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A note on spectral properties of a Dirac system with matrix coefficient

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Abstract

In this paper, we find a polynomial-type Jost solution of a self-adjoint matrix-valued discrete Dirac system. Then we investigate analytical properties and asymptotic behavior of this Jost solution. Using the Weyl compact perturbation theorem, we prove that matrix-valued discrete Dirac system has continuous spectrum filling the segment $[-2, 2]$. Finally, we examine the properties of the eigenvalues of this Dirac system and we prove that it has a finite number of simple real eigenvalues.

Keywords: Discrete Dirac system, spectral analysis, Jost solution, eigenvalue.
2010 MSC: 39A05, 39A70, 39A10, 47A05, 47A10.

1. Introduction

Consider the boundary value problem (BVP) consisting of the Sturm–Liouville equation

$$\begin{cases} -y'' + q(x)y = \lambda^2 y, & 0 \leqslant x < \infty, \\ y(0) = 0, \end{cases} \tag{1.1}$$

and the boundary condition where q is a real-valued function and λ is a spectral parameter. The bounded solution of (1.1) satisfying the condition

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geqslant 0\}$$

will be denoted by $e(.,\lambda)$. The solution $e(.,\lambda)$ is called the Jost solution of (1.1). In [17], the author presented a condition depending on the function q that guaranteed $e(.,\lambda)$ has an integral representation as

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}\, dt < \infty,$$

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where the function K is defined in terms of q. Moreover, the author showed that $e(.,\lambda)$ is analytic with respect to λ in $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > 0\}$, continuous in $\overline{\mathbb{C}}_+$ and satisfies

$$e(x,\lambda) = e^{i\lambda x}[1 + o(1)], \quad \lambda \in \overline{\mathbb{C}}_+, \quad x \to \infty.$$

The function $e(\lambda) := e(0,\lambda)$ is called Jost function of the BVP (1.1). The functions $e(.,\lambda)$ and $e(\lambda)$ play an important role in the solutions of direct and inverse problems of the quantum scattering theory [7, 15, 17, 18]. The Jost solutions are especially useful in the study of the spectral analysis of differential and difference operators [1, 5, 6, 14]. Therefore Jost solutions of Dirac systems, Schrödinger and discrete Sturm–Liouville equations have been obtained in [8, 11, 12]. Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain linear and nonlinear problems from economics, optimal control theory and other areas of study has led to the rapid development of the theory of difference equations. Also the spectral analysis of the difference equations has been treated by various authors in connection with the classical moment problem [2, 4, 13]. The spectral theory of the difference equations have also been applied to the solution of classes of nonlinear discrete Korteveg-de Vries equations and Toda lattices [19]. Let us introduce the Hilbert space $\ell_2(\mathbb{N},\mathbb{C}^{2m})$ consisting of all vector sequences $y = \{y_n\}$, $y_n = \binom{y_n^{(1)}}{y_n^{(2)}}$, where $y_n^{(i)} \in \mathbb{C}^m$, $i = 1,2$, $n \in \mathbb{N} = \{1,2,\dots\}$ and $\mathbb{C}^m$ is $m-$dimensional $(m < \infty)$ Euclidean space. In $\ell_2(\mathbb{N},\mathbb{C}^{2m})$, the norm and inner product are defined by

$$\|y\|_{\ell_2}^2 := \sum_{n=1}^{\infty} \left(\left\| y_n^{(1)} \right\|_{\mathbb{C}^m}^2 + \left\| y_n^{(2)} \right\|_{\mathbb{C}^m}^2 \right) < \infty,$$

$$\langle y,z\rangle_{\ell_2} := \sum_{n=1}^{\infty} \left[\left(y_n^{(1)}, z_n^{(1)} \right)_{\mathbb{C}^m} + \left(y_n^{(2)}, z_n^{(2)} \right)_{\mathbb{C}^m} \right],$$

where $\|.\|_{\mathbb{C}^m}$ and $(.,.)_{\mathbb{C}^m}$ denote the norm and inner product in $\mathbb{C}^m$, respectively. Now consider the matrix-valued discrete Dirac system

$$\begin{cases} A_n y_{n+1}^{(2)} + B_n y_n^{(2} + P_n y_n^{(1)} = \lambda y_n^{(1)}, \\ A_{n-1} y_{n-1}^{(1)} + B_n y_n^{(1)} + Q_n y_n^{(2)} = \lambda y_n^{(2)}, \end{cases} \quad n \in \mathbb{N}, \tag{1.2}$$

