Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 1613–1639
Research Article
ISSN: 2008-1898
Journal of Nonlinear Sciences and Applications
Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

S-γ-ϕ-φ-contractive type mappings in S-metric spaces

Mi Zhou[a], Xiao-lan Liu[b,c,*], Stojan Radenović[d]

[a]School of Science and Technology, Sanya College, 572022, Sanya, Hainan, China.

[b]College of Science, Sichuan University of Science and Engineering, 643000, Zigong, Sichuan, China.

[c]Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing, Zigong, Sichuan 643000, China.

[d]Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia.

Communicated by Y. J. Cho

Abstract

In this paper, we introduce several types of S-γ-ϕ-φ-contractive mappings which are generalizations of α-ψ-contractive mappings [B. Samet, C. Vetro, P. Vetro, Nonlinear Anal., **75** (2012), 2154–2165] in the structure of S-metric spaces. Furthermore, we prove existence and uniqueness of fixed points and common fixed points of such contractive mappings. Our results generalize, extend and improve the existing results in the literature. We also state some illustrative examples to support our results.

Keywords: S-metric space, S-γ-ϕ-φ-contractive mappings, fixed point.
2010 MSC: 47H10, 54H25, 54E50.

1. Introduction

In the last several decades, fixed point theory has attracted many researchers since 1922 with the famous Banach fixed point theorem. There exists a vast literature on this topic and this is a very active field of research at present. It is well-known that the contractive-type conditions are very indispensable in the study of fixed point theory. The first important result on fixed point for contractive-type mappings was the celebrated Banach Contractive Principle (BCP, for short) in [2, 6]. Due to application potential of the theory, many authors have directed their attentions to this field and have generalized the Banach contraction principle in various ways (see, e.g., [4, 9, 11, 13, 15, 21]). Very recently, a new, simple and unified approach in the theory of contractive mappings was given by Samet et al. [23] by using the concepts of α-ψ-contractive mappings and α-admissible mappings in metric spaces. The results obtained by Samet et al. [23] showed the Banach fixed point theorem and some other theorems in the literature became direct consequences of the results. Further, Karapinar and Samet [12] generalized the α-ψ-contractive type mappings and obtained various fixed point theorems for this generalized class of contractive mappings. On the other hand, several authors studied fixed point theory in generalized metric spaces. For details,

*Corresponding author
Email addresses: mizhou330@126.com (Mi Zhou), stellalwp@163.com (Xiao-lan Liu), radens@beotel.rs (Stojan Radenović)

doi:10.22436/jnsa.010.04.27

Received 2016-10-26

we refer readers to [5, 8, 10, 14, 26]. Not long ago, Sedghi et al. [25] introduced the concept of an S-metric space and proved some properties of an S-metric space and some fixed point theorems for a self-map on an S-metric space. After that, Sedghi and Dung [24] proved a generalized fixed point theorem in the context of S-metric spaces which is an extension of Theorem 3.1 of [25] and obtained many analogues of fixed point theorem in an S-metric space. Moreover, many authors proved some coupled common fixed point theorems and coupled coincidence point results for certain contractive contractions in a partially ordered S-metric spaces. For details, see [7, 28].

Motivated by discussions mentioned above, the purpose of this paper is first to introduce some types of S-γ-ϕ-φ-contractive mappings which are generalizations of α-ψ-contractive mappings in the structure of S-metric spaces and to prove some sufficient conditions for the existence and uniqueness of fixed points and common fixed points for some S-γ-ϕ-φ-contractions in S-metric spaces. Our main results generalize, extend and improve the existing results on this topic in the literature. Some illustrative examples are provided to demonstrate the main results and to show the genuineness of our results.

2. Preliminaries

We briefly recall some basic definitions and important results which sever a background to the following discussion.

Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers, and $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

Definition 2.1. [23] Let Ψ be the family of functions $\psi : [0,\infty) \mapsto [0,\infty)$ satisfying the following conditions:

1 ψ is nondecreasing;

2 $\Sigma_{n=1}^{\infty}\psi^n(t) < \infty$ $\forall t > 0$, where ψ^n is the n^{th} iterate of ψ.

These functions are known in the literature as (c)-comparison functions.

Lemma 2.2 ([3]). *If $\psi \in \Psi$, then the following hold:*

(1) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to 0 as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$;

(2) $\psi(t) < t$ *for any* $t \in (0,\infty)$;

(3) ψ *is continuous at 0;*

(4) *the series* $\sum_{k=1}^{\infty}\psi^k(t)$ *converges for all* $t \in \mathbb{R}^+$.

Let Φ be a family of functions $\varphi : [0,\infty) \mapsto [0,\infty)$ satisfying the following conditions:

(1) φ is lower semi-continuous;

(2) $\varphi(t) = 0$ if and only if $t = 0$.

The following definitions were introduced by Samet et al. [23].

Definition 2.3. Let (X, d) be a metric space and let $T : X \mapsto X$ be a given mapping. We say that T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \mapsto [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leqslant \psi(d(x,y)),\ \forall x,y \in X.$$

Obviously, any contractive mapping, that is, a mapping satisfying Banach contraction principle is an α-ψ-contractive mapping with $\alpha(x,y) = 1$, $\forall x,y \in X$ and $\psi(t) = kt$, $\forall t \geqslant 0$ and some $k \in [0,1)$.

Definition 2.4. Let $T : X \mapsto X$ and $\alpha : X \times X \mapsto [0,\infty)$. We say that T is α-admissible if for all $x,y \in X$, we have

$$\alpha(x,y) \geqslant 1 \Rightarrow \alpha(Tx,Ty) \geqslant 1.$$

The main results in [23] are the following fixed point theorems.

Theorem 2.5. *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *be an* α-ψ*-contractive mapping. Suppose that*

(1) T *is* α*-admissible;*

(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$*;*

(3) T *is continuous.*

Then there exists $x \in X$ *such that* $Tx = x$*.*

Theorem 2.6. *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *be an* α-ψ*-contractive mapping. Suppose that*

(1) T *is* α*-admissible;*

(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$*;*

(3) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geqslant 1$, $\forall n \in \mathbb{N}$ *and* $x_n \to x \in X$ *as* $n \to \infty$*, then* $\alpha(x_n, x) \geqslant 1$, $\forall n \in \mathbb{N}$*.*

Then there exists $x^* \in X$ *such that* $Tx^* = x^*$*.*

Samet et al. [23] added the following assumption to Theorems 2.5 and 2.6 to guarantee the uniqueness of the fixed point.

(C) For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geqslant 1$ and $\alpha(y, z) \geqslant 1$.

Recently, Karapinar and Samet [12] presented the following notion of generalized α-ψ-contractive type mappings. Further, Karapinar and Samet [12] established fixed point theorems for this new class of contractive mappings.

Definition 2.7. Let (X, d) be a metric space and let $T : X \mapsto X$ be a given mapping. We say that T is a generalized α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \mapsto [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have

$$\alpha(x, y)d(Tx, Ty) \leqslant \psi(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Sedghi et al. [25] introduced the notion of S-metric spaces as follows.

Definition 2.8. Let X be a nonempty set. An S-metric on X is a function $S : X^3 \mapsto [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$:

(S1) $S(x, y, z) = 0$ if and only if $x = y = z = 0$;

(S2) $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Immediate examples of such S-metric spaces are:

(1) Let $\mathbb{R}$ be a real line, then $S(x, y, z) = |x - z| + |y - z|$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

(2) Let $X = \mathbb{R}^+$ and $\|\cdot\|$ be a norm on X, then $S(x, y, z) = \|2x + y - 3z\| + \|x - z\|$ is an S-metric on X for all $x, y, z \in X$.

(3) Let X be a nonempty set, d is ordinary metric on X, then $S_d(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X for all $x, y, z \in X$.

Lemma 2.9 ([25]). *Let (X, S) be an S-metric space. Then*

$$S(x,x,z) \leqslant 2S(x,x,y) + S(y,y,z) \quad \text{and} \quad S(x,x,z) \leqslant 2S(x,x,y) + S(z,z,y)$$

for all $x, y, z \in X$.

Lemma 2.10 ([25]). *Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Definition 2.11. Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\} \subset X$ is said to be convergent to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $S(x_n, x_n, x) < \epsilon$.

(2) A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$, we have $S(x_n, x_n, x_m) < \epsilon$.

(3) The S-metric space (X, S) is said to be complete if every Cauchy sequence is a convergent sequence.

(4) A mapping $T : X \mapsto X$ is said to be S-continuous if $\{Tx_n\}$ is S-convergent to Tx, where $\{x_n\}$ is an S-convergent sequence converging to x.

Lemma 2.12 ([25]). *Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

Lemma 2.13 ([25]). *Let (X, S) be an S-metric space. If the sequences $\{x_n\}$ in X such that $x_n \to x$, then x is unique.*

3. Main results

We start the main results by introducing the new concept of S-γ-ψ-φ-contractive mappings as follows.

Definition 3.1. Let (X, S) be an S-metric space and let $T : X \mapsto X$ be a given mapping. We say that T is an S-γ-ψ-φ-contractive mapping of type A if there exist three functions $\gamma : X^3 \mapsto [0, \infty), \psi \in \Psi$, and $\varphi \in \Phi$ such that for all $x, y, z \in X$, we have that

$$\gamma(x,y,z)S(Tx,Ty,Tz) \leqslant \psi(S(x,y,z)) - \varphi(S(x,y,z)).$$

Definition 3.2. Let (X, S) be an S-metric space and let $T : X \mapsto X$ be a given mapping. We say that T is an S-γ-ψ-φ-contractive mapping of type B if there exist three functions $\gamma : X^3 \mapsto [0, \infty), \psi \in \Psi$, and $\varphi \in \Phi$ such that for all $x, y \in X$, we have that

$$\gamma(x,y,y)S(Tx,Ty,Ty) \leqslant \psi(S(x,y,y)) - \varphi(S(x,y,y)). \tag{3.1}$$

Definition 3.3. Let (X, S) be an S-metric space and let $T : X \mapsto X$ be a given mapping. We say that T is an S-γ-ψ-φ-contractive mapping of type C if there exist three functions $\gamma : X^3 \mapsto [0, \infty), \psi \in \Psi$, and $\varphi \in \Phi$ such that for all $x, y \in X$, we have that

$$\gamma(x,y,Tx)S(Tx,Ty,T^2x) \leqslant \psi(S(x,y,Tx)) - \varphi(S(x,y,Tx)). \tag{3.2}$$

Definition 3.4. Let $T : X \mapsto X$ and $\gamma : X^3 \mapsto [0, \infty)$. We say that T is γ-admissible if for all $x, y, z \in X$, we have that

$$\gamma(x,y,z) \geqslant 1 \Rightarrow \gamma(Tx,Ty,Tz) \geqslant 1.$$

Definition 3.5. Let $T, f : X \mapsto X$ and $\gamma : X^3 \mapsto [0, \infty)$. We say that T is f-γ-admissible if for all $x, y, z \in X$, we have that

$$\gamma(fx,fy,fz) \geqslant 1 \Rightarrow \gamma(Tx,Ty,Tz) \geqslant 1.$$

If f is the identity mapping, then T is γ-admissible.

