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Strong convergence theorems for a nonexpansive mapping and its applications for solving the split feasibility problem

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Abstract

The aim of this paper is to propose some novel algorithms and their strong convergence theorems for solving the split feasibility problem, and we obtain the corresponding strong convergence results under mild conditions. The split feasibility problem was proposed by [Y. Censor, Y. Elfving, Numer. Algorithms, **8** (1994), 221–239]. So far a lot of algorithms have been given for solving this problem due to its applications in intensity-modulated radiation therapy, signal processing, and image reconstruction. But most of these algorithms are of weak convergence. In this paper, we propose the new algorithms which can provide useful guidelines for solving the relevant problem, such as the split common fixed point problem (SCFP), multi-set split feasibility problem and so on.

Keywords: Split feasibility problem, strong convergence, nonexpansive mapping, Hilbert space.
2010 MSC: 47J25, 47H45.

1. Introduction and preliminaries

The split feasibility problem (SFP) was first introduced by Censor and Elfving [5] in 1994. The SFP is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where C is a nonempty closed convex subset of a Hilbert space H_1, Q is a nonempty closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

As we know, the SFP has received so much attention due to its applications in intensity-modulated radiation therapy, signal processing, and image reconstruction, see Byrne [1, 2], Censor [4–6], Ceng [3], Fan et al. [7], Xu [20, 21], Kraikaew and Saejung [9], Moudafi [10], Qu et al. [12–14], Qin and Yao [11], Yang et al. [16, 22, 27, 28], Yao et al. [23–26], and so on.

To solve the SFP (1.1), many algorithms have been constructed.

In 2002, the so-called CQ algorithm was proposed by Byrne [1, 2] in the following:

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \quad n \geqslant 0,$$

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where $0 < \gamma < 2/\rho$ with ρ being the spectral radius of the operator A^*A and P_C, P_Q denotes the orthogonal projection onto the sets C, Q, respectively. However, the stepsize of the CQ algorithm is fixed and related to spectral radius of the operator A^*A, and the orthogonal projection onto the sets C and Q is not easily calculated usually.

In 2004, Yang [22] constructed a relaxed CQ algorithm for solving a special case of the SFP, in which he replaced them by projections onto halfspaces C_k and Q_k. In 2005, Qu and Xiu [13] modified Yang's relaxed CQ algorithm and the CQ algorithm by adopting the Armijo-like searches to get the stepsize.

In 2008, Qu and Xiu [14] proposed a halfspace relaxation projection method for the SFP, based on a reformulation of the SFP.

Recently, Xu [21] applied Mann's algorithm to the SFP and proposed an averaged CQ algorithm which was proved to be weakly converge to a solution of the SFP. Very recently, Qu et al. [12] studied the computation of the step-size for the CQ-like algorithms for the split feasibility problem.

In this paper, based on such research results, we propose some novel algorithms for the nonexpansive mapping and construct their strong convergence theorems, and we apply these convergence theorems for solving the split feasibility problem.

We use $\to$ to denote strong convergence and $\rightharpoonup$ for weak convergence, and we use $\mathrm{Fix}(T)$ to denote the fixed point set of the operator T. Some concepts and lemmas will be useful in proving our main results as follows:

Let H be a Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Then the following inequality holds

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{1.2}$$

Definition 1.1. An operator $T : H \to H$ is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - z\|, \quad \forall x \in H.$$

(ii) ν-inverse strongly monotone (ν-ism), with $\nu > 0$, if

$$\langle x - y, Tx - Ty\rangle \geqslant \nu\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Definition 1.2. Let C be a nonempty closed convex subset of a Hilbert space H, the metric (nearest point) projection P_C from H to C is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

Lemma 1.3 ([19])**.** *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 1.4 ([15])**.** *Let C be a nonempty closed convex subset of a Hilbert space H, P_C is a nonexpansive mapping from H onto C and is characterized as: given $x \in H$, there hold the inequality*

$$\langle x - P_C x, y - P_C x\rangle \leqslant 0, \quad \forall y \in C.$$

Lemma 1.5 ([17, 18])**.** *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \gamma_n\sigma_n, \quad n \geqslant 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ are such that

(i) $\lim\limits_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n = \infty$,

(ii) *either* $\limsup\limits_{n\to\infty} \sigma_n \leqslant 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.