with the boundary condition

$$y_0^{(1)} = 0, \tag{1.3}$$

where A_n, $n \in \mathbb{N} \cup \{0\}$ and B_n, Q_n, P_n, $n \in \mathbb{N}$ are linear operators (matrices) acting in $\mathbb{C}^m$. Throughout the paper, we will assume that $\det A_n \neq 0$, $A_n = A_n^*$ $(n \in \mathbb{N} \cup \{0\})$, $\det B_n \neq 0$, $B_n = B_n^*$, $Q_n = Q_n^*$ and $P_n = P_n^*$ $(n \in \mathbb{N})$, where $*$ denotes the adjoint operator. Let L denote the operator generated in $\ell_2\left(\mathbb{N},\mathbb{C}^{2m}\right)$ by the BVP (1.2)-(1.3). The operator L is self-adjoint, i.e., $L = L^*$. In the following, we will assume that the matrix sequences $\{A_n\}$, $\{B_n\}$, $\{P_n\}$, and $\{Q_n\}$ $(n \in \mathbb{N})$, satisfy

$$\sum_{n=1}^{\infty} n \left(\|I - A_n\| + \|I + B_n\| + \|P_n\| + \|Q_n\| \right) < \infty, \tag{1.4}$$

where $\|.\|$ and I denote the matrix norm and the identity matrix in $\mathbb{C}^m$, respectively. The setup of this paper is as follows: In Section 2, we find a polynomial-type Jost solution of (1.2), investigate analytical properties, and asymptotic behavior of this Jost solution. In Section 3, we show that $\sigma_c(L) = [-2,2]$, where $\sigma_c(L)$ denotes the continuous spectrum of L. Also, we prove that under the condition (1.4), the operator L has a finite number of simple real eigenvalues. To the best of our knowledge, this paper is the first one that focuses on matrix-valued discrete Dirac system including a polynomial-type Jost solution.

2. Jost solution of (1.2)

Assume $P_n \equiv Q_n \equiv 0$, $B_n \equiv -I$ for all $n \in \mathbb{N}$, and $A_n \equiv I$ for all $n \in \mathbb{N} \cup \{0\}$ in (1.2). Then we get

$$\begin{cases} y_{n+1}^{(2)} - y_n^{(2)} = \left[-iz - (iz)^{-1}\right] y_n^{(1)}, \\ y_{n-1}^{(1)} - y_n^{(1)} = \left[-iz - (iz)^{-1}\right] y_n^{(2)} \end{cases} \tag{2.1}$$

for $\lambda = -iz - (iz)^{-1}$. It is clear that

$$e_n(z) = \begin{pmatrix} e_n^{(1)}(z) \\ e_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} z \\ -i \end{pmatrix} z^{2n}, \qquad n \in \mathbb{N},$$

is a solution of (2.1). Now, we will find the solution $\binom{F_n(z)}{G_n(z)}$, $n \in \mathbb{N}$ of (1.2) for $\lambda = -iz - (iz)^{-1}$, satisfying the condition

$$\begin{pmatrix} F_n(z) \\ G_n(z) \end{pmatrix} = [I + o(1)]\, e_n(z), \qquad |z| = 1,\ n \to \infty.$$

The solution $\binom{F_n(z)}{G_n(z)}$, $n \in \mathbb{N}$ is called the Jost solution of (1.2) for $\lambda = -iz - (iz)^{-1}$.

Theorem 2.1. *Assume that (1.4) holds. Then for $\lambda = -iz - (iz)^{-1}$ and $|z| = 1$, (1.2) has the solution $\binom{F_n(z)}{G_n(z)}$, $n \in \mathbb{N}$ having the representation*

$$\begin{pmatrix} F_n(z) \\ G_n(z) \end{pmatrix} = T_n \left(I + \sum_{m=1}^{\infty} K_{nm} z^{2m} \right) \begin{pmatrix} z \\ -i \end{pmatrix} z^{2n},\ n \in \mathbb{N}, \tag{2.2}$$

$$F_0(z) = T_0^{11} z + T_0^{11} \sum_{m=1}^{\infty} K_{0m}^{11} z^{2m+1} - i T_0^{11} \sum_{m=1}^{\infty} K_{0m}^{12} z^{2m},$$

where $K_{nm} = \begin{pmatrix} K_{nm}^{11} & K_{nm}^{12} \\ K_{nm}^{21} & K_{nm}^{22} \end{pmatrix}$, $T_n = \begin{pmatrix} T_n^{11} & T_n^{12} \\ T_n^{21} & T_n^{22} \end{pmatrix}$ *and these are expressed in terms of* $\{A_n\}$, $\{B_n\}$, $\{P_n\}$, *and* $\{Q_n\}$.

Proof. Substituting $\binom{F_n(z)}{G_n(z)}$ defined by (2.2) into (1.2) and by taking $\lambda = -iz - (iz)^{-1}$ for $|z| = 1$, we get

