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Null surfaces of null Cartan curves in Anti-de Sitter 3-space

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Abstract

In this paper, we consider the null surfaces of null Cartan curves in Anti-de Sitter 3-space and making use of singularity theory, we classify the singularities of the null surfaces and investigate the relationships between singularities of the null surfaces and differential geometric invariants of null Cartan curves in Anti-de Sitter 3-space. Finally, we give an example to illustrate our results.

Keywords: Null Cartan curve, ruled null surface, principal normal indicatrix.
2010 MSC: 51B05, 53B50, 53A04.

1. Introduction

The n-dimensional Anti-de Sitter space (AdS_n) is a maximally symmetric semi-Riemanian manifold with constant negative scalar curvature. Particular, AdS_n is an n-dimensional solution for the theory of gravitation with Einstein-Hilbert action with negative cosmological constant. It is very interesting and important to do some researches on Anti-de Sitter space. So this subject has been studied by many researchers.

It is well-known that there exist spacelike curves, timelike curves and null curves in semi-Euclidean space. More generally, from the differential geometric point of view, the study of null curves has its own geometric interest. Because the other curves (spacelike and timelike curves) of semi-Euclidean space can be studied by a similar approach to that studied in positive definite Riemannian geometry. Moreover, null curves have different properties from spacelike and timelike cuvres and the results of null curve theory are not analogues to Riemannian case. In geometry of null curves difficulties arise because the arc length vanishes, so that it is impossible to normalize the tangent vector in the usual way. Bonnor [1] gave a method for the general study of the geometry of null curves in Lorentz manifolds and more generally, in semi-Riemannian manifolds. Ferrandez et al. [6–8] have generalized the Cartan frame to semi-Riemannian space forms. They proved the fundamental existence and uniqueness theorems and

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obtained values of the Cartan curvatures in higher dimensions. Duggle [3–5] gave the existence of a canonical representation of null curves of Lorentzian manifolds.

In this paper, we consider the properties associated with the contacts of a given submanifold with null surfaces have a special relevance. By constructing extended spacelike height functions and using classical unfolding theory, Sun [9, 10] and Wang and the second author [11, 12] have classified the singularities of lightlike surface (null surface) of spacelike curve and null curve in de Sitter 3-space and investigated the geometric meanings of the singularities of such surfaces. However, to the best of author's knowledge, no literature can be found for the research of the singularity of null surfaces of null curves in Anti-de Sitter 3-space. The second author and Sun [9, 10] defined the binormal indicatrix of a non-lightlike curve and gave the relationships between singularities of such curves and geometric invariants in Minkowski 3-space. Inspired by [2, 9–11], we define a ruled null surface whose base curve is the principal normal indicatrix of a null Cartan curve. We construct the binormal indicatrix height function of a null Cartan curve, it would be quite useful to study the generic singularities of ruled null surfaces.

The main results in the present paper are stated in Theorem 2.1 and Theorem 6.3. The geometric meaning of Theorem 2.1 is described in Section 4. The proof of Theorem 2.1 is given in Section 5. Then, we consider generic properties of null Cartan curves in Section 6. In the last section, we give an example and the graphics to illustrate the singularities of null surfaces of null curves in Anti-de sitter space.

All maps and manifolds considered here are differential of class C^∞.

2. Preliminaries

Let $\mathbb{R}^4$ be a 4-dimensional vector space. For any two vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, their pseudo scalar product is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4.$$

The space $(\mathbb{R}^4, \langle , \rangle)$ is called the 4-dimensional semi-Euclidean space-time of index 2 and denoted by $\mathbb{R}^4_2$.

For three vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{y} = (y_1, y_2, y_3, y_4)$ and $\mathbf{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{R}^4_2$, we define a vector $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$ by

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \begin{vmatrix} -\mathbf{e}_1 & -\mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ is the canonical basis of $\mathbb{R}^4_2$. We have $\langle \mathbf{x}_0, \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} \rangle = \det(\mathbf{x}_0, \mathbf{x}, \mathbf{y}, \mathbf{z})$, so $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$ is pseudoorthogonal to x, y and z. A nonzero vector $x \in \mathbb{R}^4_2$ is called spacelike, null or timelike, if $\langle x, x \rangle > 0$, $\langle x, x \rangle = 0$ or $\langle x, x \rangle < 0$, respectively. The norm of $x \in \mathbb{R}^4_2$ is defined by $\|x\| = (\mathrm{sign}(x)\langle x, x \rangle)^{\frac{1}{2}}$, where $\mathrm{sign}(x)$ denotes the signature of x which is given by sign(x) =1, 0 or -1, if x is a spacelike, null or timelike vector, respectively.

Let $\gamma : I \to \mathbb{R}^4_2$ be a smooth regular curve in $\mathbb{R}^4_2$ (i.e., $\dot{\gamma}(t) \neq 0$, for any $t \in I$), where I is an open interval. For any $t \in I$, the curve γ is called spacelike curve, null (lightlike) curve or timelike curve, if all its velocity are $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$, respectively. We call γ the non-null curve, if γ is a timelike curve or a spacelike curve.