Then $\{a_n\}_{n=0}^{\infty}$ converges to zero.

2. Main results

Theorem 2.1. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H_1 *and* $\theta \in C$*, let* $T : C \to C$ *be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$*. Given* $\{\alpha_n\}_{n=1}^{\infty}$*,* $\{\beta_n\}_{n=1}^{\infty}$*, and* $\{\lambda_n\}_{n=1}^{\infty}$ *in* $(0,1)$*, the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 1$, $\lim\limits_{n\to\infty} \beta_n = 1$, $\lim\limits_{n\to\infty} \lambda_n = 1$;
(ii) $|\lambda_n - \beta_{n-1}\lambda_{n-1}| + \beta_n \leqslant 1$, $\sum_{n=0}^{\infty}(1-\beta_n)(1-\lambda_n) = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Let $\{x_n\}$ *be generated by* $x_1 \in C$ *and*

$$\begin{cases} x_{n+1} = (1-\beta_n)(\lambda_n x_n) + \beta_n y_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n. \end{cases} \tag{2.1}$$

Then the sequence $\{x_n\}$ *converges strongly to a fixed point* $\hat{x}$ *of* T*, where* $\hat{x}$ *is the minimum-norm element of* $\mathrm{Fix}(T)$*.*

Proof. First, we show the sequence $\{x_n\}$ is bounded. Indeed, taking a fixed point x^* of T, we have

$$\|y_n - x^*\| \leqslant (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|Tx_n - x^*\| \leqslant \|x_n - x^*\|,$$

so

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\beta_n)(\lambda_n x_n) + \beta_n y_n - x^*\| \\ &= \|(1-\beta_n)(\lambda_n x_n - x^*) + \beta_n(y_n - x^*)\| \\ &= \|(1-\beta_n)\lambda_n(x_n - x^*) + \beta_n(y_n - x^*) - (1-\beta_n)(1-\lambda_n)x^*\| \\ &\leqslant (1-\beta_n)\lambda_n\|x_n - x^*\| + \beta_n\|y_n - x^*\| + (1-\beta_n)(1-\lambda_n)\|x^*\| \\ &\leqslant [1-(1-\beta_n)(1-\lambda_n)]\|x_n - x^*\| + (1-\beta_n)(1-\lambda_n)\|x^*\| \\ &\leqslant \max\{\|x_n - x^*\|, \|x^*\|\} \\ &\ \ \vdots \\ &\leqslant \max\{\|x_1 - x^*\|, \|x^*\|\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{Tx_n\}$.

Second, we show $\|x_n - Tx_n\| \to 0$, as $n \to \infty$.

By condition (i) and the boundedness of $\{x_n\}$ and $\{y_n\}$, we have

$$\|x_{n+1} - y_n\| = (1-\beta_n)\|\lambda_n x_n - y_n\| \to 0, \tag{2.2}$$

and

$$\|y_n - Tx_n\| = (1-\alpha_n)\|x_n - Tx_n\| \to 0. \tag{2.3}$$

So, it suffices to show that

$$\|x_{n+1} - x_n\| \to 0.$$

Calculating $y_n - y_{n-1}$, after some manipulations we obtain

$$\begin{aligned} y_n - y_{n-1} &= (1-\alpha_n)x_n + \alpha_n T x_n - (1-\alpha_{n-1})x_{n-1} - \alpha_{n-1}Tx_{n-1} \\ &= x_n - x_{n-1} - \alpha_n x_n + \alpha_{n-1}x_{n-1} + \alpha_n Tx_n - \alpha_{n-1}Tx_{n-1} \\ &= x_n - x_{n-1} - \alpha_n x_n + \alpha_n x_{n-1} - \alpha_n x_{n-1} + \alpha_{n-1}x_{n-1} + \alpha_n Tx_n - \alpha_{n-1}Tx_{n-1} \\ &= x_n - x_{n-1} - \alpha_n(x_n - x_{n-1}) - (\alpha_n - \alpha_{n-1})x_{n-1} + \alpha_n Tx_n - \alpha_{n-1}Tx_{n-1} \\ &= (1-\alpha_n)(x_n - x_{n-1}) - (\alpha_n - \alpha_{n-1})x_{n-1} + \alpha_n Tx_n - \alpha_n Tx_{n-1} + \alpha_n Tx_{n-1} - \alpha_{n-1}Tx_{n-1} \\ &= (1-\alpha_n)(x_n - x_{n-1}) - (\alpha_n - \alpha_{n-1})x_{n-1} + \alpha_n(Tx_n - Tx_{n-1}) + (\alpha_n - \alpha_{n-1})Tx_{n-1} \\ &= (1-\alpha_n)(x_n - x_{n-1}) - (\alpha_n - \alpha_{n-1})(x_{n-1} - Tx_{n-1}) + \alpha_n(Tx_n - Tx_{n-1}). \end{aligned}$$