Example 3.6. Let $X = [1, \infty)$ and $T : X \mapsto X$. Define $Tx = x^2$ and $\gamma(x, y, z)=\begin{cases} 2, & \text{if} \quad x \geqslant y \geqslant z, \\ 0, & \text{otherwise.} \end{cases}$ Then T is γ-admissible.

The following example shows that a mapping T which is f-γ-admissible may not be γ-admissible.

Example 3.7. Let $X = [0, \infty)$. Define the mapping $\gamma(x, y, z) : X^3 \mapsto [0, \infty)$ by

$$\gamma(x, y, z) = \begin{cases} e, & \text{if} \quad x > y > z, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Also define the mappings $T, f : X \mapsto X$ by $Tx = \frac{1}{x^2}$ and $fx = e^{-x}$ for all $x \in X$.

Suppose that $\gamma(x, y, z) \geqslant 1$. This implies from the definition of γ that $x > y > z$ which further implies that $\frac{1}{x^2} < \frac{1}{y^2} < \frac{1}{z^2}$. Thus $\gamma(Tx, Ty, Tz) \ngeqslant 1$, that is T is not γ-admissible.

Now, we prove that T is f-γ-admissible. Let us assume that $\gamma(fx, fy, fz) \geqslant 1$. So

$$\gamma(fx, fy, fz) \geqslant 1 \Rightarrow fx > fy > fz \Rightarrow e^{-x} > e^{-y} > e^{-z} \Rightarrow \frac{1}{x^2} > \frac{1}{y^2} > \frac{1}{z^2} \Rightarrow \gamma(Tx, Ty, Tz) \geqslant 1.$$

Therefore, T is f-γ-admissible.

Definition 3.8. We say that:

1. a sequence $\{x_n\}$ in X is (T, γ)-orbital if $x_n = T^n x_0$ and $\gamma(x_n, x_{n+1}, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$.
2. T is γ-orbital continuous if, for every (T, γ)-orbital sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to Tx$ as $k \to \infty$.
3. X is (T, γ)-regular if, for every (T, γ)-orbital sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, there exists a subsequence $\{x_{x_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geqslant 1$ for all $k \in \mathbb{N}$.
4. X is γ-regular if, for every sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$ and $\gamma(x_n, x_{n+1}, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{x_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geqslant 1$ for all $k \in \mathbb{N}$.

Remark 3.9.

(1) If T is continuous, then T is γ-orbital continuous (for any γ).

(2) If X is γ-regular, then X is also (T, γ)-regular (for any γ).

Definition 3.10. Let $\gamma : X^3 \mapsto [0, \infty)$. We say that γ is transitive if

$$\begin{cases} \gamma(x, y, y) \geqslant 1, \\ \gamma(y, z, z) \geqslant 1, \end{cases} \quad \text{implies } \gamma(x, z, z) \geqslant 1, \text{ for all } x, y, z \in X.$$

Lemma 3.11. *Let $T : X \mapsto X$ and $\gamma : X^3 \mapsto [0, \infty)$ be γ-admissible and transitive, respectively. Assume that there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then $\gamma(x_m, x_n, x_n) \geqslant 1$, for all $m, n \in \mathbb{N}$ with $m < n$.*

Proof. Since there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$, then from the definition of γ-admissibility, we deduce that $\gamma(x_1, x_2, x_2) = \gamma(Tx_0, Tx_1, Tx_1) \geqslant 1$.

By continuing this process, we get $\gamma(x_n, x_{n+1}, x_{n+1}) \geqslant 1$, $\forall n \in \{0\} \cup \mathbb{N}$.

Suppose that $m < n$. Since

$$\begin{cases} \gamma(x_m, x_{m+1}, x_{m+1}) \geqslant 1, \\ \gamma(x_{m+1}, x_{m+2}, x_{m+2}) \geqslant 1, \end{cases}$$

by the definition of transitivity of γ, we deduce that $\gamma(x_m, x_{m+2}, x_{m+2}) \geqslant 1$. By continuing this process, we get $\gamma(x_m, x_n, x_n) \geqslant 1$, $\forall m, n \in \mathbb{N}$ with $m < n$. □

Theorem 3.12. *Let* (X, S) *be a complete* S*-metric space. Suppose that* $T : X \mapsto X$ *is an* S-γ-ψ-φ*-contractive mapping of type* B *and satisfies the following assumptions:*

(A1) T *is* γ*-admissible;*

(A2) *there exists* $x_0 \in X$ *such that* $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$*;*

(A3) T *is* γ*-orbital continuous.*

Then, there exists $x^* \in X$ *such that* $Tx^* = x^*$*.*

Proof. Let $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geqslant 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then $x^* = x_{n_0}$ is a fixed point of T. So we suppose that $x_n \neq x_{n+1}$ for all $n \in \{0\} \cup \mathbb{N}$. Since T is γ-admissible, we have that

$$\gamma(x_0, x_1, x_1) = \gamma(x_0, Tx_0, Tx_0) \geqslant 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) = \gamma(x_1, x_2, x_2) \geqslant 1.$$

By induction, we get that

$$\gamma(x_n, x_{n+1}, x_{n+1}) \geqslant 1, \quad \forall n = 0, 1, 2, \cdots . \tag{3.3}$$

From (3.2) and (3.3), it follows that for all $n \geqslant 1$, we have that

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) = S(Tx_{n-1}, Tx_n, Tx_n) &\leqslant \gamma(x_{n-1}, x_n, x_n) S(Tx_{n-1}, Tx_n, Tx_n) \\ &\leqslant \psi(S(x_{n-1}, x_n, x_n)) - \varphi(S(x_{n-1}, x_n, x_n)) \\ &\leqslant \psi(S(x_{n-1}, x_n, x_n)). \end{aligned}$$

Since ψ is nondecreasing, by induction, we have that

$$S(x_n, x_{n+1}, x_{n+1}) < \psi^n(S(x_0, x_1, x_1)), \quad \forall n \geqslant 1. \tag{3.4}$$

Using (S2) and (3.4), we have

$$\begin{aligned} S(x_n, x_m, x_m) &\leqslant 2 \sum_{k=n}^{m-2} S(x_k, x_{k+1}, x_{k+1}) + S(x_{m-1}, x_m, x_m) \\ &\leqslant 2 \sum_{k=n}^{m-2} \psi^k(S(x_0, x_1, x_1)) + \psi^{m-1}(S(x_0, x_1, x_1)). \end{aligned}$$

Since $\psi \in \Psi$ and $S(x_0, x_1, x_1) > 0$, by Lemma 2.2, we get that

$$\lim_{n,m \to \infty} S(x_n, x_m, x_m) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the S-metric space (X, S).

Since T is γ-orbital continuous, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that Tx_{n_k} converges to Tx^* as $k \to \infty$. By the uniqueness of this limit, we get $x^* = Tx^*$, that is, x^* is a fixed point of T. □

The next theorem does not require γ-orbital continuity or continuity of T.

Theorem 3.13. *Let* (X, S) *be a complete* S*-metric space. Suppose that* $T : X \mapsto X$ *is an* S-γ-ψ-φ*-contractive mapping of type* B *and satisfies the following assumptions:*

(A1) T *is* γ*-admissible;*

(A2) *there exists* $x_0 \in X$ *such that* $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$*;*

(A3) (X, S) *is* (T, γ)*-regular.*

Then, there exists $x^* \in X$ *such that* $Tx^* = x^*$.

Proof. From the proof of Theorem 3.12, it follows that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for all $n \geqslant 0$ is a Cauchy sequence in the complete S-metric space (X, S), that is convergent to $x^* \in X$.

Since $\{x_n\}$ is a (T, γ)-orbital sequence, by (A3), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\gamma(x_{n_k}, x^*, x^*) \geqslant 1, \quad \forall k \in \mathbb{N}. \tag{3.5}$$

Using (3.1) and (3.5), we have that

$$\begin{aligned} S(x_{n_k+1}, Tx^*, Tx^*) &= S(Tx_{n_k}, Tx^*, Tx^*) \\ &\leqslant \gamma(x_{n_k}, x^*, x^*) S(Tx_{n_k}, Tx^*, Tx^*) \\ &\leqslant \psi(S(x_{n_k}, x^*, x^*)) - \varphi(S(x_{n_k}, x^*, x^*)) \\ &\leqslant \psi(S(x_{n_k}, x^*, x^*)). \end{aligned}$$

Letting $k \to \infty$, since ψ is continuous at $t = 0$, it follows that $S(x^*, Tx^*, Tx^*) = 0$, then $x^* = Tx^*$. □

With the following examples, we show that the hypotheses in Theorems 3.12-3.13 do not guarantee uniqueness of fixed point.

Example 3.14. Let $X = [0, \infty)$ be an S-metric space with the S-metric defined by $S(x, y, z) = |x - z| + |y - z|, \forall x, y, z \in X$. For all $k > 1$, consider the self-mapping $T : X \mapsto X$ given by

$$Tx = \begin{cases} 2kx - \frac{8k-1}{4}, & x > 1, \\ \frac{x}{4}, & 0 \leqslant x \leqslant 1. \end{cases}$$

Also, define $\gamma : X^3 \mapsto [0, \infty)$ as

$$\gamma(x, y, z) = \begin{cases} 1, & x, y, z \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Let $\psi(t) = \frac{t}{2}, \forall t \geqslant 0$ and $\varphi(t) = \frac{t}{4k}, \forall t \geqslant 0$. Then we conclude that T is an S-γ-ψ-φ-contractive mapping of type B. In fact, for all $x, y, z \in X$, we have that

$$\gamma(x, y, y) S(Tx, Ty, Ty) \leqslant \psi(S(x, y, y)) - \varphi(S(x, y, y)).$$

On the other hand, there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$. Indeed, for $x_0 = 1$, we have

$$\gamma(1, T1, T1) = \gamma(1, \frac{1}{4}, \frac{1}{4}) = 1.$$

Notice that T is continuous. We only need to check that T is γ-admissible. For this purpose, let $x, y, z \in X$ such that $\gamma(x, y, z) \geqslant 1$, which implies that $x, y, z \in [0, 1]$. Due to the definitions of γ and T, we have that

$$Tx = \frac{x}{4} \in [0, 1], \quad Ty = \frac{y}{4} \in [0, 1], \quad Tz = \frac{z}{4} \in [0, 1].$$

Hence, $\gamma(Tx, Ty, Ty) \geqslant 1$. As a result, all the assumptions of Theorem 3.12 are satisfied. Note that Theorem 3.12 guarantees the existence of a fixed point but not the uniqueness. In fact, 0 and $\frac{8k-1}{8k-4}$ are two fixed points of T.