The Anti-de Sitter space is defined by

$$H^3_1 = \{x \in \mathbb{R}^4_2 | \langle x, x \rangle = -1\},$$

and the lightlike cone by

$$LC_p = \{x \in \mathbb{R}^4_2 | \langle x - \mathbf{p}, x - \mathbf{p} \rangle = 0\}.$$

Let $\gamma : I \to \mathbb{R}^4_2$ be a non-geodesic null curve in $\mathbb{R}^4_2$ (i.e., $\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle = 0$ for any $t \in I$). Without loss of generality, we assume a special parameter s such that $\langle \gamma''(s), \gamma''(s)\rangle = 1$. One can set up the null frame $F = \{\gamma(s), \mathbf{t}(s), \mathbf{n}(s), \boldsymbol{w}(s)\}$ of $\mathbb{R}^4_2$ which is a positively oriented tetrad of vector satisfying

$$\begin{aligned}
&\langle \gamma(s), \gamma(s)\rangle = -1, \quad \langle \boldsymbol{w}(s), \boldsymbol{w}(s)\rangle = 1,\\
&\langle \mathbf{t}(s), \mathbf{t}(s)\rangle = \langle \mathbf{n}(s), \mathbf{n}(s)\rangle = 0, \quad \langle \mathbf{t}(s), \mathbf{n}(s)\rangle = 1,\\
&\langle \gamma(s), \mathbf{t}(s)\rangle = \langle \gamma(s), \mathbf{n}(s)\rangle = \langle \gamma(s), \boldsymbol{w}(s)\rangle = \langle \mathbf{t}(s), \boldsymbol{w}(s)\rangle = \langle \mathbf{n}(s), \boldsymbol{w}(s)\rangle = 0.
\end{aligned}$$

Since $\langle \mathbf{t}'(s), \mathbf{t}'(s)\rangle = \kappa_1^2(s) > 0$ and $\langle \boldsymbol{w}, \boldsymbol{w}\rangle = 1$, then $\kappa_1(s) = \sqrt{\langle \gamma''(s), \gamma''(s)\rangle} = 1$. The Frenet formula of γ with respect to the frame F is given by [3]

$$\begin{cases}
\gamma'(s) = \mathbf{t}(s),\\
\mathbf{t}'(s) = \boldsymbol{w}(s),\\
\mathbf{n}'(s) = \gamma(s) + \kappa_2(s)\boldsymbol{w}(s),\\
\boldsymbol{w}'(s) = -\kappa_2(s)\mathbf{t}(s) - \mathbf{n}(s).
\end{cases} \tag{2.1}$$

We call (2.1) the Cartan Frenet equations and $\gamma(s)$ the null Cartan curve, where

$$\begin{aligned}
&\mathbf{t}(s) = \gamma'(s),\\
&\mathbf{n}(s) = -\kappa_2\gamma'(s) - \gamma^{(3)}(s),\\
&\boldsymbol{w}(s) = \gamma''(s),\\
&\kappa_2 = \langle \mathbf{n}'(s), \boldsymbol{w}(s)\rangle = \frac{1}{2}\langle \gamma^{(3)}(s), \gamma^{(3)}(s)\rangle.
\end{aligned}$$

The functions κ_2 is called the curvature function of γ [1]. Employing the usual terminology, the unit vector fields γ and $\boldsymbol{w}$ of F will be called 1st and 2nd principal normal vector fields, respectively. The null vector field $\mathbf{n}$ is called the binormal vector field or the transversal null vector field.

Let $\gamma : I \to H^3_1$ be a smooth null Cartan curve with $|\kappa_2(s)| > 1$. We define the principal normal indicatrix of γ as the map $\rho : I \to H^3_1$ given by

$$\rho(s) = \frac{\kappa_2}{\sqrt{(\kappa_2^2 - 1)}}\gamma(s) + \frac{1}{\sqrt{(\kappa_2^2 - 1)}}\boldsymbol{w}(s).$$

We also let binormal indicatrix of γ be the map $\mathbf{n} : I \to NC^3$.

Now we define ruled surfaces of which base curve is $\pm\rho(s)$ as follows

$$\mathcal{NS}^{\pm} : I \times \mathbb{R} \longrightarrow H^3_1, \quad \mathcal{NS}^{\pm}(s, \omega) = \pm\rho(s) + \lambda\mathbf{n}(s),$$

we call each of $\mathcal{NS}^{\pm}(s, \omega)$ the ruled null surface of principal normal indicatrix of γ.

We also define two new invariants of a null Cartan curve in H^3_1 by

$$\sigma(s) = \frac{\kappa_2'(s)}{(\kappa_2^2(s) - 1)^{\frac{3}{2}}}, \quad \text{and} \quad K(s) = \sigma'(s) - \frac{1}{\sqrt{\kappa_2^2(s) - 1}},$$

which be related to geometric meanings of the singularities of the ruled null surface.

Let $F : \mathbb{R}^4_2 \to \mathbb{R}$ be a submersion and $\gamma : I \to \mathbb{R}^4_2$ be a null Cartan curve. We say that γ and $F^{-1}(0)$ have k-point contact for $t = t_0$, if the function $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$, $g^{(k)}(t_0) \neq 0$. We also say that γ and $F^{-1}(0)$ have at least k-point contact for $t = t_0$, if the function $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$.

In this paper, we shall assume throughout the whole paper that all the maps and manifolds are C^∞ and $|\kappa_2(s)| > 1$ unless the contrary is explicitly stated. The main result in this paper is as follows:

Theorem 2.1. *Let* $\gamma : I \longrightarrow H^3_1$ *be a null Cartan curve,* $\mathbf{n}(s)$ *be the binormal indicatrix of* γ. *For* $v_0 = \mathcal{NS}^{\pm}(s_0, \omega_0)$ *and* $H^2_1(v_0) = \{u \in NC^3 | \langle u, v_0\rangle = 0\}$, *we have the followings:*

(1) $\mathbf{n}(s)$ *and* $H^2_1(v_0)$ *have at least 2-point contact for* s_0.