$$\begin{aligned}
T_n^{12} &= 0, \\
T_n^{22} &= \left(\prod_{p=n}^{\infty} (-1)^{n-p} A_p B_p \right)^{-1}, \\
T_n^{11} &= -B_n \left(\prod_{p=n}^{\infty} (-1)^{n-p} A_p B_p \right)^{-1}, \\
T_n^{21} &= -Q_n T_n^{22} - A_{n-1} T_{n-1}^{11} \left(\sum_{p=n}^{\infty} \left(T_p^{11}\right)^{-1} B_p Q_p T_p^{22} - \sum_{p=n}^{\infty} \left(T_p^{11}\right)^{-1} P_p T_p^{11} \right), \\
K_{n1}^{12} &= \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} B_p Q_p T_p^{22} - \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} P_p T_p^{11}, \\
K_{n1}^{11} &= \sum_{p=n+1}^{\infty} \left[-I + \left(T_p^{11}\right)^{-1} \left(B_p^2 T_p^{11} + A_p T_{p+1}^{22} + B_p Q_p T_p^{21} + B_p T_p^{22} + P_p T_p^{11} K_{p1}^{12} \right) \right], \\
K_{n1}^{22} &= \left(T_n^{22}\right)^{-1} \left[B_n T_n^{11} + Q_n T_n^{21} + T_n^{22} + A_{n-1} T_{n-1}^{11} K_{n-1,1}^{11} - T_n^{21} K_{n1}^{12} \right],
\end{aligned}$$

$$
\begin{aligned}
K_{n1}^{21} = & \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} T_p^{21} \left(K_{p1}^{11} - I\right) + \sum_{p=n+1}^{\infty} \left[\left(T_p^{22}\right)^{-1} \left(B_p T_p^{11} + Q_p T_p^{21}\right) K_{p1}^{12}\right] \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} Q_p T_p^{22} K_{p1}^{22} + \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} A_{p-1} T_{p-1}^{11} K_{p-1,1}^{12} \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} A_{p-1}^2 T_p^{21} + \sum_{p=n}^{\infty} \left(T_{p+1}^{22}\right)^{-1} \left[A_p P_p T_p^{11} + A_p B_p T_p^{21}\right] K_{p1}^{11}, \\
K_{n2}^{12} = & \sum_{p=n+1}^{\infty} \left[\left(T_p^{11}\right)^{-1} B_p Q_p \left(T_p^{21} K_{p1}^{12} + T_p^{22} K_{p1}^{22}\right)\right] - \sum_{p=n+1}^{\infty} K_{p1}^{12} \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p^2 T_p^{11} K_{p1}^{12} - P_p T_p^{11} K_{p1}^{11} - A_p T_{p+1}^{21} - B_p T_p^{21}\right], \\
K_{n2}^{11} = & \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p^2 T_p^{11} K_{p1}^{11} + P_p T_p^{11} K_{p2}^{12} + B_p T_p^{22} K_{p1}^{22} + B_p T_p^{21} K_{p1}^{12}\right] \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p Q_p T_p^{21} K_{p1}^{11} + B_p Q_p T_p^{22} K_{p1}^{21}\right] - \sum_{p=n+1}^{\infty} K_{p1}^{11} \\
& + \sum_{p=n+2}^{\infty} \left(T_{p-1}^{11}\right)^{-1} \left[A_{p-1} T_p^{22} K_{p1}^{22} + A_{p-1} T_p^{21} K_{p1}^{12}\right], \\
K_{n2}^{22} = & -\sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} \left[B_p T_p^{11} K_{p1}^{11} - T_p^{21} K_{p2}^{12} + Q_p T_p^{21} K_{p1}^{11} + Q_p T_p^{22} K_{p1}^{21} + T_p^{21} K_{p1}^{12}\right] \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} \left[A_{p-1} P_{p-1} T_{p-1}^{11} K_{p-1,2}^{12} + A_{p-1} B_{p-1} T_{p-1}^{21} K_{p-1,2}^{12}\right] \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} \left[-A_{p-1} T_{p-1}^{11} K_{p-1,1}^{11}\right] - \sum_{p=n+1}^{\infty} K_{p1}^{22} \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} \left[A_{p-1}^2 T_p^{22} K_{p1}^{12} + A_{p-1}^2 T_p^{21} K_{p,1}^{12}\right], \\
K_{n2}^{21} = & \sum_{p=n}^{\infty} \left(T_{p+1}^{22}\right)^{-1} \left[A_p T_p^{11} K_{p2}^{12} + A_p P_p T_p^{11} K_{p2}^{11} + A_p B_p T_p^{21} K_{p2}^{11}\right] \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} \left[A_{p-1}^2 T_p^{21} K_{p1}^{11} + A_{p-1}^2 T_p^{22} K_{p,1}^{21} + B_p T_p^{11} K_{p2}^{12}\right] - \sum_{p=n+1}^{\infty} K_{p1}^{21} \\
& + \sum_{p=n+1}^{\infty} \left(T_p^{22}\right)^{-1} \left[Q_p T_p^{21} K_{p2}^{12} + Q_p T_p^{22} K_{p2}^{22} + T_p^{21} K_{p2}^{11} - T_p^{21} K_{p1}^{11}\right],
\end{aligned}
$$

where $n \in \mathbb{N}$. Furthermore, for $m \geqslant 3$ and $n \in \mathbb{N}$, we obtain that