In the following example, T is not continuous.

Example 3.15. Let (X, S) be defined as in Example 3.14. Let $T : X \mapsto X$ be a map given by

$$Tx = \begin{cases} e^{x-1}, & x > 1, \\ \frac{3-x}{4}, & 0 \leqslant x \leqslant 1. \end{cases}$$

Also, define $\gamma : X^3 \mapsto [0, \infty)$ as

$$\gamma(x, y, z) = \begin{cases} 1, & x, y, z \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Let $\psi(t) = \frac{t}{2}, \forall t \geqslant 0$ and $\varphi(t) = \frac{t}{6}, \forall t \geqslant 0$. Then we conclude that T is an S-γ-ψ-φ-contractive mapping of type B. In fact, for all $x, y, z \in X$, we have that

$$\gamma(x, y, y)S(Tx, Ty, Ty) \leqslant \psi(S(x, y, y)) - \varphi(S(x, y, y)).$$

On the other hand, there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$. Indeed, for $x_0 = 1$, we have that

$$\gamma(1, T1, T1) = \gamma(1, \frac{1}{2}, \frac{1}{2}) = 1.$$

Let $\{x_n\}$ be a (T, γ)-orbital sequence such that $x_n \to x$ as $n \to \infty$. By the definition of γ, we have $x_n \in [0, 1)$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geqslant 1$ for all $k \in \mathbb{N}$.

To show that T satisfies all hypotheses of Theorem 3.13, it suffices to observe that T is γ-admissible. For this purpose, let $x, y, z \in X$ such that $\gamma(x, y, z) \geqslant 1$, which implies that $x, y, z \in [0, 1)$. Due to the definitions of γ and T, we have that

$$Tx = \frac{3-x}{4} \in [0, 1), \quad Ty = \frac{3-y}{4} \in [0, 1), \quad Tz = \frac{3-z}{4} \in [0, 1).$$

Hence, $\gamma(Tx, Ty, Tz) \geqslant 1$. As a result, all the assumptions of Theorem 3.13 are satisfied. In this example, T is not continuous, and 1 and $\frac{3}{5}$ are two fixed points of T.

Theorem 3.16. *Let (X, S) be a complete S-metric space. Suppose that $T : X \mapsto X$ is an S-γ-ψ-φ-contractive mapping of type C and satisfies the following assumptions:*

(A1) T *is γ-admissible;*

(A2) *there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$;*

(A3) *if $\{x_n\}$ is a sequence in X such that $\gamma(x_n, x_{n+1}, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x^*$, then $\gamma(x_n, x^*, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$.*

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Following the proof of Theorem 3.12, we only have to prove that x^* is a fixed point of T.

Since the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geqslant 0$ converges to $x^* \in X$. From (3.3) and (A3), we have that

$$\gamma(x_n, x^*, x_{n+1}) \geqslant 1, \quad \forall n \geqslant 0. \tag{3.6}$$

With (3.2) and (3.6), we have

$$\begin{aligned} S(x_{n+1}, Tx^*, x_{n+2}) &= S(Tx_n, Tx^*, T^2x_n) \\ &\leqslant \gamma(x_n, x^*, x_{n+1})S(Tx_n, Tx^*, T^2x_n) \\ &\leqslant \psi(S(x_n, x^*, Tx_n)) - \varphi(S(x_n, x^*, Tx_n)) \\ &\leqslant \psi(S(x_n, x^*, x_{n+1})). \end{aligned}$$

Letting $n \to \infty$, since ψ is continuous at $t = 0$, it follows that $S(x^*, Tx^*, x^*) = 0$, that is $x^* = Tx^*$. □

The following theorems can be derived easily from Theorems 3.12 and 3.16.

Theorem 3.17. *Let (X, S) be a complete S-metric space. Suppose that $T : X \mapsto X$ is an S-γ-ψ-φ-contractive mapping of type A and satisfies the following assumptions:*

(A1) *T is γ-admissible;*

(A2) *there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$;*

(A3) *T is γ-orbital continuous.*

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 3.18. *Let (X, S) be a complete S-metric space. Suppose that $T : X \mapsto X$ is an S-γ-ψ-φ-contractive mapping of type B and satisfies the following assumptions:*

(A1) *T is γ-admissible;*

(A2) *there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$;*

(A3) *if $\{x_n\}$ is a sequence in X such that $\gamma(x_n, x_{n+1}, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x^*$, then $\gamma(x_n, x^*, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$.*

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 3.19. *Adding the following condition to the hypotheses of Theorem 3.16 (resp. Theorems 3.12 and 3.13), we obtain the uniqueness of a fixed point T.*

(A4) *For all $x, y \in X$, there exists $z \in X$ such that $\gamma(x, z, z) \geqslant 1$ and $\gamma(y, z, z) \geqslant 1$.*

Proof. Let $u, v \in X$ be two fixed points of T. By (A4), there exists $z \in X$ such that $\gamma(u, z, z) \geqslant 1$ and $\gamma(v, z, z) \geqslant 1$.

Since T is γ-admissible, we get by induction that

$$\gamma(u, u, T^n z) \geqslant 1 \quad \text{and} \quad \gamma(v, v, T^n z) \geqslant 1, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

From (3.7) and (3.2), we have that

$$\begin{aligned} S(u, T^n z, u) &= S(Tu, T(T^{n-1}z), T^2 u) \\ &\leqslant \gamma(u, T^{n-1}z, Tu) S(Tu, T(T^{n-1}z), T^2 u) \\ &\leqslant \psi(S(u, T^{n-1}z, Tu)) - \varphi(S(u, T^{n-1}z, Tu)) \\ &\leqslant \psi(S(u, T^{n-1}z, Tu)). \end{aligned}$$

Letting $n \to \infty$, and since $\psi \in \Psi$, we have that

$$S(u, T^n z, u) \to 0.$$

This implies that $\{T^n z\}$ is convergent to u. Similarly, we can get $\{T^n z\}$ is convergent to v. By Lemma 2.13, we get $u = v$, that is, fixed point of T is unique. □

Definition 3.20. Let (X, S) be an S-metric space and let $T : X \mapsto X$ be a given mapping. We say that T is an S-γ-ψ-φ-contractive mapping of type D if there exist three functions $\gamma : X^3 \mapsto [0, \infty)$, $\psi \in \Psi$, and $\varphi \in \Phi$ such that for all $x, y \in X$, we have that

$$\gamma(x, y, y) S(Tx, Ty, Ty) \leqslant \psi(M(x, y)) - \varphi(M(x, y)), \tag{3.8}$$

where $M(x, y) = \max\{S(x, y, y), \frac{2S(x,Tx,Tx)+S(y,Ty,Ty)}{3}, \frac{2S(y,Tx,Tx)+S(x,Ty,Ty)}{3}\}$.

Theorem 3.21. *Let (X, S) be a complete S-metric space. Suppose that $T : X \mapsto X$ is an S-γ-ψ-φ-contractive mapping of type D and satisfies the following assumptions:*

(A1) *T is γ-admissible;*

(A2) *there exists $x_0 \in X$ such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$;*

(A3) *(X, S) is (T, γ)-regular.*

Then, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. In view of assumption (A2), let $x_0 \in X$ be such that $\gamma(x_0, Tx_0, Tx_0) \geqslant 1$. Define the sequence $\{x_n\}$ in X as follows

$$x_{n+1} = Tx_n, \quad \forall n \geqslant 0.$$

Since T is γ-admissible, we have that

$$\gamma(x_0, x_1, x_1) = \gamma(x_0, Tx_0, Tx_0) \geqslant 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) \geqslant 1.$$

Using mathematical induction, we get that

$$\gamma(x_n, x_{n+1}, x_{n+1}) \geqslant 1, \quad \forall n = 0, 1, 2, \dots. \tag{3.9}$$

If $x_{n_0} = x_{n_0+1}$ for some n_0, then $x^* = x_{n_0}$ is a fixed point T, and so we have finished the proof. For this, we assume that $S(x_n, x_{n+1}, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. Now, from (3.8) and (3.9), we have that

$$\begin{aligned} S(Tx_n, Tx_{n+1}, Tx_{n+1}) &\leqslant \gamma(x_n, x_{n+1}, x_{n+1}) S(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leqslant \psi(M(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &\leqslant \psi(M(x_n, x_{n+1})) \end{aligned}$$

for all $n \in \mathbb{N}$. On the other hand, we have that

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{S(x_n, x_{n+1}, x_{n+1}), \frac{2S(x_n, Tx_n, Tx_n) + S(x_{n+1}, Tx_{n+1}, Tx_{n+1})}{3} \\ &\quad , \frac{2S(x_{n+1}, Tx_n, Tx_n) + S(x_n, Tx_{n+1}, Tx_{n+1})}{3}\} \\ &= \max\{S(x_n, x_{n+1}, x_{n+1}), \frac{2S(x_n, x_{n+1}, x_{n+1}) + S(x_{n+1}, x_{n+2}, x_{n+2})}{3} \\ &\quad , \frac{2S(x_{n+1}, x_{n+1}, x_{n+1}) + S(x_n, x_{n+2}, x_{n+2})}{3}\} \\ &\leqslant \max\{S(x_n, x_{n+1}, x_{n+1}), S(x_{n+1}, x_{n+2}, x_{n+2})\}. \end{aligned}$$

Hence, we have that

$$S(Tx_n, Tx_{n+1}, Tx_{n+1}) \leqslant \psi(\max\{S(x_n, x_{n+1}, x_{n+1}), S(x_{n+1}, x_{n+2}, x_{n+2})\}), \quad \forall n \in \mathbb{N}. \tag{3.10}$$

If for some $n \geqslant 1$ we have $S(x_n, x_{n+1}, x_{n+1}) \leqslant S(x_{n+1}, x_{n+2}, x_{n+2})$, from (3.10), we have that

$$S(x_{n+1}, x_{n+2}, x_{n+2}) \leqslant \psi(S(x_{n+1}, x_{n+2}, x_{n+2})) < S(x_{n+1}, x_{n+2}, x_{n+2}),$$

which is a contradiction. Thus, for all $n \geqslant 1$, we conclude that

$$\max\{S(x_n, x_{n+1}, x_{n+1}), S(x_{n+1}, x_{n+2}, x_{n+2})\} = S(x_n, x_{n+1}, x_{n+1}).$$

So, we have that

$$S(Tx_n, Tx_{n+1}, Tx_{n+1}) = S(x_{n+1}, x_{n+2}, x_{n+2}) \leqslant \psi(S(x_n, x_{n+1}, x_{n+1})).$$

Continuing this process inductively, we obtain that

$$S(x_n, x_{n+1}, x_{n+1}) \leqslant \psi^n(S(x_0, x_1, x_1)), \quad \forall n \geqslant 1. \tag{3.11}$$

From (3.11) and Lemma 2.9, for all $k \geqslant 1$, we have that

$$\begin{aligned} S(x_n, x_{n+k}, x_{n+k}) &\leqslant 2 \sum_{i=n}^{n+k-2} S(x_i, x_{i+1}, x_{i+1}) + S(x_{n+k-1}, x_{n+k-1}, x_{n+k}) \\ &\leqslant 2 \sum_{i=n}^{n+k-2} \psi^i(S(x_0, x_1, x_1)) + \psi^{n+k-1}(S(x_0, x_1, x_1)). \end{aligned}$$

Since $\psi \in \Psi$ and $S(x_0, x_1, x_1) > 0$, by Lemma 2.2, we have that

$$\lim_{n,k\to\infty} S(x_n, x_{n+k}, x_{n+k}) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since (X, S) is complete, then there exists $x^* \in X$ such that $x_n \to x^*$, as $n \to \infty$.