(2) $\mathbf{n}(s)$ *and* $H^2_1(v_0)$ *have 3-point contact for* s_0, *iff* $v_0 = \pm(\rho(s_0) + \sigma(s_0)\mathbf{n}(s_0))$ *and* $K(s_0) \neq 0$. *Under the condition, the ruled null surface* $\mathcal{NS}^{\pm}$ *at* $\mathcal{NS}^{\pm}(s_0, \omega_0)$ *is locally diffeomorphic to the cuspidal edge.*

(3) $\mathbf{n}(s)$ *and* $H^2_1(v_0)$ *have 4-point contact for* s_0, *iff* $v_0 = \pm(\rho(s_0) + \sigma(s_0)\mathbf{n}(s_0))$ *and* $K(s_0) = 0$, $K'(s_0) \neq 0$. *Under the condition, the ruled null surface* $\mathcal{NS}^{\pm}$ *at* $\mathcal{NS}^{\pm}(s_0, \omega_0)$ *is locally diffeomorphic to the swallow tail (Figure 1).*

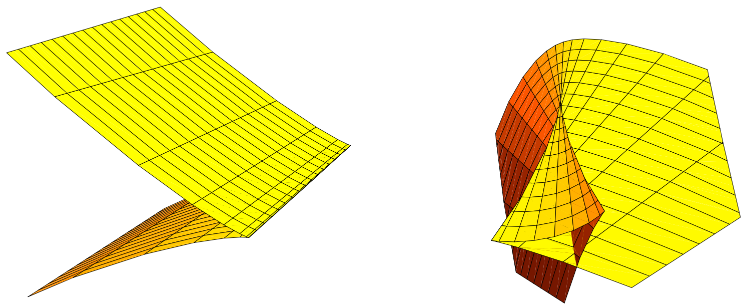


Figure 1: Cuspidal edge and swallow tail.

3. Geometric invariants of null Cartan curves in Anti-de Sitter 3-space

The purpose of this section is to obtain two geometric invariants of null Cartan curves by introducing a family of functions on a null Cartan curve.

Let $\gamma : I \to H^3_1$ be a null Cartan curve and $\mathbf{n}(s)$ be the binormal indicatrix of $\gamma(s)$, we define the function $H : I \times H^3_1 \to \mathbb{R}$ by $H(s, v) = \langle \mathbf{n}(s), v\rangle$. It be called the binormal indicatrix height functions of null Cartan curve $\gamma(s) \in H^3_1$. For any fixed vector v in H^3_1, denote $h_v(s) = H(s, v)$. Then we have the following proposition.

Proposition 3.1. *Let* $\gamma : I \to H^3_1$ *be a regular null Cartan curve. Then*

(1) $h_v(s) = 0$, *iff there exist real numbers* μ, ν, λ *such that* $v = \mu\gamma(s) + \nu\mathbf{w}(s) + \lambda\mathbf{n}(s)$ *and* $\mu^2 - \nu^2 = 1$.

(2) $h_v(s) = h'_v(s) = 0$, *iff* $v = \pm\rho(s) + \lambda\mathbf{n}(s)$.

(3) $h_v(s) = h'_v(s) = h''_v(s) = 0$, *iff* $v = \pm(\rho(s) + \sigma(s)\mathbf{n}(s))$.

(4) $h'_v(s) = h''_v(s) = h^{(3)}_v(s) = 0$, *iff* $v = \pm(\rho(s) + \sigma(s)\mathbf{n}(s))$ *and* $K(s) = 0$.

(5) $h'_v(s) = h''_v(s) = h^{(3)}_v(s) = h^{(4)}_v(s) = 0$, *iff* $v = \pm(\rho(s) + \sigma(s)\mathbf{n}(s))$ *and* $K(s) = K'(s) = 0$.

4. Null Cartan curve and its principal normal indicatrix in Anti-de Sitter 3-space

The purpose of this section is to study the geometric properties of the ruled null surface of principal normal indicatrix to a null Cartan curve in H^3_1. By these properties, one can recognize the functions $\sigma(s)$ and $K(s) = \sigma'(s) - \frac{1}{\sqrt{\kappa_2^2 - 1}}$ have special meanings, one also obtain the relationship between a null Cartan curve and its principal normal indicatrix. These properties will be stated in the following.

Proposition 4.1. *Let $\gamma : I \to H^3_1$ be a regular null Cartan curve. Then*

(1) *The singular set of $\mathcal{NS}^{\pm}$ is $\{(s, \omega)|\ \omega = \pm\sigma(s), s \in I\}$.*

(2) *If $\mathcal{NS}^{\pm}(s, \pm\sigma(s)) = \nu_0^{\pm}$ is a constant vector, then $\pm\rho(s)$ is in $NC^2(\nu_0^{\pm}) \subset H^3_1$ and $\mathbf{n}(s)$ is in $H^2_1(\nu_0^{\pm}) \subset NC^3$ for any s in I and $K(s) = \sigma'(s) - \frac{1}{\sqrt{\kappa_2^2-1}} \equiv 0$. Then image $\mathcal{NS}^{\pm} \subset NC^2(\nu_0^{\pm})$.*

Proof. (1). By the straightforward calculations, we have

$$\begin{aligned}\partial\mathcal{NS}^{\pm}/\partial s &= \left(\lambda \pm \frac{\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}}\right)\gamma(s) \\ &\quad + \left(\lambda\kappa_2 \pm \frac{\kappa_2\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}}\right)\boldsymbol{w}(s) \pm \frac{1}{\sqrt{\kappa_2^2-1}}\mathbf{n}(s), \\ \partial\mathcal{NS}^{\pm}/\partial\lambda &= \mathbf{n}(s).\end{aligned}$$

The two equalities above imply that $\partial\mathcal{NS}^{\pm}/\partial s$ and $\partial\mathcal{NS}^{\pm}/\partial\lambda$ are linearly dependent, if and only if $\lambda = \pm\frac{\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}}$. This completes the proof of the assertion (1).