It follows that

$$\begin{aligned}\|y_n - y_{n-1}\| &\leqslant (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1} - Tx_{n-1}\| + \alpha_n\|Tx_n - Tx_{n-1}\| \\ &\leqslant \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1} - Tx_{n-1}\|.\end{aligned} \tag{2.4}$$

Calculating $x_{n+1} - x_n$, after some manipulations we obtain

$$\begin{aligned}x_{n+1} - x_n &= (1-\beta_n)(\lambda_n x_n) + \beta_n y_n - (1-\beta_{n-1})(\lambda_{n-1}x_{n-1}) - \beta_{n-1}y_{n-1} \\ &= \lambda_n x_n - \beta_n\lambda_n x_n + \beta_n y_n - \lambda_{n-1}x_{n-1} + \beta_{n-1}\lambda_{n-1}x_{n-1} - \beta_{n-1}y_{n-1} \\ &= \lambda_n x_n - \lambda_n x_{n-1} + \lambda_n x_{n-1} - \lambda_{n-1}x_{n-1} - \beta_n\lambda_n x_n \\ &\quad + \beta_{n-1}\lambda_{n-1}x_n - \beta_{n-1}\lambda_{n-1}x_n + \beta_{n-1}\lambda_{n-1}x_{n-1} \\ &\quad + \beta_n y_n - \beta_n y_{n-1} + \beta_n y_{n-1} - \beta_{n-1}y_{n-1} \\ &= \lambda_n(x_n - x_{n-1}) + (\lambda_n - \lambda_{n-1})x_{n-1} - (\beta_n\lambda_n - \beta_{n-1}\lambda_{n-1})x_n \\ &\quad - \beta_{n-1}\lambda_{n-1}(x_n - x_{n-1}) + \beta_n(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})y_{n-1} \\ &= (\lambda_n - \beta_{n-1}\lambda_{n-1})(x_n - x_{n-1}) + (\lambda_n - \lambda_{n-1})x_{n-1} \\ &\quad - (\beta_n\lambda_n - \beta_{n-1}\lambda_{n-1})x_n + \beta_n(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})y_{n-1},\end{aligned} \tag{2.5}$$

Then it follows from (2.5) and (2.4) that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leqslant |\lambda_n - \beta_{n-1}\lambda_{n-1}|\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|x_{n-1}\| \\ &\quad + |\beta_n\lambda_n - \beta_{n-1}\lambda_{n-1}|\|x_n\| + \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1}\| \\ &\leqslant |\lambda_n - \beta_{n-1}\lambda_{n-1}|\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|x_{n-1}\| + |\beta_n\lambda_n - \beta_{n-1}\lambda_{n-1}|\|x_n\| \\ &\quad + \beta_n(\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1} - Tx_{n-1}\|) + |\beta_n - \beta_{n-1}|\|y_{n-1}\| \\ &\leqslant (\beta_n + |\lambda_n - \beta_{n-1}\lambda_{n-1}|)\|x_n - x_{n-1}\| \\ &\quad + M(|\lambda_n - \lambda_{n-1}| + |\beta_n\lambda_n - \beta_{n-1}\lambda_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\leqslant [1 - (1 - \beta_n - |\lambda_n - \beta_{n-1}\lambda_{n-1}|)]\|x_n - x_{n-1}\| \\ &\quad + M(|\lambda_n - \lambda_{n-1}| + |\beta_n\lambda_n - \beta_{n-1}\lambda_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\leqslant (1-\sigma_n)\|x_n - x_{n-1}\| + M(|\lambda_n - \lambda_{n-1}| + |\beta_n\lambda_n - \beta_{n-1}\lambda_{n-1}| \\ &\quad + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\leqslant (1-\sigma_n)\|x_n - x_{n-1}\| + M(2|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|),\end{aligned} \tag{2.6}$$