$$
\begin{aligned}
K_{nm}^{12} = & -\sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[P_p T_p^{11} K_{p,m-1}^{11} - B_p^2 T_p^{11} K_{p,m-1}^{12} + B_p T_p^{21} K_{p,m-2}^{11}\right] \\
& + \sum_{p=n+1}^{\infty} \left(K_{p,m-2}^{21} - K_{p,m-1}^{12}\right) + \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p Q_p^2 T_p^{21} K_{p,m-2}^{11}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[A_p T_{p+1}^{21} K_{p+1,m-2}^{11} + A_p T_{p+1}^{22} K_{p+1,m-2}^{21}\right] \\
&+ \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p Q_p A_{p-1} T_{p-1}^{11} K_{p-1,m-1}^{11} + B_p Q_p T_p^{11} K_{p,m-2}^{11}\right] \\
&+ \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p Q_p^2 T_p^{22} K_{p,m-2}^{21} + B_p Q_p T_p^{21} K_{p,m-2}^{12} + B_p Q_p T_p^{22} K_{p,m-2}^{22}\right],
\end{aligned}
$$

$$
\begin{aligned}
K_{nm}^{11} = &\sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[P_p T_p^{11} K_{pm}^{12} + B_p^2 T_p^{11} K_{p,m-1}^{11} + B_p Q_p T_p^{21} K_{p,m-1}^{11}\right] \\
&- \sum_{p=n+1}^{\infty} K_{p,m-1}^{11} + \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p Q_p T_p^{22} K_{p,m-1}^{21}\right] \\
&+ \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[A_p T_{p+1}^{22} K_{p+1,m-1}^{22} + A_p T_{p+1}^{21} K_{p+1,m-1}^{12}\right] \\
&+ \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p T_p^{21} K_{p,m-1}^{12} + B_p T_p^{22} K_{p,m-1}^{22}\right],
\end{aligned}
$$

$$
\begin{aligned}
K_{nm}^{22} = &K_{n,m-1}^{22} + \left(T_n^{22}\right)^{-1} \left[A_{n-1} T_{n-1}^{11} K_{n-1,m}^{11} - T_n^{21} K_{nm}^{21} + B_n T_n^{11} K_{n,m-1}^{11}\right] \\
&+ \left(T_n^{22}\right)^{-1} \left[Q_n T_n^{21} K_{n,m-1}^{11} + Q_n T_n^{22} K_{n,m-1}^{21} + T_n^{21} K_{n,m-1}^{12}\right], \\
K_{nm}^{21} = &K_{n,m-1}^{21} + \left(T_n^{22}\right)^{-1} \left[-A_{n-1} T_{n-1}^{11} K_{n-1,m+1}^{12} - B_n T_n^{11} K_{nm}^{12}\right] \\
&+ \left(T_n^{22}\right)^{-1} \left[-Q_n T_n^{21} K_{nm}^{12} + T_n^{21} K_{n,m-1}^{11} - Q_n T_n^{22} K_{nm}^{22} - T_n^{21} K_{nm}^{11}\right].
\end{aligned}
$$

By the condition (1.4), the infinite products and the series in the definition of T_n^{ij} and K_{nm}^{ij} $(i, j = 1, 2)$ are absolutely convergent. Therefore, T_n^{ij} and K_{nm}^{ij} $(i, j = 1, 2)$ can uniquely be defined by $\{A_n\}$, $n \in \mathbb{N} \cup \{0\}$, $\{B_n\}$, $\{P_n\}$, and $\{Q_n\}$, $n \in \mathbb{N}$, i.e., the system (1.2) for $\lambda = -iz - (iz)^{-1}$ has the solution $\binom{F_n(z)}{G_n(z)}$ given by (2.2). □

Theorem 2.2. *If the condition* (1.4) *holds, then*

$$
\left\|K_{nm}^{ij}\right\| \leqslant C \sum_{p=n+\lfloor \frac{m}{2} \rfloor}^{\infty} \left(\|I - A_p\| + \|I + B_p\| + \|Q_p\| + \|P_p\|\right), \quad i, j = 1, 2, \tag{2.3}
$$

where $\lfloor \frac{m}{2} \rfloor$ *is the integer part of* $\frac{m}{2}$ *and* $C > 0$ *is a constant.*

Proof. We will use the method of induction to prove the theorem. For $m = 1$, we get that

$$
\begin{aligned}
\left\|K_{n1}^{12}\right\| &= \left\| \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left[B_p Q_p T_p^{22} - P_p T_p^{11}\right] \right\| \leqslant A \sum_{p=n+1}^{\infty} \left\|B_p Q_p T_p^{22} - P_p T_p^{11}\right\| \\
&\leqslant A' \sum_{p=n+1}^{\infty} \|Q_p\| + \sum_{p=n+1}^{\infty} \|P_p\| \leqslant C \sum_{p=n+1}^{\infty} \left(\|Q_p\| + \|P_p\|\right) \\
&\leqslant C \sum_{p=n}^{\infty} \left(\|I - A_p\| + \|I + B_p\| + \|Q_p\| + \|P_p\|\right) \\
&= C \sum_{p=n+\lfloor \frac{1}{2} \rfloor}^{\infty} \left(\|I - A_p\| + \|I + B_p\| + \|Q_p\| + \|P_p\|\right),
\end{aligned}
$$