(2). For a smooth function $\lambda : I \to \mathbb{R}$, define

$$f_\lambda^{\pm} : I \to H^3_1, \quad f_\lambda^{\pm}(s) = \pm\rho(s) + \lambda(s)\mathbf{n}(s).$$

If $f_\lambda^{\pm}(s) = \nu_0^{\pm}$ is a constant, then

$$\begin{aligned}\frac{df_\lambda^{\pm}(s)}{ds} &= \left(\lambda(s) \pm \frac{\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}}\right)\gamma(s) \\ &\quad + \left(\kappa_2\lambda(s) \pm \frac{\kappa_2\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}}\right)\boldsymbol{w}(s) \\ &\quad + \left(\lambda'(s) \pm \frac{1}{\sqrt{\kappa_2^2-1}}\right)\mathbf{n}(s) \\ &= 0.\end{aligned}$$

Since the singularities of $\mathcal{NS}^{\pm}$ are $\lambda(s) = \pm\frac{\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}} = \pm\sigma(s), \mu'(s) = \pm\sigma'(s)$, substituting these relations into the above equality, we have $K(s) = \sigma'(s) + \frac{1}{\sqrt{\kappa_2^2-1}} = 0$. Under the assumption that $\pm\frac{1}{\sqrt{\kappa_2^2-1}}[\kappa_2\gamma(s) - \boldsymbol{w}(s) + \frac{\kappa_2'}{\kappa_2^2-1}\mathbf{n}(s)] = \nu_0^{\pm}$ is constant, $\rho(s)$ satisfies

$$\begin{aligned}\langle \pm\rho(s) - \nu_0^{\pm}, \pm\rho(s) - \nu_0^{\pm}\rangle &= \langle \pm\frac{1}{\sqrt{\kappa_2^2-1}}(\kappa_2\gamma(s) + \boldsymbol{w}(s)) - \nu_0^{\pm}, \\ &\qquad \pm\frac{1}{\sqrt{\kappa_2^2-1}}(\kappa_2\gamma(s) + \boldsymbol{w}(s)) - \nu_0^{\pm}\rangle \\ &= 0,\end{aligned}$$

and $\mathbf{n}(s)$ satisfies

$$\langle \mathbf{n}(s) - \nu_0^{\pm}, \mathbf{n}(s) - \nu_0^{\pm}\rangle = -1.$$

Then we also have

$$\begin{aligned}\langle \mathcal{NS}^{\pm}(s,u)-v_0^{\pm}, \mathcal{NS}^{\pm}(s,u)-v_0^{\pm}\rangle =&\langle(\lambda \pm \frac{\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}})\mathbf{n}(s), (\lambda \pm \frac{\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}})\mathbf{n}(s)\rangle\\ =&0.\end{aligned}$$

□

It is well-known in semi-Euclidean 4-space of index 2 that null Cartan curves which have constant k_1 and k_2 (not both zero) are called null Cartan helices.

Proposition 4.2. *Let $\gamma : I \to H_1^3$ be a null Cartan curve. Then*

(1) *$\rho(s)$ is a spacelike or null curve.*

(2) *$\rho(s)$ is a null curve if and only if $\gamma(s)$ is a null Cartan helix.*

(3) *If $\gamma(s)$ is a null Cartan helix, then singular locus of $\rho(s)+\lambda\mathbf{n}(s)$ is $\rho(s)$.*

Proof. (1) By (2.1), we have

$$\rho'(s) = \sigma(s)(-\gamma(s) - \kappa_2 \boldsymbol{w}(s) - \frac{1}{\sqrt{\kappa^2-1}})\mathbf{n}(s).$$

Then

$$\langle \rho'(s), \rho'(s)\rangle = \sigma^2(s)(\kappa_2^2-1).$$

This means that $\langle \rho'(s), \rho'(s)\rangle$ is non-negative and hence the desired result.

(2) It is easy to see from (1) that $\rho(s)$ is a null curve if and only if $\sigma^2(s) = 0$, which is equivalent to $\kappa_2' = 0$, that is κ_2 is equal to a constant, the assertion (2) follows.

(3) Let $\gamma : I \to H_1^3$ be a null Cartan helix, the Cartan curvature k_2 are constants. Then by Proposition 4.1, the singular locus of $\rho(s)+\lambda\mathbf{n}(s)$ is $\rho(s)+\frac{\kappa_2'}{(\kappa_2-1)\sqrt{\kappa_2^2-1}}\mathbf{n}(s)$, hence the assertion follows from $\kappa_2' = 0$.

□

5. Versal unfolding of binormal indicatrix height function

In this section we use some general results on the singularity theory for families of function germs.

Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be a function germ. We call F an r-parameter unfolding of f, where $f(s) = F_{x_0}(s, \boldsymbol{x}_0)$. We say that $f(s)$ has A_k-singularity at s_0, if $f^{(p)}(s_0) = 0$ for all $1 \leqslant p \leqslant k$ and $f^{(k+1)}(s_0) \neq 0$. We also say that $f(s)$ has $A_{\geqslant k}$-singularity at s_0, if $f^{(p)}(s_0) = 0$ for all $1 \leqslant p \leqslant k$. Let F be an unfolding of f and $f(s)$ has A_k-singularity $(k \geqslant 1)$ at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, \boldsymbol{x}_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}(s-s_0)^j$, for $i = 1, \cdots, r$. Then F is called a (p) versal unfolding, if the $(k-1)\times r$ matrix of coefficients (α_{ji}) has rank $k-1$ $(k-1 \leqslant r)$. Under the same as the above, F is called a versal unfolding, if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank k $(k \leqslant r)$, where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$.

We now introduce several important sets concerning the unfolding.

In this section we shall apply A_k-singularity and the unfolding theory of function germ to prove Theorem 2.1.

Let function germ $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an unfolding of f, where $f(s) = F_{x_0}(s, x_0)$. We now introduce an important set concerning the unfolding. The discriminant set of F is given by

$$\mathfrak{D}_F = \{x \in \mathbb{R}^r |\ \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x)\}.$$

By Proposition 3.1, the discriminant set of the binormal indicatrix height function $H(s,\nu)$ is given as follows

$$\mathfrak{D}_H = \{\nu = \pm\rho + \lambda \mathbf{n}(s) | s, \lambda \in \mathbb{R}\}.$$

For the binormal indicatrix height function $H(s,\nu) = \langle \mathbf{n}(s), \nu\rangle$, one can prove the following interesting result.