Now, we will show that x^* is a fixed point of T. We assume on contrary that $S(x^*, Tx^*, Tx^*) > 0$. By (A3), we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x^*, x^*) \geqslant 1$, for all $k \in \mathbb{N}$. Then by Lemma 2.9, Lemma 2.10, and (3.8), we have that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leqslant 2S(x^*, Tx_{n_k}, Tx_{n_k}) + S(Tx_{n_k}, Tx^*, Tx^*) \\ &\leqslant 2S(x^*, Tx_{n_k}, Tx_{n_{(k)}}) + \gamma(x_{n_k}, x^*, x^*)S(Tx_{n_k}, Tx^*, Tx^*) \\ &\leqslant 2S(x^*, Tx_{n_k}, Tx_{n_{(k)}}) + \psi(M(x_{n_k}, x^*)) - \varphi(M(x_{n_k}, x^*)) \\ &\leqslant 2S(x^*, Tx_{n_k}, Tx_{n_{(k)}}) + \psi(M(x_{n_k}, x^*)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} M(x_{n_k}, x^*) &= \max\{S(x_{n_k}, x^*, x^*), \frac{2S(x_{n_k}, Tx_{n_k}, Tx_{n_k}) + S(x^*, Tx^*, Tx^*)}{3} \\ &\quad, \frac{2S(x^*, Tx_{n_k}, Tx_{n_k}) + S(x_{n_k}, Tx^*, Tx^*)}{3}\} \\ &= \max\{S(x_{n_k}, x^*, x^*), \frac{2S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x^*, Tx^*, Tx^*)}{3} \\ &\quad, \frac{2S(x^*, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k}, Tx^*, Tx^*)}{3}\}. \end{aligned}$$

From above inequality and equality, we get that

$$\begin{aligned} S(x^*, Tx^*, Tx^*) &\leqslant 2S(x^*, Tx_{n_k}, Tx_{n_k}) + \psi(M(x_{n_k}, x^*)) \\ &= 2S(x^*, x_{n_k+1}, x_{n_k+1}) + \psi(\max\{S(x_{n_k}, x^*, x^*), \frac{2S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + S(x^*, Tx^*, Tx^*)}{3} \\ &\quad, \frac{2S(x^*, x_{n_k+1}, x_{n_k+1}) + S(x_{n_k}, Tx^*, Tx^*)}{3}\}). \end{aligned}$$

Letting $k \to \infty$ in above inequality, it yields that

$$S(x^*, Tx^*, Tx^*) \leqslant \psi(\frac{S(x^*, Tx^*, Tx^*)}{3}) < \frac{S(x^*, Tx^*, Tx^*)}{3},$$

which is a contradiction. Hence, we have that $S(x^*, Tx^*, Tx^*) = 0$, that is $x^* = Tx^*$. This shows that x^* is a fixed point of T. □

Theorem 3.22. *In addition to the hypotheses of Theorem 3.21, suppose that for all $x, y \in X$, there exists $z \in X$ such that $\gamma(x, z, z) \geqslant 1$ and $\gamma(y, z, z) \geqslant 1$. Then T has a unique fixed point.*

Proof. Let $u, v \in X$ be two fixed points of T. By hypotheses, then there exists $z \in X$ such that

$$\gamma(u, z, z) \geqslant 1 \quad \text{and} \quad \gamma(v, z, z) \geqslant 1. \tag{3.12}$$

Define the sequence $\{z_n\}$ in X by $z_n = T^n z$ for all $n \geqslant 0$ and $z_0 = z$. Since T is γ-admissible, we have from (3.12) that

$$\gamma(u, z_n, z_n) \geqslant 1 \quad \text{and} \quad \gamma(v, z_n, z_n) \geqslant 1, \quad \forall n \geqslant 0. \tag{3.13}$$

Applying (3.8) and (3.13), we obtain that for all $n \geqslant 0$,

$$\begin{aligned} S(u, z_{n+1}, z_{n+1}) = S(Tu, Tz_n, Tz_n) &\leqslant \gamma(u, z_n, z_n) S(Tu, Tz_n, Tz_n) \\ &\leqslant \psi(M(u, z_n)) - \varphi(M(u, z_n)) \\ &\leqslant \psi(M(u, z_n)). \end{aligned}$$

On the other hand, we have that for all $n \geqslant 0$

$$\begin{aligned} M(u, z_n) &= \max\{S(u, z_n, z_n), \frac{2S(u, Tu, Tu) + S(z_n, Tz_n, Tz_n)}{3}, \frac{2S(z_n, Tu, Tu) + S(u, Tz_n, Tz_n)}{3}\} \\ &\leqslant \max\{S(u, z_n, z_n), S(u, z_{n+1}, z_{n+1})\}. \end{aligned}$$

Using above two inequalities, owing to the monotone property of ψ, we obtain that

$$S(u, z_{n+1}, z_{n+1}) \leqslant \psi(\max\{S(u, z_n, z_n), S(u, z_{n+1}, z_{n+1})\}), \quad \forall n \geqslant 0. \tag{3.14}$$

If $\max\{S(u, z_n, z_n), S(u, z_{n+1}, z_{n+1})\} = S(u, z_{n+1}, z_{n+1})$, we have from (3.14) and Lemma 2.2 that

$$S(u, z_{n+1}, z_{n+1}) \leqslant \psi(S(u, z_{n+1}, z_{n+1})) < S(u, z_{n+1}, z_{n+1}),$$

which is a contradiction. Hence, $\max\{S(u, z_n, z_n), S(u, z_{n+1}, z_{n+1})\} = S(u, z_n, z_n)$ and $S(u, z_{n+1}, z_{n+1}) \leqslant \psi(S(u, z_n, z_n))$ for all $n \geqslant 0$. This implies that

$$S(u, z_{n+1}, z_{n+1}) \leqslant \psi^n(S(u, z, z)), \forall n \geqslant 0.$$

Letting $n \to \infty$ in above inequality, we can infer that

$$\lim_{n\to\infty} S(u, z_n, z_n) = 0. \tag{3.15}$$

Similarly, we also have that

$$\lim_{n\to\infty} S(v, z_n, z_n) = 0. \tag{3.16}$$

It follows from (3.15) and (3.16) that $u = v$. □

In what follows, we present an illustrative example to show the validity of Theorem 3.21.

Example 3.23. Consider $X = \mathbb{R}$ and $S(x, y, z) = |x - y| + |x - z|$. Then (X, S) is a complete S-metric space. Let $x_0 = 1$ and $r = 2$, then

$$B_S[x_0, r] = B_S[1, 2] = \{y \in X | S(x_0, y, y) \leqslant 2\} = [0, 2].$$

Now, let $T : B_S[1,2] \mapsto X, Tx = \frac{x}{2}$ and define $\gamma : X^3 \mapsto [0,\infty)$ as

$$\gamma(x,y,z) = \begin{cases} 1, & \text{if } x,y,z \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(t) = t, \forall t \geqslant 0$ and $\varphi(t) = \frac{t}{4}, \forall t \geqslant 0$. In fact, for all $x, y \in B_S[1,2]$,

$$S(Tx,Ty,Ty) = 2|Tx-Ty| = |x-y| = \frac{1}{2}(2|x-y|) = \frac{1}{2}S(x,y,y),$$

and

$$M(x,y) = \max\{S(x,y,y), \frac{2S(x,Tx,Tx)+S(y,Ty,Ty)}{3}, \frac{2S(y,Tx,Tx)+S(x,Ty,Ty)}{3}\}.$$

so, we have that

$$\gamma(x,y,y)S(Tx,Ty,Ty) \leqslant \psi(M(x,y)) - \varphi(M(x,y)).$$

On the other hand, there exists $x_0 \in X$ such that $\gamma(x_0,Tx_0,Tx_0) \geqslant 1$. Indeed, for $x_0 = 1$, we have $\gamma(1,T1,T1) = \gamma(1,\frac{1}{2},\frac{1}{2}) = 1$.

Let $\{x_n\}$ be a (T,γ)-orbital sequence such that $x_n \to x$ as $n \to \infty$. By the definition of γ, we have that $x_n \in [0,1]$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k},x,x) \geqslant 1$ for all $k \in \mathbb{N}$. Now, we only need to show that T is γ-admissible. For this purpose, let $x,y,z \in X$ such that $\gamma(x,y,z) \geqslant 1$ which further implies that $x,y,z \in [0,1]$. Due to the definition of T and γ, we have that

$$Tx = \frac{x}{2} \in [0,1], \quad Ty = \frac{y}{2} \in [0,1], \quad Tz = \frac{z}{2} \in [0,1].$$

Hence, $\gamma(Tx,Ty,Tz) \geqslant 1$.

As a result, all the hypotheses of Theorem 3.21 are satisfied. In fact, $x^* = 0$ is a fixed point of T.

Definition 3.24. Let (X,S) be an S-metric space and let $T : X \mapsto X$ be a given mapping. We say that T is an S-γ-ψ-φ-contractive mapping of type E if there exist three functions $\gamma : X^3 \mapsto [0,\infty), \psi \in \Psi$, and $\varphi \in \Phi$ such that for all $x,y \in X$, we have that

$$\psi(\gamma(x,y,y)S(Tx,Ty,Ty)) \leqslant \psi(M_1(x,y)) - \varphi(M_1(x,y)), \tag{3.17}$$

where $M_1(x,y) = \max\{S(x,y,y), S(x,Tx,Tx), S(y,Ty,Ty)\}$.