where $\sigma_n = 1 - \beta_n - |\lambda_n - \beta_{n-1}\lambda_{n-1}|$ and $M > 0$ is a constant such that $M \geqslant \max\{\|x_{n-1}\|, \|x_{n-1} - Tx_{n-1}\|, \|y_{n-1}\|\}$ for all n. By the assumption (i)-(iii), we have $\lim_{n\to\infty}\sigma_n = 0$, $\sum_{n=1}^{\infty}\sigma_n = \infty$, and $\sum_{n=1}^{\infty} 2|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}| < \infty$. Hence, applying Lemma 1.5 to (2.6), we obtain

$$\|x_{n+1} - x_n\| \to 0. \tag{2.7}$$

By (2.2), (2.3), and (2.7), we get

$$\|x_n - Tx_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - Tx_n\| \to 0, \tag{2.8}$$

as $n \to \infty$.

Since $\{x_n\}$ is bounded, there exists a subsequence x_{n_j} of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \hat{x} \in H_1$. By (2.8) and the demiclosedness principle of $T - I$ at zero in Lemma 1.3, we have that $z \in F(T)$.

At last, we prove $\{x_n\}$ converges strongly to $\hat{x}$. Setting $w_n = (1-\beta_n)x_n + \beta_n y_n$, $n \geqslant 1$, then from (2.1) we have

$$x_{n+1} = w_n - (1-\beta_n)(1-\lambda_n)x_n.$$

By the boundedness of $\{x_n\}$, we have,

$$\|x_{n+1} - w_n\| = (1-\beta_n)(1-\lambda_n)\|x_n\| \to 0. \tag{2.9}$$

Using the fact $x_{n_j} \rightharpoonup z$ and (2.9), we conclude that $w_{n_j} \rightharpoonup z$. It follows that

$$\begin{aligned} x_{n+1} &= [1-(1-\beta_n)(1-\lambda_n)]w_n - (1-\beta_n)(1-\lambda_n)(x_n - w_n) \\ &= [1-(1-\beta_n)(1-\lambda_n)]w_n - (1-\beta_n)(1-\lambda_n)\beta_n(x_n - y_n). \end{aligned} \tag{2.10}$$

Also we have

$$\|w_n - \hat{x}\|^2 = \|x_n - \hat{x} - \beta_n(x_n - y_n)\|^2 \leqslant \|x_n - \hat{x}\|^2 - 2\beta_n\langle x_n - y_n, w_n - \hat{x}\rangle. \tag{2.11}$$

By (2.10), (2.11), and (1.2), we obtain

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &= \|[1-(1-\beta_n)(1-\lambda_n)](w_n - \hat{x}) \\ &\quad - (1-\beta_n)(1-\lambda_n)\beta_n(x_n - y_n) - (1-\beta_n)(1-\lambda_n)\hat{x}\|^2 \\ &\leqslant [1-(1-\beta_n)(1-\lambda_n)]^2\|w_n - \hat{x}\|^2 - 2(1-\beta_n)(1-\lambda_n)\langle \beta_n(x_n - y_n) + \hat{x}, x_{n+1} - \hat{x}\rangle \\ &= [1-(1-\beta_n)(1-\lambda_n)]^2\|w_n - \hat{x}\|^2 - 2(1-\beta_n)(1-\lambda_n)\beta_n\langle (x_n - y_n), x_{n+1} - \hat{x}\rangle \\ &\quad - 2(1-\beta_n)(1-\lambda_n)\langle \hat{x}, x_{n+1} - \hat{x}\rangle \\ &\leqslant [1-(1-\beta_n)(1-\lambda_n)](\|x_n - \hat{x}\|^2 - 2\beta_n\langle x_n - y_n, w_n - \hat{x}\rangle) \\ &\quad - 2(1-\beta_n)(1-\lambda_n)\beta_n\langle (x_n - y_n), x_{n+1} - \hat{x}\rangle - 2(1-\beta_n)(1-\lambda_n)\langle \hat{x}, x_{n+1} - \hat{x}\rangle \\ &= [1-(1-\beta_n)(1-\lambda_n)]\|x_n - \hat{x}\|^2 - 2(1-\beta_n)(1-\lambda_n)\beta_n\langle x_n - y_n, w_n - \hat{x}\rangle \\ &\quad - 2(1-\beta_n)(1-\lambda_n)\beta_n\langle (x_n - y_n), x_{n+1} - \hat{x}\rangle - 2(1-\beta_n)(1-\lambda_n)\langle \hat{x}, x_{n+1} - \hat{x}\rangle \\ &= (1-\gamma_n)\|x_n - \hat{x}\|^2 + \gamma_n(-2\beta_n\langle x_n - y_n, w_n - \hat{x}\rangle \\ &\quad - 2\beta_n\langle (x_n - y_n), x_{n+1} - \hat{x}\rangle - 2\langle \hat{x}, x_{n+1} - \hat{x}\rangle), \end{aligned} \tag{2.12}$$