Theorem 5.1. *Suppose that* $H : I \times H^3_1 \to \mathbb{R}$ *is the binormal indicatrix height function on a null Cartan curve* $\gamma(s)$ *and* ν *is in* $\mathfrak{D}_H$. *If* h_ν *has* A_k*-singularity at* s_0 $(k = 1, 2, 3)$, *then* H *is a versal unfolding of* h_ν.

Proof. Let

$$N(s) = (N_1(s), N_2(s), N_3(s), N_4(s)),$$

and

$$\nu = \Big(\pm\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1}, \nu_2, \nu_3, \nu_4\Big) \in H^3_1.$$

Then

$$H(s,\nu) = \mp N_1(s)\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1} - N_2(s)\nu_2 + N_3(s)\nu_3 + N_4(s)\nu_4,$$

$$\frac{\partial H}{\partial \nu_i}(s,\nu) = \mp\frac{N_1(s)\nu_i}{\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1}} + \eta_i N_i(s),$$

$$\frac{\partial}{\partial s}\frac{\partial H}{\partial \nu_i}(s,\nu) = \mp\frac{N_1'(s)\nu_i}{\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1}} + \eta_i N_i'(s),$$

$$\frac{\partial}{\partial s^2}\frac{\partial H}{\partial \nu_i}(s,\nu) = \mp\frac{N_1''(s)\nu_i}{\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1}} + \eta_i N_i''(s),$$

where $\eta_2 = 1$, $\eta_3 = \eta_4 = -1$, $i = 2, 3, 4$.

Let $j^{k-1}\frac{\partial H}{\partial \nu_i}(s,\nu_0)(s_0)$ be the $(k-1)$-jet of $\frac{\partial H}{\partial \nu_i}(s,\nu)$ $(i = 2, 3, 4)$ at s_0, then one can show that

$$\begin{aligned}\frac{\partial H}{\partial \nu_i}(s_0,\nu_0) + j^2\Big(\frac{\partial H}{\partial \nu_i}(s,\nu_0)\Big)(s_0) &= \frac{\partial H}{\partial \nu_i}(s_0,\nu_0) + \frac{\partial}{\partial s}\frac{\partial H}{\partial \nu_i}(s_0,\nu_0)(s-s_0) + \frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial H}{\partial \nu_i}(s_0,\nu_0)(s-s_0)^2 \\ &= \alpha_{0,i} + \alpha_{1,i}(s-s_0) + \frac{1}{2}\alpha_{2,i}(s-s_0)^2.\end{aligned}$$

We denote that

$$A = \begin{pmatrix} \alpha_{0,2} & \alpha_{0,3} & \alpha_{0,4} \\ \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \end{pmatrix},$$

$$A(i,j,k) = \det\begin{pmatrix} N_i(s) & N_j(s) & N_k(s) \\ N_i'(s) & N_j'(s) & N_k'(s) \\ N_i''(s) & N_j''(s) & N_k''(s) \end{pmatrix}.$$

Then

$$\begin{aligned}\det A &= A(2,3,4) \mp \frac{\nu_2}{\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1}}A(1,3,4) \\ &\quad \mp \frac{\nu_3}{\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1}}A(2,1,4) \mp \frac{\nu_4}{\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1}}A(2,3,1) \\ &= \pm\frac{1}{\sqrt{-\nu_2^2 + \nu_3^2 + \nu_4^2 + 1}}\langle \nu, \mathbf{n}(s) \wedge \mathbf{n}'(s) \wedge \mathbf{n}''(s)\rangle.\end{aligned}$$

Since $\nu \in \mathfrak{D}_H$ is a singular point,

$$\begin{aligned}\nu &= \pm(\rho(s) + \frac{\kappa_2'}{(\kappa_2^2-1)\sqrt{\kappa_2^2-1}}\mathfrak{n}(s))\\ &= \pm\frac{1}{\sqrt{(\kappa_2^2-1)}}((\kappa_2\gamma(s)+\boldsymbol{w}(s)) + \frac{\kappa_2'}{\kappa_2^2-1}\mathfrak{n}(s)),\end{aligned}$$

and

$$\mathfrak{n}(s)\wedge\mathfrak{n}'(s)\wedge\mathfrak{n}''(s) = (1-\kappa_2^2)\boldsymbol{w}(s) + \kappa_2'\mathbf{t}(s) + \kappa_2(1-\kappa_2^2)\gamma(s).$$

Therefore

$$\det A = \pm\frac{(\kappa_2^2-1)^2}{(-\nu_2^2+\nu_3^2+\nu_4^2+1)\sqrt{\kappa_2^2-1}} \neq 0,$$

which implies the rank A is 3 and desired results. □

Proof of Theorem 2.1. Let $\gamma : I \to H_1^3$ be a null Cartan curve. For $\nu_0^{\pm} = \mathcal{NS}^{\pm}(s_0, u_0)$, we define a function $\mathfrak{H} : NC^{3^*} \to \mathbb{R}$ by $\mathfrak{H}(u) = \langle u, \nu_0^{\pm}\rangle$. Thus we have $h_{\nu_0^{\pm}}(s) = \mathfrak{H}(\mathfrak{n}(s))$. By Proposition 3.1, the discriminant set of H is $\mathfrak{D}_H = \{\nu = \pm\rho(s) + \lambda\mathfrak{n}(s) | \ s, \omega \in \mathbb{R}\}$. Since $NC^{3^*} \supset H_1^2(\nu_0^{\pm}) = \mathfrak{H}^{-1}(0)$ and 0 is a regular value of $\mathfrak{H}$, $h_{\nu_0^{\pm}}$ has the A_k-singularity at s_0, if and only if $N(s)$ and $H_1^2(\nu_0^{\pm})$ have $(k+1)$-point contact for s_0, using the results of the singularity theory for families of function germs [10] and combining Proposition 3.1, Theorem 5.1 and so we have desired results. □