where $A = \left\| \left(T_p^{11}\right)^{-1} \right\|$, $A' = A \|B_p\| \|T_p^{22}\|$, $C = \max\left\{1, A'\right\}$. Similar to this inequality, we can get (2.3) for K_{n1}^{11}, K_{n1}^{22}, and K_{n1}^{21}. Now, if we suppose that (2.3) is correct for $m = k$, then we can write

$$
\begin{aligned}
\|K_{n,k+1}^{12}\| \leqslant & \left\| \sum_{p=n+1}^{\infty} K_{p,k-1}^{21} - \sum_{p=n+1}^{\infty} K_{p,k}^{12} \right\| \\
& + \left\| \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left\{ -P_p T_p^{11} K_{pk}^{11} + B_p^2 T_p^{11} K_{pk}^{12} - B_p T_p^{21} K_{p,k-1}^{11} \right\} \right\| \\
& + \left\| \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left\{ -A_p T_{p+1}^{21} K_{p+1,k-1}^{11} - A_p T_{p+1}^{22} K_{p+1,k-1}^{21} \right\} \right\| \\
& + \left\| \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left\{ B_p Q_p A_{p-1} T_{p-1}^{11} K_{p-1,k}^{11} + B_p Q_p B_p T_p^{11} K_{p,k-1}^{11} \right\} \right\| \\
& + \left\| \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left\{ B_p Q_p^2 T_p^{21} K_{p,k-1}^{11} + B_p Q_p T_p^{22} K_{p,k-1}^{21} \right\} \right\| \\
& + \left\| \sum_{p=n+1}^{\infty} \left(T_p^{11}\right)^{-1} \left\{ B_p Q_p T_p^{21} K_{p,k-1}^{12} + B_p Q_p T_p^{22} K_{p,k-1}^{22} \right\} \right\| .
\end{aligned}
$$

If we use $T_{p+1}^{22} = A_p T_p^{11}$ for last inequality, we find

$$
\begin{aligned}
\|K_{n,k+1}^{12}\| \leqslant & \sum_{p=n+1}^{\infty} \|K_{p,k-1}^{21} - K_{p+1,k-1}^{21}\| + \sum_{p=n+1}^{\infty} \left\| \left(T_p^{11}\right)^{-1} \left(I - A_p^2\right) T_p^{11} K_{p+1,k-1}^{21} \right\| \\
& + \sum_{p=n+1}^{\infty} \left\| \left(T_p^{11}\right)^{-1} \left(-B_p - I\right) T_p^{21} K_{p,k-1}^{11} \right\| + \sum_{p=n+1}^{\infty} \left\| \left(T_p^{11}\right)^{-1} \left(I - A_p\right) T_{p+1}^{21} K_{p+1,k-1}^{21} \right\| \\
& + \sum_{p=n+1}^{\infty} \left\| \left(T_p^{11}\right)^{-1} \left(T_p^{21} K_{p,k-1}^{11} - T_{p+1}^{21} K_{p+1,k-1}^{11}\right) \right\| \\
& + C'' \sum_{p=n+1}^{\infty} \|B_p + I\| \sum_{s=p+\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} \|N_s\| + \sum_{p=n+1}^{\infty} \|P_p\| \sum_{s=p+\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} \|N_s\| \\
& + B \sum_{p=n+1}^{\infty} \|Q_p\| \sum_{s=p+\left\lfloor \frac{k-2}{2} \right\rfloor}^{\infty} \|N_s\| + D \sum_{p=n+1}^{\infty} \|Q_p\| \sum_{s=p+\left\lfloor \frac{k-1}{2} \right\rfloor}^{\infty} \|N_s\| ,
\end{aligned}
$$

where $\|N_s\| = \|I - A_s\| + \|I + B_s\| + \|Q_s\| + \|P_s\|$ and C'', B, D are constants. It follows from that

$$
\begin{aligned}
\|K_{n,k+1}^{12}\| \leqslant & \|K_{n+1,k-1}^{21}\| + D' \sum_{p=n+1}^{\infty} \|I - A_p\| \sum_{s=p+1+\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} \|N_s\| + D'' \sum_{p=n+1}^{\infty} \|B_p + I\| \sum_{s=p+\left\lfloor \frac{k-1}{2} \right\rfloor}^{\infty} \|N_s\| \\
& + D''' \sum_{p=n+1}^{\infty} \|I - A_p\| \sum_{s=p+1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{\infty} \|N_s\| + T \|K_{n+1,k-1}^{11}\| + C'' \sum_{p=n+1}^{\infty} \|B_p + I\| \sum_{s=p+\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} \|N_s\| \\
& + \|P_p\| \sum_{s=p+\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} \|N_s\| + B \sum_{p=n+1}^{\infty} \|Q_p\| \sum_{s=p+\left\lfloor \frac{k-2}{2} \right\rfloor}^{\infty} \|N_s\| + D \sum_{p=n+1}^{\infty} \|Q_p\| \sum_{s=p+\left\lfloor \frac{k-1}{2} \right\rfloor}^{\infty} \|N_s\| ,
\end{aligned}
$$