Theorem 3.25. *Let (X,S) be a complete S-metric space. Suppose that $T : X \mapsto X$ is an S-γ-ψ-φ-contractive mapping of type E and satisfies the following assumptions:*

(A1) *T is γ-admissible and transitive;*

(A2) *there exists $x_0 \in X$ such that $\gamma(x_0,Tx_0,Tx_0) \geqslant 1$;*

(A3) *(X,S) is (T,γ)-regular;*

(A4) *either $\gamma(u,v,v) \geqslant 1$ or $\gamma(v,u,u) \geqslant 1$, whenever $u = Tu$ and $v = Tv$.*

Then, T has a unique fixed point in X, that is, there exists a unique $x^* \in X$ such that $x^* = Tx^*$.

Proof. In view of assumption (A2), let $x_0 \in X$ such that $\gamma(x_0,Tx_0,Tx_0) \geqslant 1$. Define the sequence $\{x_n\}$ in X as follows

$$x_{n+1} = Tx_n, \quad \forall n \geqslant 0.$$

Since T is γ-admissible, we have that

$$\gamma(x_0, x_1, x_1) = \gamma(x_0, Tx_0, Tx_0) \geqslant 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) = \gamma(x_1, x_2, x_2) \geqslant 1.$$

By induction, we get that

$$\gamma(Tx_n, Tx_{n+1}, Tx_{n+1}) \geqslant 1, \quad \forall n = 0, 1, 2, \cdots. \tag{3.18}$$

If $x_{n_0} = x_{n_0+1}$ for some n_0, then $x^* = x_{n_0}$ is a fixed point of T and so we have finished the proof. For this, we assume that $S(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \geqslant 1$.

Now, from (3.17) and (3.18), we have that

$$\begin{aligned}\psi(S(x_{n+1}, x_{n+2}, x_{n+2})) &\leqslant \psi(\gamma(x_n, x_{n+1}, x_{n+1})S(Tx_n, Tx_{n+1}, Tx_{n+1}))\\ &\leqslant \psi(M_1(x_n, x_{n+1})) - \varphi(M_1(x_n, x_{n+1})), \quad \forall n \geqslant 1.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}M_1(x_n, x_{n+1}) &= \max\{S(x_n, x_{n+1}, x_{n+1}), S(x_n, Tx_n, Tx_n), S(x_{n+1}, Tx_{n+1}, Tx_{n+1})\}\\ &= \max\{S(x_n, x_{n+1}, x_{n+1}), S(x_{n+1}, x_{n+2}, x_{n+2})\}.\end{aligned}$$

Now, if $M_1(x_n, x_{n+1}) = S(x_{n+1}, x_{n+2}, x_{n+2})$, from above inequality, for all $n \in \mathbb{N}$, we deduce that

$$\psi(S(x_{n+1}, x_{n+2}, x_{n+2})) \leqslant \psi(S(x_{n+1}, x_{n+2}, x_{n+2})) - \varphi(S(x_{n+1}, x_{n+2}, x_{n+2})) < \psi(S(x_{n+1}, x_{n+2}, x_{n+2}))$$

and $S(x_{n+1}, x_{n+2}, x_{n+2}) = 0$ for all $n \in \mathbb{N}$, which is a contradiction. Thus,

$$M_1(x_n, x_{n+1}) = S(x_n, x_{n+1}, x_{n+1}) > 0$$

for all $n \in \mathbb{N}$, we get that

$$\psi(S(x_{n+1}, x_{n+2}, x_{n+2})) \leqslant \psi(S(x_n, x_{n+1}, x_{n+1})) - \varphi(S(x_n, x_{n+1}, x_{n+1})) < \psi(S(x_n, x_{n+1}, x_{n+1})).$$

Since ψ is nondecreasing, by induction, we have that

$$\psi(S(x_n, x_{n+1}, x_{n+1})) \leqslant \psi^n(S(x_0, x_1, x_1)), \quad \forall n \geqslant 0. \tag{3.19}$$

Letting $n \to \infty$ in (3.19) and by (1), (3) of Lemma 2.2, we have

$$\lim_{n\to\infty} \psi(S(x_n, x_{n+1}, x_{n+1})) \leqslant \lim_{n\to\infty} \psi^n(S(x_0, x_1, x_1)) = 0,$$

which implies that

$$\lim_{n\to\infty} S(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.20}$$

Next, we will prove $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which $n_k > m_k \geqslant k$,

$$S(x_{m_k}, x_{n_k}, x_{n_k}) \geqslant \epsilon.$$

This means that

$$S(x_{m_k}, x_{n_k-1}, x_{n_k-1}) < \epsilon.$$

From (3.19), (3.20) and $(S2)$, we obtain that

$$\epsilon \leqslant S(x_{m_k}, x_{n_k}, x_{n_k}) \leqslant 2S(x_{m_k}, x_{n_k-1}, x_{n_k-1}) + S(x_{n_k-1}, x_{n_k}, x_{n_k}) < 2\epsilon + S(x_{n_k-1}, x_{n_k}, x_{n_k}).$$

Letting $k \to \infty$ in above inequality, we obtain that

$$\epsilon \leqslant \lim_{k\to\infty} S(x_{m_k}, x_{n_k}, x_{n_k}) = \epsilon^+ < 2\epsilon. \tag{3.21}$$

From (3.17) and Lemma 3.11, with $x = x_{m_k}$, $y = x_{n_k}$, we get that

$$\begin{aligned}\psi(S(Tx_{m_k}, Tx_{n_k}, Tx_{n_k})) &\leqslant \psi(\gamma(x_{m_k}, x_{n_k}, x_{n_k})S(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}))\\ &\leqslant \psi(M_1(x_{m_k}, x_{n_k})) - \varphi(M_1(x_{m_k}, x_{n_k}))\\ &< \psi(M_1(x_{m_k}, x_{n_k})),\end{aligned}$$

where $M_1(x_{m_k}, x_{n_k}) = \max\{S(x_{m_k}, x_{n_k}, x_{n_k}), S(x_{m_k}, x_{m_k+1}, x_{m_k+1}), S(x_{n_k}, x_{n_k+1}, x_{n_k+1})\}$. Since ψ is nondecreasing, we have that

$$S(x_{m_k+1}, x_{n_k+1}, x_{n_k+1}) < M_1(x_{m_k}, x_{n_k}).$$

Letting $k \to \infty$ in above inequality, by (3.20) and (3.21), we get that

$$\epsilon^+ = \lim_{k\to\infty} S(x_{m_k+1}, x_{n_k+1}, x_{n_k+1}) < \lim_{k\to\infty} M_1(x_{m_k}, x_{n_k}) = \epsilon^+,$$

which implies $\epsilon = 0$, a contradiction with $\epsilon > 0$.

Hence, $\{x_n\}$ is a Cauchy sequence. Since (X, S) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Since X is (T, γ)-regular, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x^*, x^*) \geqslant 1$, for all $k \in \mathbb{N}$. If $x^* \neq Tx^*$, applying contractive mapping assumption (3.17) with $x = x_{n_k}$, $y = x^*$, we obtain that

$$\psi(S(Tx_{n_k}, Tx^*, Tx^*)) \leqslant \psi(\gamma(x_{n_k}, x^*, x^*)S(Tx_{n_k}, Tx^*, Tx^*)) \leqslant \psi(M_1(x_{n_k}, x^*)) - \varphi(M_1(x_{n_k}, x^*)),$$

where $M_1(x_{n_k}, x^*) = \max\{S(x_{n_k}, x^*, x^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}), S(x^*, Tx^*, Tx^*)\}$. Now, from

$$S(x_{n_k}, x^*, x^*), S(x_{n_k}, x_{n_k+1}, x_{n_k+1}) \to 0, \quad \text{as} \quad k \to \infty.$$

We deduce that $\lim\limits_{k\to\infty} M_1(x_{n_k}, x^*) = S(x^*, Tx^*, Tx^*)$. On the other hand, we have that

$$\begin{aligned}S(x^*, Tx^*, Tx^*) &\leqslant 2S(x^*, x_{n_k}, x_{n_k}) + S(x_{n_k}, Tx^*, Tx^*)\\ &\leqslant 2S(x^*, x_{n_k}, x_{n_k}) + 2S(x_{n_k}, Tx_{n_k}, Tx_{n_k}) + S(Tx_{n_k}, Tx^*, Tx^*),\end{aligned}$$

which implies

$$S(x^*, Tx^*, Tx^*) \leqslant \liminf_{k\to\infty} S(Tx_{n_k}, Tx^*, Tx^*).$$

Since ψ is nondecreasing, we get that

$$\psi(S(x^*, Tx^*, Tx^*)) \leqslant \psi(\liminf_{k\to\infty} S(Tx_{n_k}, Tx^*, Tx^*)) \leqslant \psi(S(x^*, Tx^*, Tx^*)) - \varphi(S(x^*, Tx^*, Tx^*)),$$

which implies $S(x^*, Tx^*, Tx^*) = 0$, that is $x^* = Tx^*$ and x^* is a fixed point of T. The uniqueness of the fixed point is a direct consequence of the assumptions of (A1) and (A4), so we omit the details. □

Example 3.26. Let $X = [0, \infty)$ and $S(x, y, z) = |x - y| + |x - z|, \forall x, y, z \in X$. Then (X, S) is a complete S-metric space.

Define $T : X \mapsto X$ and $\gamma : X^3 \mapsto [0, \infty)$ as follows:

$$Tx = \begin{cases} kx - (k-1), & k > 1, x > 1,\\ \frac{x}{4}, & x \in [0, 1]. \end{cases} \quad \text{and } \gamma(x, y, z) = \begin{cases} 1, & \text{if } x, y, z \in [0, 1],\\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(t) = t, \forall t \geqslant 0$ and $\varphi(t) = \frac{t}{4}, \forall t \geqslant 0$.

We first verify that the contractive condition of type E holds true, that is (3.17) is satisfied. If $\gamma(x, y, y) = 0$, it obviously follows that (3.17) holds true. If $\gamma(x, y, y) \neq 0$, it follows that $x, y \in [0, 1]$ and $\gamma(x, y, y) = 1$. Then, we have that $S(Tx, Ty, Ty) = S(\frac{x}{4}, \frac{y}{4}, \frac{y}{4}) = \frac{1}{2}|x - y|$, $\psi(S(Tx, Ty, Ty)) = \frac{1}{2}|x - y|$ and $M_1(x, y) = \max\{2|x - y|, \frac{3}{2}|x|, \frac{3}{2}|y|\}$.