6. Generic properties of null Cartan curves in Anti-de Sitter 3-space

In this section we consider generic properties of null Cartan curves in H_1^3. The main tool is a kind of transversality theorems. Let $\mathrm{Emb}_{sp}(I, H_1^3)$ be the space of null embedding $\gamma : I \to H_1^3$ with equipped with Whitney C^∞-topology. We also consider the function $\mathcal{H} : H_1^3 \times H_1^3 \to \mathbb{R}$ defined by $\mathcal{H}(u,\nu) = \langle u^{(3)}(s) + \frac{1}{2}\langle u^{(3)}(s), u^{(3)}(s)\rangle u'(s), \nu\rangle$. We claim that $\mathcal{H}_\nu$ is a submersion for any ν in H_1^3, where $\mathcal{H}_\nu(u) = \langle u^{(3)}(s) + \frac{1}{2}\langle u^{(3)}(s), u^{(3)}(s)\rangle u'(s), \nu\rangle$. For any γ in $\mathrm{Emb}_{sp}(I, H_1^3)$, we have $H = \mathcal{H} \circ (\gamma \times id_{H_1^3})$. We also have the l-jet extension

$$j_1^l H : I \times H_1^3 \to J^l(I, \mathbb{R}),$$

defined by $j_1^l\ H(s,\nu) = j^l h_\nu(s,\nu)$. We consider the trivialization $J^l(I,\mathbb{R}) \equiv I \times \mathbb{R} \times J^l(1,1)$. For any submanifold $\mathcal{O} \subset J^l(1,1)$, we denote that $\tilde{\mathcal{O}} = I \times \{0\} \times \mathcal{O}$. It is evident that $j_1^l H$ is submersion and $\tilde{\mathcal{O}}$ is an immersed submanifold of $J^l(I,\mathbb{R})$. Then $J_1^l H$ is transversal to $\tilde{\mathcal{O}}$. We have the following proposition as a corollary of Lemma 6 in Wassermann [13].

Proposition 6.1. *Let $\mathcal{O}$ be submanifolds of $J^l(1,1)$. Then the set*

$$T_{\mathcal{O}} = \{\gamma \in \mathrm{Emb}_{sp}(I, H_1^3) | \ j_1^l H \textit{ is transversal to } \tilde{\mathcal{O}}\},$$

is residual subset of $\mathrm{Emb}_{sp}(I, H_1^3)$. If $\mathcal{O}$ is a closed subset, then $T_{\mathcal{O}}$ is open .

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ which has an A_k-singularity at 0. It is well-known that there exists a diffeomorphism germ $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $f \circ \phi(s) = \pm s^{k+1}$. This is the classification of A_k-singularities. For any $z = j^l f(0)$ in $J^l(1,1)$, we have the orbit $L^l(z)$ given by the action of the Lie group of l-jet diffeomorphism germs. If f has an A_k-singularity, then the codimension of the orbit is k. There is another characterization of versal unfoldings as follows.

Proposition 6.2. *Let $F : (\mathbb{R} \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be an r-parameter unfolding of $f : (\mathbb{R}, 0) \to \underset{\frown}{(\mathbb{R}, 0)}$ which has an A_k-singularity at 0. Then F is a versal unfolding if and only if $j_1^l F$ is transversal to the orbit $L^l(j^l f(0))$ for $l \geqslant k+1$. Here, $j_1^l F : (\mathbb{R} \times \mathbb{R}^r, 0) \to J^l(\mathbb{R}, \mathbb{R})$ is the l-jet extension of F given by $j_1^l F(s,x) = j^l F_x(s)$.*

The generic classification theorem is given as follows.

Theorem 6.3. *There exists an open and dense subset $T_{L_k^l} \subset \mathrm{Emb}_{sp}(I, H_1^3)$ such that for any $\gamma \in T_{L_k^l}$, the ruled null surface of principal normal indicatrix of γ is locally diffeomorphic to the cuspidal edge or the swallowtail at any singular point.*

Proof. (1). For $l \geqslant 4$, we consider the decomposition of the jet space $J^l(1,1)$ into $L^l(1)$ orbits. We now define a semi-algebraic set by

$$\Sigma^l = \{z = j^l f(0) \in J^l(1,1) |\ f \text{ has an } A_{\geqslant 4}\text{-singularity}\}.$$

Then the codimension of Σ^l is 4. Therefore, the codimension of $\widetilde{\Sigma}_0^l = I \times \{0\} \times \Sigma^l$ is 5. We have the orbit decomposition of $J^l(1,1) - \Sigma^l$ into

$$J^l(1,1) - \Sigma^l = L_0^l \cup L_1^l \cup L_2^l \cup L_3^l,$$

where L_k^l is the orbit through an A_k-singularity. Thus, the codimension of $\widetilde{L_k^l}$ is $k+1$. We consider the l- jet extension $j_1^l H$ of the binormal indicatrix height function H. By Proposition 6.2, there exists an open and dense subset $T_{L_k^l} \subset \mathrm{Emb}(I, H_1^3)$ such that $j_1^l H$ is transversal to $\widetilde{L_k^l}(k = 0,1,2,3)$ and the orbit decomposition of $\widetilde{\Sigma^l}$. This means that $j_1^l H(I \times S_1^3) \bigcap \widetilde{\Sigma^l} = \emptyset$ and H is a versal unfolding of h at any point (s_0, v_0). By Theorem 5.1, the discriminant set of H (i.e., the ruled null surface of principal normal indicatrix of γ) is locally diffeomorphic to cuspidal edge or the swallowtail if the point is singular. $\square$

7. Example

In this section, we give an example to illustrate the idea of Theorem 2.1.

