, \quad \forall y \in C.$$

The solution set of VIP is denoted by $VI(B, C)$.

The mixed variational inequality is to find $x \in C$ such that

$$\langle Bx, y-x\rangle + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C. \tag{4.1}$$

We denote the solution set of (4.1) with $VI(B, C, \varphi)$.

If $\Theta = \varphi \equiv 0$, the generalized mixed equilibrium problem reduces a variational inequality problem.

If $\Theta \equiv 0$, the generalized mixed equilibrium problem reduces the mixed variational inequality.

Putting $F(x, y) = \langle Bx, y-x\rangle$, if B is an α-inverse-strongly monotone mapping, we can easily show that F satisfies conditions (A1)-(A4). Then, the following theorems can be obtained from Theorem 3.1.

Theorem 4.3. *Let C be a nonempty closed convex subset of an infinite dimensional real Hilbert space H and let $\{T_i\}_{1\leqslant i\leqslant N}$ be a finite family of $\{s_i\}_{1\leqslant i\leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence $\{s_i\} \subseteq [0, 1)$ and $\{k_{n,i}\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ and $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $B : C \to H$ be an α-inverse-strongly monotone mapping. Assume that $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ and $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap VI(B, C) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ x_n \in C, \quad \textit{such that} \quad \langle Bx_n, y-x_n\rangle + \frac{1}{r_{n-1}}\langle y-x_n, x_n - z_{n-1}\rangle \geqslant 0, \quad \forall y \in C, \ \ n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{r_n\}$ satisfying the following conditions:

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;

(ii) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;

(iii) $\liminf_{n\to\infty} g_{n-1} > 0$, $\forall n \geqslant 1$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1} - s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$.

Then sequence $\{x_n\}$ converges weakly to a point in Γ.

Theorem 4.4. *Let C be a nonempty closed convex subset of an infinite dimensional real Hilbert space H and let $\{T_i\}_{1\leqslant i\leqslant N}$ be a finite family of $\{s_i\}_{1\leqslant i\leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence $\{s_i\} \subseteq [0, 1)$ and $\{k_{n,i}\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ and $\sum_{n=0}^{\infty}(k_{n,i} - 1) < \infty$. Let $\varphi : C \to R$ be a proper lower semi-continuous and convex functional and $B : C \to H$ be an α-inverse-strongly monotone mapping. Assume that $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ and $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap VI(B, C, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ x_n \in C, \quad \textit{such that} \quad \langle Bx_n, y-x_n\rangle + \varphi(y) - \varphi(x_n) + \frac{1}{r_{n-1}}\langle y-x_n, x_n - z_{n-1}\rangle \geqslant 0, \quad \forall y \in C, \ \ n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{r_n\}$ satisfying the following conditions:

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;
(ii) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;
(iii) $\liminf_{n\to\infty} g_{n-1} > 0$, $\forall n \geqslant 1$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1} - s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$.

Then sequence $\{x_n\}$ converges weakly to a point in Γ.

4.3. Application to convex minimization problem

In (1.2), if $\Theta = B \equiv 0$, the generalized mixed equilibrium problem reduces a convex minimization problem, that is, to find $x \in C$ such that

$$\varphi(y) \geqslant \varphi(x), \quad \forall y \in C. \tag{4.2}$$

We denote the solution set of (4.2) with $CMP(\varphi)$.

Therefore, Theorem 3.1 can reduce the following theorem about convex minimization problem.

Theorem 4.5. *Let C be a nonempty closed convex subset of an infinite dimensional real Hilbert space H and let $\{T_i\}_{1\leqslant i\leqslant N}$ be a finite family of $\{s_i\}_{1\leqslant i\leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence $\{s_i\} \subseteq [0,1)$ and $\{k_{n,i}\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ and $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. Let $\varphi : C \to R$ be a proper lower semi-continuous and convex functional. Assume that $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ and $\Gamma = \bigcap_{i=1}^{N} F(T_i) \bigcap CMP(\varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \;\; \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ x_n \in C, \quad \textit{such that} \quad \varphi(y) - \varphi(x_n) + \frac{1}{r_{n-1}}\langle y - x_n, x_n - z_{n-1}\rangle \geqslant 0, \quad \forall y \in C, \;\; n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\}$ satisfying the following conditions:

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;
(ii) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;
(iii) $\liminf_{n\to\infty} g_{n-1} > 0$, $\forall n \geqslant 1$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1}-s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$.

Then sequence $\{x_n\}$ converges weakly to a point in Γ.

4.4. Application to convex feasibility problem

The convex feasibility problem for a family of mappings $\{T_i\}_{1\leqslant i\leqslant N}$ is to find a point x such that $x \in \bigcap_{i=1}^{N} F(T_i)$.