where $D^{'}$, $D^{''}$, $D^{'''}$ and T are also constants. Using last inequality, we obtain

$$\begin{aligned}\left\|K^{12}_{n,k+1}\right\| \leqslant C \sum_{p=n+1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{\infty} \|N_p\| + TC \sum_{p=n+1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{\infty} \|N_p\| + \left(D^{'}+D^{'''}\right) \sum_{p=n+1}^{\infty} \|I-A_p\| \sum_{s=p+\left\lfloor\frac{k+1}{2}\right\rfloor}^{\infty} \|N_s\| \\ + \max\left\{D^{''},D\right\} \sum_{p=n+1}^{\infty} \left(\|B_p+I\|+\|Q_p\|\right) \sum_{s=p+\left\lfloor\frac{k-1}{2}\right\rfloor}^{\infty} \|N_s\| \\ + \max\left\{C^{''},1\right\} \sum_{p=n+1}^{\infty} \left(\|B_p+I\|+\|P_p\|\right) \sum_{s=p+\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} \|N_s\| + B \sum_{p=n+1}^{\infty} \|N_p\| \sum_{s=p+\left\lfloor\frac{k-2}{2}\right\rfloor}^{\infty} \|N_s\|\end{aligned}$$

and

$$\begin{aligned}\left\|K^{12}_{n,k+1}\right\| &\leqslant Z \sum_{p=n+\left\lfloor\frac{k+1}{2}\right\rfloor}^{\infty} \|N_p\| + Y \sum_{p=n+1}^{\infty} \|N_p\| \sum_{s=p+\left\lfloor\frac{k-2}{2}\right\rfloor}^{\infty} \|N_s\| \\ &\leqslant Z \sum_{p=n+\left\lfloor\frac{k+1}{2}\right\rfloor}^{\infty} \|N_p\| + Y \left\{ \sum_{p=n+1}^{\infty} \|N_p\| \sum_{s=p+\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} \|N_s\| \right\} \\ &\leqslant Z \sum_{p=n+\left\lfloor\frac{k+1}{2}\right\rfloor}^{\infty} \|N_p\| + Y^{'} \sum_{p=n+\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} \|N_p\| \\ &\leqslant 2Z \sum_{p=n+\left\lfloor\frac{k+1}{2}\right\rfloor}^{\infty} \|N_p\| + Y^{'} \sum_{p=n+\left\lfloor\frac{k+1}{2}\right\rfloor}^{\infty} \|N_p\| \leqslant G \sum_{p=n+\left\lfloor\frac{k+1}{2}\right\rfloor}^{\infty} \|N_p\|,\end{aligned}$$

where $C+TC=Z$, $Y=D^{'}+D^{'''}+B+\max\left\{C^{''},1\right\}+\max\left\{D^{''},D\right\}$, $Y^{'}=Y\sum_{p=n+1}^{\infty}\|N_p\|$, and $2Z+Y^{'}=G$. Similar to $K^{12}_{n,k+1}$, we can easily obtain (2.3) for $K^{11}_{n,k+1}$, $K^{21}_{n,k+1}$, and $K^{22}_{n,k+1}$. □

It follows from (2.2) and (2.3) that $\binom{F_n(z)}{G_n(z)}$ $n \in \mathbb{N}\cup\{0\}$ has analytic continuation from

$$D_0 := \{z\in\mathbb{C} : |z|=1\} \text{ to } \{z\in\mathbb{C} : |z|<1\}\backslash\{0\}.$$

Theorem 2.3. *Assume that (1.4) holds. Then the Jost solution satisfies*

$$\begin{pmatrix}F_n(z)\\G_n(z)\end{pmatrix} = [I+o(1)]\begin{pmatrix}z\\-i\end{pmatrix}z^{2n},\ n\to\infty \tag{2.4}$$

for $z\in D := \{z\in\mathbb{C} : |z|\leqslant 1\}\backslash\{0\}$.

Proof. It follows from (2.2) that

$$\begin{pmatrix}F_n(z)\\G_n(z)\end{pmatrix} = \begin{pmatrix}T^{11}_n & T^{12}_n\\ T^{21}_n & T^{22}_n\end{pmatrix}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix} + \sum_{m=1}^{\infty}\begin{pmatrix}K^{11}_{nm} & K^{12}_{nm}\\ K^{21}_{nm} & K^{22}_{nm}\end{pmatrix}z^{2m}\right]\begin{pmatrix}z^{2n+1}\\-iz^{2n}\end{pmatrix},$$

then using (1.4), (2.3), and the definition of T^{ij}_n for $i,j=1,2$, we get

$$\begin{pmatrix}T^{11}_n & T^{12}_n\\ T^{21}_n & T^{22}_n\end{pmatrix} \to I,\ n\to\infty, \tag{2.5}$$

and

$$\sum_{m=1}^{\infty}\begin{pmatrix}K^{11}_{nm} & K^{12}_{nm}\\ K^{21}_{nm} & K^{22}_{nm}\end{pmatrix}z^{2m} = o(1),\ z\in D,\ n\to\infty. \tag{2.6}$$

From (2.2), (2.5), and (2.6), we find (2.4). □

3. Continuous and discrete spectrum of L

Theorem 3.1. *Under the condition* (1.4), $\sigma_c(L) = [-2, 2]$.