Example 7.1. Let $\gamma(s)$ be a null Cartan curve (Figure 2) of $\mathbb{R}_2^4$ defined by

$$\gamma(s) = \left(\sqrt{2}\cosh(\frac{1}{\sqrt[4]{2}}s), \sinh(\sqrt[4]{2}s), \sqrt{2}\sinh(\frac{1}{\sqrt[4]{2}}s), \cosh(\sqrt[4]{2}s)\right),$$

with respect to a distinguished parameter s.

The Cartan Frenet frame is

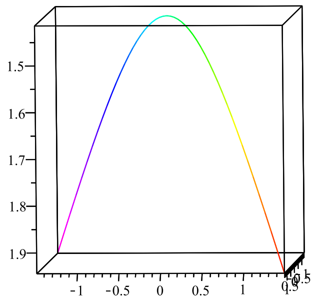
$$F = \{\gamma(s), \mathbf{t}(s), \mathbf{n}(s), \boldsymbol{w}(s)\},$$

where

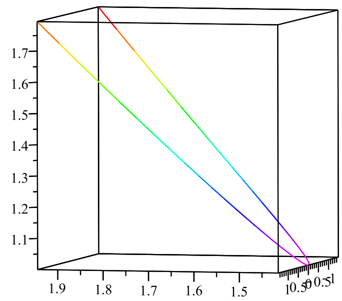
$$\begin{aligned}
\gamma(s) &= \left(\sqrt{2}\cosh(\frac{1}{\sqrt[4]{2}}s), \sinh(\sqrt[4]{2}s), \sqrt{2}\sinh(\frac{1}{\sqrt[4]{2}}s), \cosh(\sqrt[4]{2}s)\right),\\
\mathbf{t}(s) &= \left(\sqrt[4]{2}\sinh(\frac{1}{\sqrt[4]{2}}s), \sqrt[4]{2}\cosh(\sqrt[4]{2}s), \sqrt[4]{2}\cosh(\frac{1}{\sqrt[4]{2}}s), \sqrt[4]{2}\sinh(\sqrt[4]{2}s)\right),\\
\mathbf{n}(s) &= \left(\frac{1}{2\sqrt[4]{2}}\sinh(\frac{1}{\sqrt[4]{2}}s), -\frac{1}{2\sqrt[4]{2}}\cosh(\sqrt[4]{2}s), \frac{1}{2\sqrt[4]{2}}\cosh(\frac{1}{\sqrt[4]{2}}s), -\frac{1}{2\sqrt[4]{2}}\sqrt[4]{2}\sinh(\sqrt[4]{2}s)\right),\\
\boldsymbol{w}(s) &= \left(\cosh(\frac{1}{\sqrt[4]{2}}s), \sqrt{2}\sinh(\sqrt[4]{2}s), \sinh(\frac{1}{\sqrt[4]{2}}s), \sqrt{2}\cosh(\sqrt[4]{2}s)\right).
\end{aligned}$$

Then we can calculate

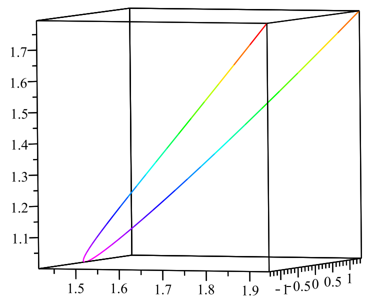
$$\kappa_2(s) = -\frac{3\sqrt{2}}{4}.$$



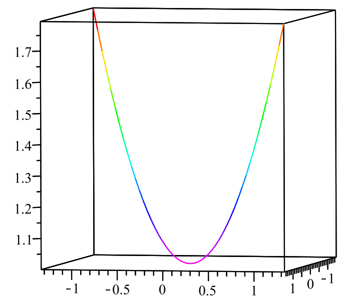
(a) Null Cartan curve projection on $x^1x^2x^3$-space.



(b) Null Cartan curve projection on $x^1x^2x^4$-space.

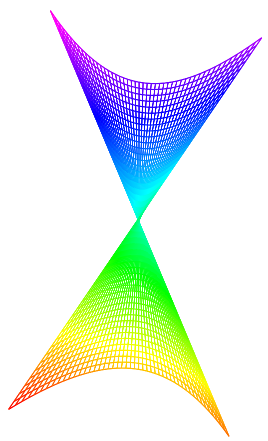


(c) Null Cartan curve projection on $x^1x^3x^4$-space.

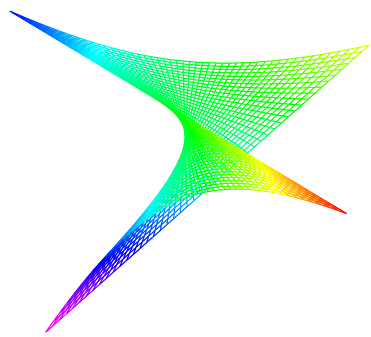


(d) Null Cartan curve projection on $x^2x^3x^4$-space.

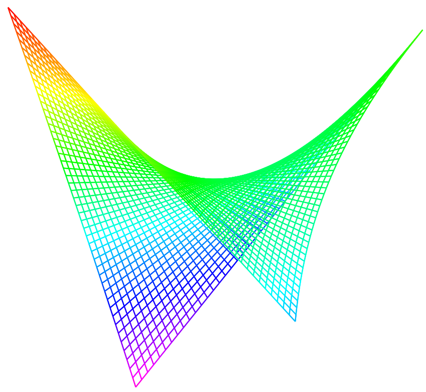
Figure 2: Null Cartan curve projection on $x^1x^2x^3$-space, $x^1x^2x^4$-space,$x^1x^3x^4$-space, $x^2x^3x^4$-space respectively.