Therefore, Theorem 3.1 can also reduce the following theorem about convex feasibility problem.

Theorem 4.6. *Let C be a nonempty closed convex subset of an infinite dimensional real Hilbert space H and let $\{T_i\}_{1\leqslant i\leqslant N}$ be a finite family of $\{s_i\}_{1\leqslant i\leqslant N}$-strictly asymptotically pseudo-contractive mappings with sequence $\{s_i\} \subseteq [0,1)$ and $\{k_{n,i}\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_{n,i} = 1$ and $\sum_{n=0}^{\infty}(k_{n,i}-1) < \infty$. Assume that $s = \max\{s_i : 1 \leqslant i \leqslant N\}$, $\{k_n\} = \max\{k_{n,i} : 1 \leqslant i \leqslant N\}$ and $\Gamma = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in C \;\; \textit{chosen arbitrarily}, \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ \quad x_n = P_C z_{n-1}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\}$ satisfying the following conditions:

(i) $\frac{\alpha_n}{\beta_n} \geqslant s$, $\forall n \geqslant 0$;
(ii) $\liminf_{n\to\infty} r_n > 0$, $\forall n \geqslant 0$;
(iii) $\liminf_{n\to\infty} g_{n-1} > 0$, $\forall n \geqslant 1$,
where $g_{n-1} = (1-\alpha_{n-1})(1-\beta_{n-1})[k_{h(n)}^2(\beta_{n-1}-s) - (1-\beta_{n-1})^2L^2s + \beta_{n-1}s]$.

Then sequence $\{x_n\}$ converges weakly to a point in Γ.

5. Numerical experiments

In this section, respectively, we give the corresponding numerical examples of Theorem 3.1 and Theorem 3.2.

Example 5.1. Let $H := R$ and $C := [-1, 1] \subset R$. For all $x, y \in C$, $\Theta(x, y) = y^2 + xy - 2x^2$, $Bx = 2x$, $\varphi(x) = \frac{1}{2}x^2$. It is obvious that $\Theta : C \times C \to R$ is a real-valued bifunction satisfying the following conditions (A1)–(A4), $B : C \to H$ is a monotone mapping and $\varphi : C \to R$ is a continuous and convex functional.

Then, for given $r > 0$ and $x \in H$, by Lemma 2.1, there exists a unique $u \in C$ such that

$$\Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geqslant 0, \quad \forall y \in C.$$

$$\Updownarrow$$

$$y^2 + uy - 2u^2 + 2u(y - u) + \frac{1}{2}y^2 - \frac{1}{2}u^2 + \frac{1}{r}(y - u)(u - x) \geqslant 0, \quad \forall y \in C.$$

$$\Updownarrow$$

$$3ry^2 + (6ru + 2u - 2x)y + 2ux - 2u - 9ru^2 \geqslant 0, \quad \forall y \in C.$$

Let $F(y) = 3ry^2 + (6ru + 2u - 2x)y + 2ux - 2u - 9ru^2$. Then discriminant Δ of $F(y)$ is

$$\Delta = (6ru + 2u - 2x)^2 - 4 \times 3ry(2ux - 2u - 9ru^2) = [2x - 2u(1 + 6r)]^2.$$

Taking $\Delta \leqslant 0$, then for all $y \in C, F(y) \geqslant 0$. Again from uniqueness of u, we have that

$$u = K_r x = \frac{1}{1 + 6r}x.$$

From algorithm (3.1), we can obtain that

$$\begin{cases} y_{n-1} = \beta_{n-1}x_{n-1} + (1 - \beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ \quad x_n = \frac{1}{1+6r_{n-1}}z_{n-1}. \end{cases}$$

Let $N = 2$, $T_1x = x$, for all $x \in [-1, 1]$,

$$T_2x := \begin{cases} x, & x \in [-1, 0); \\ \frac{1}{2}x, & x \in [0, 1], \end{cases}$$

and taking $x_0 = 1$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2(n+1)}$, $r_n = \frac{n}{4(n+1)}$. We have the following numerical results:

Table 1: numerical examples of Theorem 3.1.

n	0	1	2	3	4	5	6	...
x_n	1	1	0.3750	0.1875	0.0308	0.0140	0.0014	...