Indeed, if $M_1(x, y) = \max\{2|x - y|, \frac{3}{2}|x|, \frac{3}{2}|y|\} = 2|x - y|$, then $\frac{1}{2}|x - y| \leqslant 2|x - y| - \frac{1}{4}2|x - y| = \frac{3}{2}|x - y|$. If $M_1(x, y) = \max\{2|x - y|, \frac{3}{2}|x|, \frac{3}{2}|y|\} = \frac{3}{2}|x|$, then $\frac{1}{2}|x - y| \leqslant \frac{3}{2}|x| - \frac{1}{4}\frac{3}{2}|x| = \frac{9}{8}|x|$. If $M_1(x, y) = \max\{2|x - y|, \frac{3}{2}|x|, \frac{3}{2}|y|\} = \frac{3}{2}|y|$, then $\frac{1}{2}|x - y| \leqslant \frac{3}{2}|y| - \frac{1}{4}\frac{3}{2}|y| = \frac{9}{8}|y|$. Hence, the contractive condition of type E is satisfied. Next, we will show that T is γ-admissible. For this, let $x, y, z \in X$ such that $\gamma(x, y, z) \geqslant 1$ which further implies that $x, y, z \in [0, 1]$. Due to the definition of T and γ, we have that

$$Tx = \frac{x}{4} \in [0, 1], \quad Ty = \frac{y}{4} \in [0, 1], \quad Tz = \frac{z}{4} \in [0, 1].$$

Then, $\gamma(Tx, Ty, Tz) \geqslant 1$. Set $x_0 = 1$, then $\gamma(1, T1, T1) = \gamma(1, \frac{1}{4}, \frac{1}{4}) = 1$. Hence, assumption (A1) is satisfied.

Let $\{x_n\}$ be a (T, γ)-orbital sequence such that $x_n \to x$ as $n \to \infty$. By the definition of γ, we have that $x_n \in [0, 1]$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(x_{n_k}, x, x) \geqslant 1$ for all $k \in \mathbb{N}$. Therefore, (A3) holds true. Hence, all hypotheses except (A4) of Theorem 3.25 are satisfied. In fact $x^* = 0$ and $x^* = 1$ are two fixed points of T.

Next, we prove some common fixed point results for two self-mappings satisfying certain S-γ-ψ-φ-contractive condition.

Now, we first present the new notion of S-γ-ψ-φ-contractive pair mappings as follows:

Definition 3.27. Let (X, S) be an S-metric space and let $f, T : X \mapsto X$ be two given mappings. We say that the pair (f, T) is an S-γ-ψ-φ-contractive pair of mappings of type D' if there exist three functions $\gamma : X^3 \mapsto [0, \infty), \psi \in \Psi$, and $\varphi \in \Phi$ such that for all $x, y \in X$, we have that

$$\gamma(fx, fy, fy)S(Tx, Ty, Ty) \leqslant \psi(M_2(x, y)) - \varphi(M_2(x, y)), \tag{3.22}$$

where $M_2(x, y) = \max\{S(fx, fy, fy), \frac{2S(fx,Tx,Tx)+S(fy,Ty,Ty)}{3}, \frac{2S(fy,Tx,Tx)+S(fx,Ty,Ty)}{3}\}$.

Theorem 3.28. *Let (X, S) be a complete S-metric space and $f, T : X \mapsto X$ be such that $T(X) \subseteq f(X)$. Assume that the pair (f, T) is an S-γ-ψ-φ-contractive pair of mappings of type D' and satisfies the following assumptions:*

(A1) *T is f-γ-admissible;*

(A2) *there exists $x_0 \in X$ such that $\gamma(fx_0, Tx_0, Tx_0) \geqslant 1$;*

(A3) *if $\{fx_n\}$ is a sequence in X such that $\gamma(fx_n, fx_{n+1}, fx_{n+1}) \geqslant 1$, for all $n \in \{0\} \cup \mathbb{N}$ and $fx_n \to fx \in f(X)$ as $n \to \infty$, then there exists a subsequence $\{fx_{n_k}\}$ of $\{fx_n\}$ such that $\gamma(fx_{n_k}, fx, fx) \geqslant 1$ for all $k \in \mathbb{N}$.*

Also suppose that $f(X)$ is closed. Then, T and f have a coincidence point, that is, there exists $x^* \in X$ such that $Tx^* = fx^*$.

Proof. In view of assumption (A2), let $x_0 \in X$ be such that $\gamma(fx_0, Tx_0, Tx_0) \geqslant 1$.

Since $T(X) \subseteq f(X)$, we can choose a point $x_1 \in X$ such that $Tx_0 = fx_1$. Continuing this process having chosen $x_1, x_2, \cdots, x_n$, we choose $x_{n+1} \in X$ such that

$$fx_{n+1} = Tx_n, \quad \forall n = 0, 1, 2, \cdots. \tag{3.23}$$

Since T is f-γ-admissible, we have that

$$\gamma(fx_0, Tx_0, Tx_0) = \gamma(fx_0, fx_1, fx_1) \geqslant 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) = \gamma(fx_1, fx_2, fx_2) \geqslant 1.$$

Using mathematical induction, we get that

$$\gamma(fx_n, fx_{n+1}, fx_{n+1}) \geqslant 1, \quad \forall n = 0, 1, 2 \cdots . \tag{3.24}$$

If $Tx_{n_0} = Tx_{n_0+1}$ for some n_0, then by (3.23), $Tx_{n_0} = fx_{n_0}$, that is T and f have a coincidence point at $x^* = x_{n_0}$, and so we have finished the proof. For this, we assume that $S(Tx_n, Tx_{n+1}, Tx_{n+1}) > 0$, for all $n \in \mathbb{N}$.

Now, from (3.22) and (3.24), we have that

$$\begin{aligned} S(Tx_n, Tx_{n+1}, Tx_{n+1}) &\leqslant \gamma(fx_n, fx_{n+1}, fx_{n+1}) S(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\leqslant \psi(M_2(x_n, x_{n+1})) - \varphi(M_2(x_n, x_{n+1})), \quad \forall n \in \mathbb{N}. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} M_2(x_n, x_{n+1}) &= \max\{S(fx_n, fx_{n+1}, fx_{n+1}), \frac{2S(fx_n, Tx_n, Tx_n) + S(fx_{n+1}, Tx_{n+1}, Tx_{n+1})}{3} \\ &\quad , \frac{2S(fx_{n+1}, Tx_n, Tx_n) + S(fx_n, Tx_{n+1}, Tx_{n+1})}{3}\} \\ &= \max\{S(Tx_{n-1}, Tx_n, Tx_n), \frac{2S(Tx_{n-1}, Tx_n, Tx_n) + S(Tx_n, Tx_{n+1}, Tx_{n+1})}{3} \\ &\quad , \frac{2S(Tx_n, Tx_n, Tx_n) + S(Tx_{n-1}, Tx_{n+1}, Tx_{n+1})}{3}\} \\ &\leqslant \max\{S(Tx_{n-1}, Tx_n, Tx_n), S(Tx_n, Tx_{n+1}, Tx_{n+1})\}. \end{aligned}$$

Hence, we have that

$$S(Tx_n, Tx_{n+1}, Tx_{n+1}) \leqslant \psi(\max\{S(Tx_{n-1}, Tx_n, Tx_n), S(Tx_n, Tx_{n+1}, Tx_{n+1})\}), \quad \forall n \in \mathbb{N}. \tag{3.25}$$

If for some $n \geqslant 1$, we have $S(Tx_{n-1}, Tx_n, Tx_n) \leqslant S(Tx_n, Tx_{n+1}, Tx_{n+1})$, from (3.25), we have that

$$S(Tx_n, Tx_{n+1}, Tx_{n+1}) \leqslant \psi(S(Tx_n, Tx_{n+1}, Tx_{n+1})) < S(Tx_n, Tx_{n+1}, Tx_{n+1}),$$

which is a contradiction. Thus, for all $n \geqslant 1$, we conclude that

$$\max\{S(Tx_{n-1}, Tx_n, Tx_n), S(Tx_n, Tx_{n+1}, Tx_{n+1})\} = S(Tx_{n-1}, Tx_n, Tx_n).$$

So, we have that

$$S(Tx_n, Tx_{n+1}, Tx_{n+1}) \leqslant \psi(S(Tx_{n-1}, Tx_n, Tx_n)).$$

Continuing this process inductively, we obtain that

$$S(Tx_n, Tx_{n+1}, Tx_{n+1}) \leqslant \psi^n(S(Tx_0, Tx_1, Tx_1)), \quad \forall n \geqslant 1. \tag{3.26}$$

From (3.26) and Lemma 2.9, for all $k \geqslant 1$, we have that

$$\begin{aligned} S(Tx_n, Tx_{n+k}, Tx_{n+k}) &\leqslant 2 \sum_{i=n}^{n+k-2} S(Tx_i, Tx_{i+1}, Tx_{i+1}) + S(Tx_{n+k-1}, Tx_{n+k-1}, Tx_{n+k}) \\ &\leqslant 2 \sum_{i=n}^{n+k-2} \psi^i(S(Tx_0, Tx_1, Tx_1)) + \psi^{n+k-1}(S(Tx_0, Tx_1, Tx_1)). \end{aligned}$$

Since $\psi \in \Psi$ and $S(Tx_0, Tx_1, Tx_1) > 0$, by Lemma 2.2, we have that

$$\lim_{n,k\to\infty} S(Tx_n, Tx_{n+k}, Tx_{n+k}) = 0.$$

This implies that $\{Tx_n\}$ is a Cauchy sequence. Since (X, S) is complete, by (3.23) we have $\{Tx_n\} = \{fx_{n+1}\} \subseteq f(X)$ and $f(X)$ is closed, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} fx_{n+1} = fx^*. \tag{3.27}$$

Now, we will show that x^* is a coincidence point of T and f. We assume on contrary that $S(fx^*, Tx^*, Tx^*) > 0$.

By (A3) and (3.27), we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\gamma(fx_{n_k}, fx^*, fx^*) \geqslant 1$, for all $k \in \mathbb{N}$. Then by Lemma 2.9, Lemma 2.10, and (3.22), we have that

$$\begin{aligned} S(fx^*, Tx^*, Tx^*) &\leqslant 2S(fx^*, Tx_{n_k}, Tx_{n_k}) + S(Tx_{n_k}, Tx^*, Tx^*) \\ &\leqslant 2S(fx^*, Tx_{n_k}, Tx_{n_k}) + \gamma(fx_{n_k}, fx^*, fx^*)S(Tx_{n_k}, Tx^*, Tx^*) \\ &\leqslant 2S(fx^*, Tx_{n_k}, Tx_{n_k}) + \psi(M_2(x_{n_k}, x^*)) - \varphi(M_2(x_{n_k}, x^*)) \\ &\leqslant 2S(fx^*, Tx_{n_k}, Tx_{n_k}) + \psi(M_2(x_{n_k}, x^*)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} M_2(x_{n_k}, x^*) = \max\{&S(fx_{n_k}, fx^*, fx^*), \frac{2S(fx_{n_k}, Tx_{n_k}, Tx_{n_k}) + S(fx^*, Tx^*, Tx^*)}{3} \\ &, \frac{2S(fx^*, Tx_{n_k}, Tx_{n_k}) + S(fx_{n_k}, Tx^*, Tx^*)}{3}\}. \end{aligned}$$

From above inequality and equality, we get that

$$\begin{aligned} S(fx^*, Tx^*, Tx^*) &\leqslant 2S(fx^*, Tx_{n_k}, Tx_{n_k}) + \psi(M_2(x_{n_k}, x^*)) \\ &= 2S(fx^*, fx_{n_k+1}, fx_{n_k+1}) + \psi(\max\{S(fx_{n_k}, fx^*, fx^*), \frac{2S(fx_{n_k}, Tx_{n_k}, Tx_{n_k}) + S(fx^*, Tx^*, Tx^*)}{3} \\ &\quad , \frac{2S(fx^*, Tx_{n_k}, Tx_{n_k}) + S(fx_{n_k}, Tx^*, Tx^*)}{3}\}). \end{aligned}$$

Letting $k \to \infty$ in above inequality, it yields that

$$S(fx^*, Tx^*, Tx^*) \leqslant \psi(\frac{S(fx^*, Tx^*, Tx^*)}{3}) < \frac{S(fx^*, Tx^*, Tx^*)}{3},$$

which is a contradiction. Hence, we have that $S(fx^*, Tx^*, Tx^*) = 0$, that is $fx^* = Tx^*$. This shows that x^* is a coincidence point of T and f. □

The next theorem shows that under additional hypotheses we can obtain the existence and uniqueness of a common fixed point.