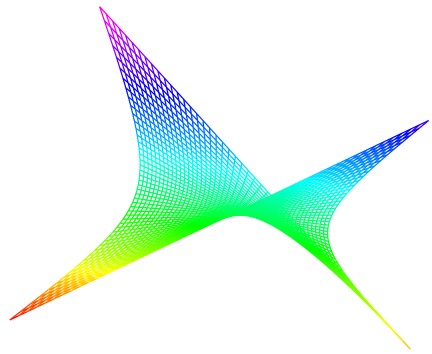
(a) Ruled null surface projection on $x^1x^2x^3$-space.



(b) Ruled null surface projection on $x^1x^2x^4$-space.



(c) Ruled null surface projection on $x^1x^3x^4$-space.



(d) Ruled null surface projection on $x^2x^3x^4$-space.

Figure 3: Ruled null surface projection on $x^1x^2x^3$-space, $x^1x^2x^4$-space, $x^1x^3x^4$-space, $x^2x^3x^4$-space respectively.

The principal normal indicatrix of $\gamma(s)$ is

$$\begin{aligned}\rho(s) = \{&-3\sqrt{2}\cosh(\frac{1}{\sqrt[4]{2}}s) + \frac{4}{\sqrt[4]{2}}(s)\sinh(\frac{1}{\sqrt[4]{2}}s), -3\sqrt{2}\sinh(\sqrt[4]{2}s) + \frac{4}{\sqrt[4]{2}}(s)\cosh(\sqrt[4]{2}s),\\ &-3\sqrt{2}\sinh(\frac{1}{\sqrt[4]{2}s}) + \frac{4}{\sqrt[4]{2}(s)}\cosh(\frac{1}{\sqrt[4]{2}}s), -3\sqrt{2}\cosh(\sqrt[4]{2}s) + \frac{4}{\sqrt[4]{2}}(s)\sinh(\sqrt[4]{2}s)\},\end{aligned}$$

and the null surface (Figure 3) of principal normal indicatrix of $\gamma(s)$ is

$$\mathcal{NS}(s,\lambda) = \{\mathcal{NS}_1, \mathcal{NS}_2, \mathcal{NS}_3, \mathcal{NS}_4\},$$

where

$$\begin{aligned}\mathcal{NS}_1 &= \frac{1}{2\sqrt[4]{2}}\sinh(\frac{1}{\sqrt[4]{2}}s)\lambda - 3\sqrt{2}\cosh(\frac{1}{\sqrt[4]{2}}s) + \frac{4}{\sqrt[4]{2}}\sinh(\frac{1}{\sqrt[4]{2}}s),\\ \mathcal{NS}_2 &= -\frac{1}{2\sqrt[4]{2}}\cosh(\sqrt[4]{2}s)\lambda - 3\sqrt{2}\sinh(\sqrt[4]{2}s) + \frac{4}{\sqrt[4]{2}}\cosh(\sqrt[4]{2}s),\\ \mathcal{NS}_3 &= \frac{1}{2\sqrt[4]{2}}\cosh(\frac{1}{\sqrt[4]{2}}s)\lambda - 3\sqrt{2}\sinh(\frac{1}{\sqrt[4]{2}}s) + \frac{4}{\sqrt[4]{2}}\cosh(\frac{1}{\sqrt[4]{2}}s),\\ \mathcal{NS}_4 &= -\frac{1}{2\sqrt[4]{2}}\sinh(\sqrt[4]{2}s)\lambda - 3\sqrt{2}\cosh(\sqrt[4]{2}s) + \frac{4}{\sqrt[4]{2}}\sinh(\sqrt[4]{2}s).\end{aligned}$$

On the other hand, we can calculate the geometric invariant $\sigma(s) = 0$. By Theorem 2.1 and Theorem 6.3, we have the null surface is locally diffeomorphic to cuspidal edge at singularity points.

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References

[1] W. B. Bonnor, *Null curves in a Minkowski space-time*, Tensor (N.S.), **20** (1969), 229–242. 1, 2

[2] J. W. Bruce, P. J. Giblin, *Curves and singularities*, A geometrical introduction to singularity theory, Second edition, Cambridge University Press, Cambridge, (1992). 1
[3] K. L. Duggal, *Constant scalar curvature and warped product globally null manifolds*, J. Geom. Phys., **43** (2002), 327–340. 1, 2
[4] K. L. Duggal, *A report on canonical null curves and screen distributions for lightlike geometry*, Acta Appl. Math., **95** (2007), 135–149.
[5] K. L. Duggal, *On scalar curvature in lightlike geometry*, J. Geom. Phys., **57** (2007), 473–481. 1
[6] A. Ferrández, A. Giménez, P. Lucas, *Geometrical particle models on 3D null curves*, Phys. Lett. B, **543** (2002), 311–317. 1
[7] Y.-L. Li, D.-H. Pei, *Evolutes of dual spherical curves for ruled surfaces*, Math. Methods Appl. Sci., **39** (2016), 3005–3015.
[8] D.-H. Pei, T. Sano, *The focal developable and the binormal indicatrix of a nonlightlike curve in Minkowski 3-space*, Tokyo J. Math., **23** (2000), 211–225. 1
[9] J.-G. Sun, D.-H. Pei, *Null Cartan Bertrand curves of AW(k)-type in Minkowski 4-space*, Phys. Lett. A, **376** (2012), 2230–2233. 1
[10] J.-G. Sun, D.-H. Pei, *Null surfaces of null curves on 3-null cone*, Phys. Lett. A, **378** (2014), 1010–1016. 1, 5
[11] Z.-G. Wang, D.-H. Pei, *Singularities of ruled null surfaces of the principal normal indicatrix to a null Cartan curve in de Sitter 3-space*, Phys. Lett. B., **689** (2010), 101–106. 1
[12] Z.-G. Wang, D.-H. Pei, *Null Darboux developable and pseudo-spherical Darboux image of null Cartan curve in Minkowski 3-space*, Hokkaido Math. J., **41** (2011), 219–240. 1
[13] G. Wassermann, *Stability of caustics*, Math. Ann., **216** (1975), 43–50. 6