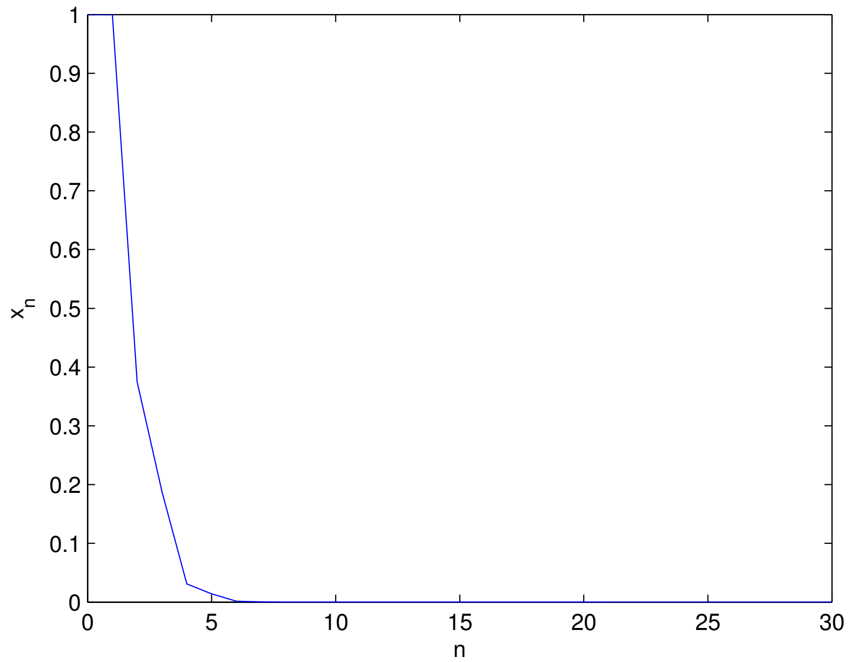

Figure 1: numerical examples of Theorem 3.1.

Example 5.2. Let $H := R$ and $C := [-1, 1] \subset R$. For all $x, y \in C$, $\Theta(x, y) = y^2 + xy - 2x^2$, $Bx = 2x$. Then, for given $r > 0$ and $x \in H$, by Lemma 2.1, there exists a unique $u \in C$ such that

$$\Theta(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geqslant 0, \quad \forall y \in C.$$

Similar to the above method, we can obtain that

$$u = T_r x = \frac{1}{1 + 3r}.$$

From algorithm (3.20), we can obtain that

$$\begin{cases} y_{n-1} = \beta_{n-1}x_{n-1} + (1 - \beta_{n-1})T_{i(n)}^{h(n)}x_{n-1}, \\ z_{n-1} = \alpha_{n-1}x_{n-1} + (1 - \alpha_{n-1})T_{i(n)}^{h(n)}y_{n-1}, \\ \quad x_n = T_{r_{n-1}}(z_{n-1} - r_{n-1}Bz_{n-1}) = \frac{1-2r_{n-1}}{1+3r_{n-1}}z_{n-1}. \end{cases}$$

Let N, T_1, T_2, α_n, β_n, r_n, x_0 are the same as before.

Table 2: numerical examples of Theorem 3.2.

n	0	1	2	3	4	5	6	...
x_n	1	1	0.3580	0.1591	0.0222	0.0083	6.5189×10^{-4}	...

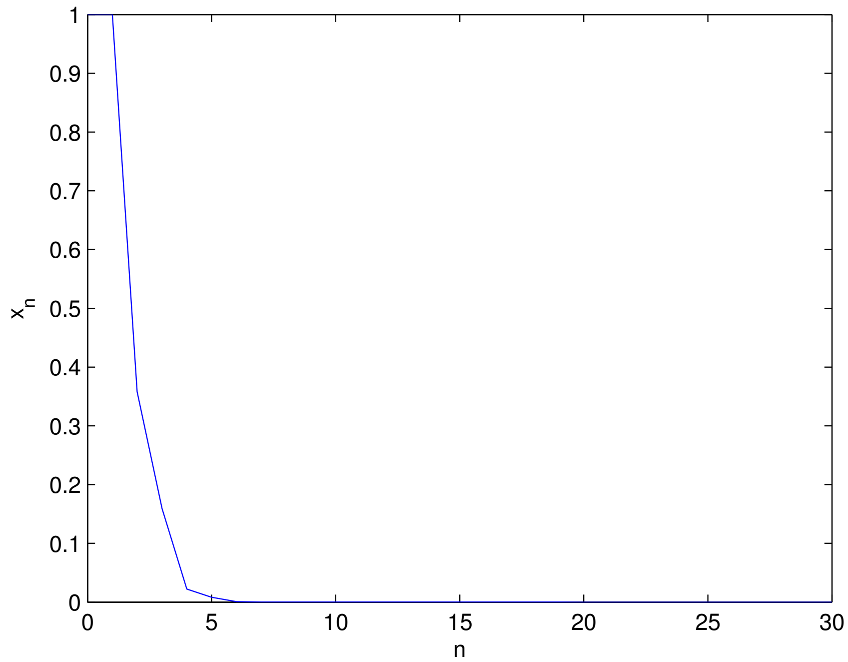

Figure 2: numerical examples of Theorem 3.2.

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