Proof. Let L_0 denote the operator generated in $\ell_2(\mathbb{N}, \mathbb{C}^{2m})$ by the difference expression

$$(l_1 y)_n := \begin{cases} y_{n+1}^{(2)} - y_n^{(2)}, \\ y_{n-1}^{(1)} - y_n^{(1)}, \end{cases}$$

with the boundary condition $y_0^{(1)} = 0$. We also define the operator J in $\ell_2(\mathbb{N}, \mathbb{C}^{2m})$ by

$$J\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} := \begin{pmatrix} P_n & 0 \\ 0 & Q_n \end{pmatrix}\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} + \begin{pmatrix} I + B_n & 0 \\ 0 & I + B_n \end{pmatrix}\begin{pmatrix} y_n^{(2)} \\ y_n^{(1)} \end{pmatrix} + \begin{pmatrix} A_n - I & 0 \\ 0 & A_{n-1} - I \end{pmatrix}\begin{pmatrix} y_{n+1}^{(2)} \\ y_{n-1}^{(1)} \end{pmatrix}$$
$$= \begin{pmatrix} (A_n - I)\, y_{n+1}^{(2)} + (I + B_n)\, y_n^{(2)} + P_n y_n^{(1)} \\ (A_{n-1} - I)\, y_n^{(1)} + (I + B_n)\, y_n^{(1)} + Q_n y_n^{(2)} \end{pmatrix}.$$

It is clear that $L_0 = L_0^*$ and $L = L_0 + J$. Since L_0 is self-adjoint, its spectrum contains its eigenvalues and continuous spectrum, but the operator L_0 has no eigenvalues. Moreover, we easily prove that $\sigma(L_0) = [-2, 2]$, where $\sigma(L_0)$ shows the spectrum of the operator L_0. So we can write that

$$\sigma(L_0) = \sigma_c(L_0) = [-2, 2].$$

Using (1.4), we also get that the operator J is compact in $\ell_2(\mathbb{N}, \mathbb{C}^{2m})$ [16]. By the Weyl theorem [9, p. 13] of a compact perturbation, we obtain

$$\sigma_c(L) = \sigma_c(L_0) = [-2, 2].$$

This completes the proof. □

Since the operator L is self-adjoint, the eigenvalues of L are real. From the definition of the eigenvalues, we can write

$$\sigma_d(L) = \left\{\lambda \in \mathbb{R} : \lambda = -iz - (iz)^{-1},\ iz \in (-1, 0) \cup (0, 1),\ \det F_0(z) = 0\right\},$$

where $\sigma_d(L)$ denotes the set of all eigenvalues of L.

Definition 3.2. The multiplicity of a zero of the function $\det F_0(z)$ is called the multiplicity of the corresponding eigenvalue of L.

Theorem 3.3. *Assume that* (1.4) *holds. Then the operator* L *has a finite number of simple real eigenvalues.*

Proof. To prove the theorem, we have to show that the function $\det F_0(z)$ has a finite number of simple zeros. Let z_0 be one of the zeros of $\det F_0(z)$. Hence $\det F_0(z_0) = 0$, there is a non-zero vector u such that $F_0(z_0)\, u = 0$ [3]. As we know, $\binom{F_n(z)}{G_n(z)}$ is the Jost solution of (1.2) for $\lambda = -iz - (iz)^{-1}$, i.e.,

$$\begin{cases} A_n G_{n+1}(z) + B_n G_n(z) + P_n F_n(z) = \left[-iz - (iz)^{-1}\right] F_n(z), \\ A_{n-1} F_{n-1}(z) + B_n F_n(z) + Q_n G_n(z) = \left[-iz - (iz)^{-1}\right] G_n(z). \end{cases} \tag{3.1}$$

Differentiating (3.1) with respect to z, we have

$$\begin{aligned} A_n \frac{d}{dz} G_{n+1}(z) + B_n \frac{d}{dz} G_n(z) + P_n \frac{d}{dz} F_n(z) &= \left[-iz - (iz)^{-1}\right] \frac{d}{dz} F_n(z) - i\left(1 - z^{-2}\right) F_n(z), \\ A_{n-1} \frac{d}{dz} F_{n-1}(z) + B_n \frac{d}{dz} F_n(z) + Q_n \frac{d}{dz} G_n(z) &= \left[-iz - (iz)^{-1}\right] \frac{d}{dz} G_n(z) - i\left(1 - z^{-2}\right) G_n(z). \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we obtain