Theorem 3.29. *In addition to the hypotheses of Theorem 3.28, suppose that for all $u, v \in C(f, T)$, where $C(f, T)$ denotes the set of coincidence points of T and f, there exists $w \in X$ such that $\gamma(fu, fw, fw) \geqslant 1$ and $\gamma(fv, fw, fw) \geqslant 1$, and f and T commute at their coincidence points. Then T and f have a unique common fixed point.*

Proof. Let $u, v \in C(f, T)$. By hypotheses, then, there exists $w \in X$ such that

$$\gamma(fu, fw, fw) \geqslant 1 \quad \text{and} \quad \gamma(fv, fw, fw) \geqslant 1. \tag{3.28}$$

According to the fact $T(X) \subseteq f(X)$, define the sequence $\{w_n\}$ in X by $fw_{n+1} = Tw_n$ for all $n \geqslant 0$ and $w_0 = w$.

Since T is f-γ-admissible, we have from (3.28) that

$$\gamma(fu, fw_n, fw_n) \geqslant 1 \quad \text{and} \quad \gamma(fv, fw_n, fw_n) \geqslant 1, \quad \forall n \geqslant 0.$$

Applying (3.22) and (3.28), we obtain that for all $n \geqslant 0$,

$$\begin{aligned} S(fu, fw_{n+1}, fw_{n+1}) &= S(Tu, Tw_n, Tw_n) \\ &\leqslant \gamma(fu, fw_n, fw_n) S(Tu, Tw_n, Tw_n) \\ &\leqslant \psi(M_2(u, w_n)) - \varphi(M_2(u, w_n)) \\ &\leqslant \psi(M_2(u, w_n)). \end{aligned}$$

On the other hand, we have that for all $n \geqslant 0$,

$$\begin{aligned} M_2(u, w_n) &= \max\{S(fu, fw_n, fw_n), \frac{2S(fu, Tu, Tu) + S(fw_n, Tw_n, Tw_n)}{3} \\ &\quad , \frac{2S(fw_n, Tu, Tu) + S(fu, Tw_n, Tw_n)}{3}\} \\ &\leqslant \max\{S(fu, fw_n, fw_n), S(fu, Tw_n, Tw_n)\} \\ &= \max\{S(fu, fw_n, fw_n), S(fu, fw_{n+1}, fw_{n+1})\}. \end{aligned}$$

Using above two inequalities, owing to the monotone property of ψ, we obtain that

$$S(fu, fw_{n+1}, fw_{n+1}) \leqslant \psi(\max\{S(fu, fw_n, fw_n), S(fu, fw_{n+1}, fw_{n+1})\}), \quad \forall n \geqslant 0. \tag{3.29}$$

If $\max\{S(fu, fw_n, fw_n), S(fu, fw_{n+1}, fw_{n+1})\} = S(fu, fw_{n+1}, fw_{n+1})$, we have from (3.29) and Lemma 2.2 that

$$S(fu, fw_{n+1}, fw_{n+1}) \leqslant \psi(S(fu, fw_{n+1}, fw_{n+1})) < S(fu, fw_{n+1}, fw_{n+1}),$$

which is a contradiction. Hence,

$$\max\{S(fu, fw_n, fw_n), S(fu, fw_{n+1}, fw_{n+1})\} = S(fu, fw_n, fw_n),$$

and

$$S(fu, fw_{n+1}, fw_{n+1}) \leqslant \psi(S(fu, fw_n, fw_n)), \forall n \geqslant 0.$$

This implies that

$$S(fu, fw_{n+1}, fw_{n+1}) \leqslant \psi^n(S(fu, fw, fw)), \forall n \geqslant 0.$$

Letting $n \to \infty$ in above inequality, we can infer that

$$\lim_{n\to\infty} S(fu, fw_n, fw_n) = 0. \tag{3.30}$$

Similarly, we also have that

$$\lim_{n\to\infty} S(fv, fw_n, fw_n) = 0. \tag{3.31}$$

It follows from (3.30) and (3.31) that $fu = fv$. Next, we prove the existence of a common fixed point.

Let $u \in C(f, T)$, that is $fu = Tu$. Owing to the commutativity of f and T at their coincidence points, we have that

$$f^2u = fTu = Tfu. \tag{3.32}$$

Let us denote $fu = x^*$, then from (3.32), $fx^* = Tx^*$. Thus, x^* is a coincidence of point of f and T. Then we have that $fu = fx^* = x^* = Tx^*$. Hence, x^* is a common fixed point of f and T. Suppose that x' is another common fixed point of f and T. Then $x' \in C(f, T)$, so we have $x' = fx' = fx^* = x^*$. □

In what follows, we present an illustrative example to demonstrate Theorem 3.29 on the existence and uniqueness of a common fixed point.

Example 3.30. Let $X = [0, \infty)$ equipped with S-metric $S(x, y, z) = |x - y| + |x - z|$ for all $x, y, z \in X$. Define the mappings $T : X \mapsto X$ and $f : X \mapsto X$ by

$$T(x) = \begin{cases} 2x - \frac{3}{2}, & \text{if } x > 2; \\ \frac{x}{5}, & \text{if } 0 \leqslant x \leqslant 2. \end{cases} \quad \text{and } f(x) = \frac{x}{2}, \forall x \in X.$$

Now, we also define the mapping $\gamma : X^3 \mapsto [0, \infty)$ by

$$\gamma(x, y, z) = \begin{cases} 1, & \text{if } x, y, z \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the pair (T, f) is an S-γ-ψ-φ-contractive pair of mappings of type D' with $\psi(t) = \frac{4}{5}t, \forall t \geqslant 0$, $\varphi(t) = \frac{t}{5}, \forall t \geqslant 0$. In fact, for all $x, y \in [0, 1]$,

$$\gamma(fx, fy, fy)S(Tx, Ty, Ty) = 1 \cdot 2|\frac{x}{5} - \frac{y}{5}| = \frac{2}{5}S(fx, fy, fy) < \frac{3}{5}S(fx, fy, fy).$$

It follows from above that

$$\gamma(fx, fy, fy)S(Tx, Ty, Ty) \leqslant \psi(M_2(x, y)) - \varphi(M_2(x, y)), \quad \forall x, y \in X.$$

Moreover, there exists $x_0 \in X$ such that $\gamma(fx_0, Tx_0, Tx_0) \geqslant 1$. Indeed, for $x_0 = 1$, we have $\gamma(\frac{1}{2}, \frac{1}{5}, \frac{1}{5}) = 1$.

Next, we will show that T is f-γ-admissible. For this, let $x, y, z \in X$ such that $\gamma(fx, fy, fz) \geqslant 1$. This implies that $fx, fy, fz \in [0, 1]$ and by the definition of f, we have $x, y, z \in [0, 2]$. Therefore, by the definition of T and f, we have

$$Tx = \frac{x}{5} \in [0, 1], \quad Ty = \frac{y}{5} \in [0, 1], \quad Tz = \frac{z}{5} \in [0, 1], \quad \gamma(Tx, Ty, Tz) = 1.$$

Thus, T is f-γ-admissible. Clearly $T(X) \subseteq f(X)$ and $f(X)$ is closed.

At last, let $\{fx_n\}$ be a sequence in X such that $\gamma(fx_n, fx_{n+1}, fx_{n+1}) \geqslant 1$ for all $n \in \{0\} \cup \mathbb{N}$ and $fx_n \to fx \in f(X)$ as $n \to \infty$. Since $\gamma(fx_n, fx_{n+1}, fx_{n+1}) \geqslant 1$ for all $n \in \{0\} \cup \mathbb{N}$, by the definition of γ, we have $fx_n \in [0, 1]$ for all $n \in \{0\} \cup \mathbb{N}$ and $fx \in [0, 1]$. Then, $\gamma(fx_n, fx, fx) \geqslant 1$. Hence all the hypotheses of Theorem 3.28 are satisfied. Consequently, f and T have a coincidence point. Furthermore, all the hypotheses of Theorem 3.29 are also satisfied, here 0 is the unique common fixed point of f and T.

Definition 3.31. Let (X, S) be an S-metric space and let $f, T : X \mapsto X$ be two given mappings. We say that the pair (f, T) is an S-γ-ψ-φ-contractive pair of mappings of type E' if there exist three functions $\gamma : X^3 \mapsto [0, \infty), \psi \in \Psi$, and $\varphi \in \Phi$ such that for all $x, y \in X$, we have that

$$\psi(\gamma(fx, fy, fy)S(Tx, Ty, Ty)) \leqslant \psi(M_3(x, y)) - \varphi(M_3(x, y)), \tag{3.33}$$

where $M_3(x, y) = \max\{S(fx, fy, fy), S(fx, Tx, Tx), S(fy, Ty, Ty)\}$.

Theorem 3.32. *Let (X, S) be a complete S-metric space and $f, T : X \mapsto X$ be such that $T(X) \subseteq f(X)$. Assume that the pair (f, T) is an S-γ-ψ-φ-contractive pair of mappings of type E' and satisfies the following assumptions:*

(A1) *T is f-γ-admissible and transitive;*

(A2) *there exists $x_0 \in X$ such that $\gamma(fx_0, Tx_0, Tx_0) \geqslant 1$;*

(A3) *(X, S) is (T, γ)-regular;*

(A4) *either* $\gamma(fu, fv, fv) \geqslant 1$ *or* $\gamma(fv, fu, fu) \geqslant 1$, *whenever* $fu = Tu$ *and* $fv = Tv$.