where $\gamma_n = (1-\beta_n)(1-\lambda_n)$.

By conditions (i) and (ii), we have that $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$. Clearly,

$$\begin{aligned} &\limsup_{n\to\infty} -2\beta_n\langle x_n - y_n, w_n - \hat{x}\rangle = 0, \\ &\limsup_{n\to\infty} -2\beta_n\langle (x_n - y_n), x_{n+1} - \hat{x}\rangle = 0, \end{aligned}$$

and

$$\limsup_{n\to\infty} -2\langle \hat{x}, x_{n+1} - \hat{x}\rangle = \lim_{j\to\infty} -2\langle \hat{x}, x_{n_j} - \hat{x}\rangle = -2\langle \hat{x}, z - \hat{x}\rangle \leqslant 0.$$

Hence, applying Lemma 1.5 to (2.12), we obtain that $\|x_n - \hat{x}\| \to 0$.

The proof is completed. □

3. Applications

Lemma 3.1 ([21]). *Given $x^* \in H$, then x^* solves the SFP (1.1) if and only if x^* is the solution of the fixed point equation $x = P_C(I - \gamma A^*(I - P_Q)A)x$.*

Proposition 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H_1, Q be a nonempty closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator. Let P_C, P_Q denote the orthogonal projections onto the sets C, Q, respectively. Let $0 < \gamma < \frac{2}{\rho}$, ρ is the spectral radius of A^*A, and A^* is the adjoint of A. Then the operator $T \triangleq P_C(I - \gamma A^*(I - P_Q)A)$ is nonexpansive on C.*

Proof. This proof is divided into 4 steps in the following.

Step 1. We show that P_Q is 1-ism.

$$\langle x-y, P_Q x - P_Q y\rangle - \|P_Q x - P_Q y\|^2 = \langle x - P_Q x, P_Q x - P_Q y\rangle + \langle y - P_Q y, P_Q y - P_Q x\rangle \geqslant 0.$$

Step 2. We show that $I - P_Q$ is 1-ism.

$$\begin{aligned}&\langle x-y, (I-P_Q)x - (I-P_Q)y\rangle - \|(I-P_Q)x - (I-P_Q)y\|^2\\ &= \|x-y\|^2 - \langle x-y, P_Q x - P_Q y\rangle - \|x-y\|^2 - \|P_Q x - P_Q y\|^2 + 2\langle x - Y, P_Q x - P_Q y\rangle\\ &= \langle x - Y, P_Q x - P_Q y\rangle - \|P_Q x - P_Q y\|^2 \geqslant 0.\end{aligned}$$

Step 3. We show $U \triangleq A^*(I-P_Q)A$ is $\frac{1}{\rho}$-ism.

Since $I - P_Q$ is 1-ism and from the property of adjoint operator, we get

$$\begin{aligned}\langle x-y, Ux - Uy\rangle &= \langle x-y, A^*(I-P_Q)Ax - A^*(I-P_Q)Ay\rangle\\ &= \langle Ax - Ay, (I-P_Q)Ax - (I-P_Q)Ay\rangle\\ &\geqslant \|(I-P_Q)Ax - (I-P_Q)Ay\|^2\\ &= \frac{\|A^*\|^2}{\|A\|^2}\|(I-P_Q)Ax - (I-P_Q)Ay\|^2\\ &\geqslant \frac{1}{\rho}\|A^*(I-P_Q)Ax - A^*(I-P_Q)Ay\|^2 = \frac{1}{\rho}\|Ux - Uy\|^2.\end{aligned}$$

It follows from the above inequality that γU is $\frac{1}{\gamma\rho}$-ism.