$$\begin{aligned} &\left(\frac{d}{dz}F_n(z)\right)^* A_n G_{n+1}(z) + \left(\frac{d}{dz}F_n(z)\right)^* B_n G_n(z) \\ &\quad - \left(\frac{d}{dz}G_{n+1}(z)\right)^* A_n F_n(z) - \left(\frac{d}{dz}G_n(z)\right)^* B_n F_n(z) \\ &= \left[-iz-(iz)^{-1}\right]\left(\frac{d}{dz}F_n(z)\right)^* F_n(z) \\ &\quad - \overline{\left[-iz-(iz)^{-1}\right]}\left(\frac{d}{dz}F_n(z)\right)^* F_n(z) + \overline{i\,(1-z^{-2})}F_n^*(z)\,F_n(z) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\left(\frac{d}{dz}G_n(z)\right)^* A_{n-1} F_{n-1}(z) + \left(\frac{d}{dz}G_n(z)\right)^* B_n F_n(z) \\ &\quad - \left(\frac{d}{dz}F_{n-1}(z)\right)^* A_{n-1} G_n(z) - \left(\frac{d}{dz}F_n(z)\right)^* B_n G_n(z) \\ &= \left[-iz-(iz)^{-1}\right]\left(\frac{d}{dz}G_n(z)\right)^* G_n(z) \\ &\quad - \overline{\left[-iz-(iz)^{-1}\right]}\left(\frac{d}{dz}G_n(z)\right)^* G_n(z) + \overline{i\,(1-z^{-2})}G_n^*(z)\,G_n(z). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned} &\left(\frac{d}{dz}G_1(z)\right)^* A_0 F_0(z) - \left(\frac{d}{dz}F_0(z)\right)^* A_0 G_1(z) \\ &= \left[-iz-(iz)^{-1}\right]\sum_{n=1}^{\infty}\left[\left(\frac{d}{dz}F_n(z)\right)^* F_n(z) + \left(\frac{d}{dz}G_n(z)\right)^* G_n(z)\right] \\ &\quad - \overline{\left(iz-(iz)^{-1}\right)}\sum_{n=1}^{\infty}\left[\left(\frac{d}{dz}F_n(z)\right)^* F_n(z) + \left(\frac{d}{dz}G_n(z)\right)^* G_n(z)\right] \\ &\quad + i\overline{(1-z^{-2})}\sum_{n=1}^{\infty}\left[F_n^*(z)\,F_n(z) + G_n^*(z)\,G_n(z)\right]. \end{aligned} \tag{3.5}$$

If we write (3.5) for $z = z_0$, we obtain

$$\begin{aligned} &\left(\frac{d}{dz}G_1(z_0)\right)^* A_0 F_0(z_0) - \left(\frac{d}{dz}F_0(z_0)\right)^* A_0 G_1(z_0) \\ &= -i\left(1-\overline{z_0^{-2}}\right)\sum_{n=1}^{\infty}\left[F_n^*(z_0)\,F_n(z_0) + G_n^*(z_0)\,G_n(z_0)\right] \end{aligned} \tag{3.6}$$

using $iz_0 \in (-1,0)\cup(0,1)$. Then if we multiply (3.6) with the vector u on the right side ($u \in \ell_2(\mathbb{N},\mathbb{C}^{2m})$), ($u \neq 0$), we get

$$\left\langle A_0 G_1(z_0)\,u, \frac{d}{dz}F_0(z_0)\,u\right\rangle = \left(i - \frac{i}{(iz_0)^2}\right)\left\{\sum_{n=1}^{\infty}\|F_n(z_0)\,u\|^2 + \sum_{n=1}^{\infty}\|G_n(z_0)\,u\|^2\right\}.$$

Since $iz_0 \neq 0$ and $iz_0 \neq 1$, we can write $i - \dfrac{i}{(iz_0)^2} \neq 0$. Also we can write $\|F_n(z_0)\,u\| \neq 0$ and $\|G_n(z_0)\,u\| \neq 0$ for all $n \in \mathbb{N}$, so

$$\left\langle A_0 G_1(z_0)\,u, \frac{d}{dz}F_0(z_0)\,u\right\rangle \neq 0.$$

This shows that $\frac{d}{dz}F_0(z_0)u \neq 0$, that is, all zeros of $\det F_0(z)$ are simple. To complete the proof of theorem, we have to show that the function $\det F_0(z)$ has a finite number of zeros. Let us take the function

$$M(z) = z^{-1}\left(T_0^{11}\right)^{-1} F_0(z) = I + A(z),$$

where $A(z) = \sum_{m=1}^{\infty} K_{0m}^{11} z^{2m} - i\sum_{m=1}^{\infty} K_{0m}^{12} z^{2m-1}$. Since $A(z)$ is matrix-valued analytic function on D, the function M has inverse on the boundary of D [10, Theorem 5.1], i.e., the set of limit points of the set of zeros of

$$\det F_0(z) = 0 \tag{3.7}$$

is empty. Therefore, the set of zeros of (3.7) in D is finite, i.e., the operator L has a finite number of eigenvalues. □

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