Step 4. We show $V \triangleq I - \gamma U$ is nonexpansive. By $0 < \gamma < \frac{2}{\rho}$, we obtain

$$\begin{aligned}\|Vx - Vy\|^2 &= \langle (I-\gamma U)x - (I-\gamma U)y, (I-\gamma U)x - (I-\gamma U)y\rangle\\ &= \|x-y\|^2 + \gamma[\gamma\|Ux - Uy\|^2 - 2\langle x-y, Ux - Uy\rangle]\\ &\leqslant \|x-y\|^2.\end{aligned}$$

Hence, $\|Vx - Vy\| \leqslant \|x-y\|$. Then $T \triangleq P_C(I - \gamma A^*(I-P_Q)A)$ is nonexpansive on C. □

Theorem 3.3. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, and $A^* : H_2 \to H_1$ be a adjoint operator of A. Assume the SFP (1.1) is consistent, $0 < \gamma < \frac{2}{\rho}$, ρ is the spectral radius of A^*A, $S \neq \emptyset$, and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, and $\{\lambda_n\}_{n=1}^{\infty}$ in $(0,1)$, the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 1$, $\lim\limits_{n\to\infty} \beta_n = 1$, $\lim\limits_{n\to\infty} \lambda_n = 1$;

(ii) $|\lambda_n - \beta_{n-1}\lambda_{n-1}| + \beta_n \leqslant 1$, $\sum_{n=0}^{\infty}(1-\beta_n)(1-\lambda_n) = \infty$;

(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$.

Let $\{x_n\}$ be generated by $x_1 \in H_1$ and

$$\begin{cases} x_{n+1} = (1-\beta_n)(\lambda_n x_n) + \beta_n y_n,\\ y_n = (1-\alpha_n)x_n + \alpha_n P_C(x_n - \gamma A^*(I-P_Q)Ax_n).\end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in S$, where $\hat{x}$ is the minimum-norm solution of (1.1).

Proof. From Lemma 3.1, we know $x \in S$ if and only if $x = P_C(I - \gamma A^*(I-P_Q)A)x$.

From Proposition 3.2, we know the operator $T \triangleq P_C(I - \gamma UA^*(I-P_Q)A)$ is nonexpansive.

Based on Theorem 2.1, we can obtain the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in S$, where $\hat{x}$ is the minimum-norm solution of (1.1). □

Theorem 3.4. *Let* H_1 *and* H_2 *be two real Hilbert spaces,* $A : H_1 \to H_2$ *be a bounded linear operator, and* $A^* : H_2 \to H_1$ *be a adjoint operator of* A*. Assume the SFP (1.1) is consistent,* $0 < \gamma < \frac{2}{\rho}$*,* ρ *is the spectral radius of* A^*A*,* $S \neq \emptyset$*, and* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *in* $(0, 1)$*, the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 1$, $\lim_{n\to\infty} \beta_n = 1$;
(ii) $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$, $\sum_{n=0}^{\infty}(1-\beta_n) = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

Let $\{x_n\}$ *be generated by* $x_1 \in H_1$ *and*

$$\begin{cases} x_{n+1} = (1-\beta_n)u + \beta_n y_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n P_C(x_n - \gamma A^*(I - P_Q)Ax_n). \end{cases}$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $\hat{x} \in S$*, and the solution* $\hat{x}$ *is the nearest point to* u*.*

Proof. From Lemma 3.1, we know $x \in S$ if and only if $x = P_C(I - \gamma A^*(I - P_Q)A)x$.

From Proposition 3.2, we know the operator $T \triangleq P_C(I - \gamma U A^*(I - P_Q)A)$ is nonexpansive.

Based on Theorem 1 of [8], we can obtain the sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in S$, and the solution $\hat{x}$ is the nearest point to u. □

4. Conclusions

In this paper, we propose two strong convergence algorithms for solving the split feasibility problem and obtain corresponding strong convergence theorems. This method can be applied in solving the relevant problem, such as the split common fixed point problem (SCFP), multi-set split feasibility problem, and so on.

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