Then, T *and* f *have a unique coincidence point in* X*. Moreover, if* f *and* T *commute at their coincidence points, then* f *and* T *have a unique common fixed point.*

Proof. In view of assumption (A2), let $x_0 \in X$ such that $\gamma(fx_0, Tx_0, Tx_0) \geqslant 1$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_{n+1} = Tx_n, \quad \forall n \geqslant 0.$$

Moreover, we assume that if $y_n = Tx_n = Tx_{n+k} = y_{n+k}$, then we choose $x_{n+k+1} = x_{n+1}$. The proof is completed, since $T(X) \subseteq f(X)$. In particular, if $y_n = y_{n+1}$, then x_{n+1} is a point of coincidence of T and f. So we can suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$.

By assumption (A2), we have $\gamma(fx_0, Tx_0, Tx_0) = \gamma(fx_0, fx_1, fx_1) \geqslant 1$. Since T is f-γ-admissible, we have that

$$\gamma(fx_0, Tx_0, Tx_0) = \gamma(fx_0, fx_1, fx_1) \geqslant 1 \Rightarrow \gamma(Tx_0, Tx_1, Tx_1) = \gamma(fx_1, fx_2, fx_2) \geqslant 1.$$

By induction, we get that

$$\gamma(fx_n, fx_{n+1}, fx_{n+1}) \geqslant 1, \quad \forall n = 0, 1, 2, \cdots . \tag{3.34}$$

Now, from (3.33) and (3.34), we have that

$$\begin{aligned} \psi(S(fx_{n+1}, fx_{n+2}, fx_{n+2})) &\leqslant \psi(\gamma(fx_n, fx_{n+1}, fx_{n+1})S(Tx_n, Tx_{n+1}, Tx_{n+1})) \\ &\leqslant \psi(M_3(x_n, x_{n+1})) - \varphi(M_3(x_n, x_{n+1})), \quad \forall n \geqslant 1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} M_3(x_n, x_{n+1}) &= \max\{S(fx_n, fx_{n+1}, fx_{n+1}), S(fx_n, Tx_n, Tx_n), S(fx_{n+1}, Tx_{n+1}, Tx_{n+1})\} \\ &= \max\{S(y_{n-1}, y_n, y_n), S(y_n, y_{n+1}, y_{n+1})\}. \end{aligned}$$

Now, if $M_3(x_n, x_{n+1}) = S(y_n, y_{n+1}, y_{n+1})$, from above inequality, for all $n \in \mathbb{N}$, we deduce that

$$\psi(S(fx_{n+1}, fx_{n+2}, fx_{n+2})) \leqslant \psi(S(y_n, y_{n+1}, y_{n+1})) - \varphi(S(y_n, y_{n+1}, y_{n+1})) < \psi(S(y_n, y_{n+1}, y_{n+1}))$$

and $S(y_n, y_{n+1}, y_{n+1}) = 0$ for all $n \in \mathbb{N}$, which is a contradiction. Thus,

$$M_3(x_n, x_{n+1}) = S(y_{n-1}, y_n, y_n) > 0$$

for all $n \in \mathbb{N}$, we get that

$$\psi(S(y_n, y_{n+1}, y_{n+1})) \leqslant \psi(S(y_{n-1}, y_n, y_n)) - \varphi(S(y_{n-1}, y_n, y_n)) < \psi(S(y_{n-1}, y_n, y_n)).$$

Since ψ is nondecreasing, by induction, we have that

$$\psi(S(y_n, y_{n+1}, y_{n+1})) \leqslant \psi^n(S(y_0, y_1, y_1)), \quad \forall n \geqslant 0. \tag{3.35}$$

Letting $n \to \infty$ in (3.35) and by (1), (3) of Lemma 2.2, we have

$$\lim_{n\to\infty} \psi(S(y_n, y_{n+1}, y_{n+1})) \leqslant \lim_{n\to\infty} \psi^n(S(y_0, y_1, y_1)) = 0,$$

which implies that

$$\lim_{n\to\infty} S(y_n, y_{n+1}, y_{n+1}) = 0. \tag{3.36}$$

Next, we will prove $\{y_n\}$ is a Cauchy sequence. Suppose to the contrary, that is, $\{y_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that n_k is the smallest index for which $n_k > m_k \geqslant k$,

$$S(y_{m_k}, y_{n_k}, y_{n_k}) \geqslant \epsilon.$$

This means that

$$S(y_{m_k}, y_{n_k-1}, y_{n_k-1}) < \epsilon.$$

From (3.35), (3.36), and (S2), we obtain that

$$\epsilon \leqslant S(y_{m_k}, y_{n_k}, y_{n_k}) \leqslant 2S(y_{m_k}, y_{n_k-1}, y_{n_k-1}) + S(y_{n_k-1}, y_{n_k}, y_{n_k}) < 2\epsilon + S(y_{n_k-1}, y_{n_k}, y_{n_k}).$$

Letting $k \to \infty$ in above inequality, we obtain that

$$\epsilon \leqslant \lim_{k\to\infty} S(y_{m_k}, y_{n_k}, y_{n_k}) = \epsilon^+ < 2\epsilon. \tag{3.37}$$

From (3.33) and Lemma 3.11, with $x = x_{m_k}$, $y = x_{n_k}$, we get that

$$\begin{aligned}\psi(S(Tx_{m_k}, Tx_{n_k}, Tx_{n_k})) &\leqslant \psi(\gamma(fx_{m_k}, fx_{n_k}, fx_{n_k})S(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}))\\ &\leqslant \psi(M_3(x_{m_k}, x_{n_k})) - \varphi(M_3(x_{m_k}, x_{n_k}))\\ &< \psi(M_3(x_{m_k}, x_{n_k})),\end{aligned}$$

where $M_3(x_{m_k}, x_{n_k}) = \max\{S(fx_{m_k}, fx_{n_k}, fx_{n_k}), S(fx_{m_k}, Tx_{m_k}, Tx_{m_k}), S(fx_{n_k}, Tx_{n_k}, Tx_{n_k})\}$. Since ψ is nondecreasing, we have that

$$S(Tx_{m_k+1}, Tx_{n_k}, Tx_{n_k}) < M_3(x_{m_k}, x_{n_k}).$$

Letting $k \to \infty$ in above inequality, by (3.36) and (3.37), we get that

$$\epsilon^+ = \lim_{k\to\infty} S(y_{m_k}, y_{n_k}, y_{n_k}) < \lim_{k\to\infty} M_3(x_{m_k}, x_{n_k}) = \epsilon^+,$$

which implies $\epsilon = 0$, a contradiction with $\epsilon > 0$. Hence, $\{y_n\}$ is a Cauchy sequence. Since (X, S) is complete, there exists $z \in X$ such that $y_n \to z$. Let $y' \in X$ be such that $fy' = z$. Since X is (T, γ)-regular, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\gamma(y_{n_k}, fy', fy') \geqslant 1$ for all $k \in \mathbb{N}$. If $fy' \neq Ty'$, applying contractive mapping assumption (3.33) with $x = x_{n_k}$, $y = y'$, we obtain that

$$\psi(S(Tx_{n_k}, Ty', Ty')) \leqslant \psi(\gamma(fx_{n_k}, fy', fy')S(Tx_{n_k}, Ty', Ty')) \leqslant \psi(M_3(x_{n_k}, y')) - \varphi(M_3(x_{n_k}, y')),$$

where $M_3(x_{n_k}, y') = \max\{S(fx_{n_k}, fy', fy'), S(fx_{n_k}, Tx_{n_k}, Tx_{n_k}), S(fy', Ty', Ty')\}$.

Now, from

$$S(fx_{n_k}, fy', fy'), S(fx_{n_k}, Tx_{n_k}, Tx_{n_k}) \to 0, \quad \text{as} \quad k \to \infty,$$

we deduce that $\lim_{k\to\infty} M_3(x_{n_k}, y') = S(fy', Ty', Ty')$. On the other hand, we have that

$$\begin{aligned}S(fy', Ty', Ty') &\leqslant 2S(y', fx_{n_k-1}, fx_{n_k-1}) + S(fx_{n_k-1}, Ty', Ty')\\ &\leqslant 2S(fy', fx_{n_k-1}, fx_{n_k-1}) + 2S(fx_{n_k-1}, Tx_{n_k}, Tx_{n_k}) + S(Tx_{n_k}, Ty', Ty'),\end{aligned}$$

which implies

$$S(fy', Ty', Ty') \leqslant \liminf_{k\to\infty} S(Tx_{n_k}, Ty', Ty').$$

Since ψ is nondecreasing, we get that

$$\psi(S(fy', Ty', Ty')) \leqslant \psi(\liminf_{k\to\infty} S(Tx_{n_k}, Ty', Ty')) \leqslant \psi(S(fy', Ty', Ty')) - \varphi(S(fy', Ty', Ty')),$$

which implies $S(fy', Ty', Ty') = 0$, that is $fy' = Ty'$ and y' is a coincidence point of T and f.

The uniqueness of the coincidence point is a direct consequence of the assumptions of (A1) and (A4).

Now, if z is the point of coincidence of f and T, since T and f commute at their coincidence points, so $z = fz = Tz$. Consequently, z is the unique common fixed point of T and f. □

Example 3.33. Let $X = [0, \infty)$ equipped with S-metric $S(x, y, z) = |x - y| + |x - z|$ for all $x, y, z \in X$. Define the mappings $T, f : X \mapsto X$ by

$$T(x) = \frac{x}{5}, \forall x \in X \text{ and } f(x) = \frac{x}{3}, \forall x \in X.$$

Now, we also define the mapping $\gamma : X^3 \mapsto [0, \infty)$ by

$$\gamma(x, y, z) = \begin{cases} 1, & \text{if } x, y, z \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the pair (T, f) is an S-γ-ψ-φ-contractive pair of mappings of type E' with $\psi(t) = \frac{4}{5}t$, $\varphi(t) = \frac{t}{5}, \forall t \geqslant 0$. In fact, for all $x, y \in [0, 3]$,

$$\begin{aligned}
&\gamma(fx, fy, fz) = \gamma(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}) = 1, \\
&S(Tx, Ty, Ty) = 2|Tx - Ty| = 2|\frac{x}{5} - \frac{y}{5}| = \frac{2}{5}|x - y|, \\
&\psi(\gamma(fx, fy, fy)S(Tx, Ty, Ty)) = \psi(S(Tx, Ty, Ty)) = \frac{4}{5} \cdot \frac{2}{5}|x - y| = \frac{8}{25}|x - y|, \\
&M_3(x, y) = \max\{S(fx, fy, fy), S(fx, Tx, Tx), S(fy, Ty, Ty)\} = \max\{\frac{2}{3}|x - y|, \frac{4}{15}x, \frac{4}{15}y\}.
\end{aligned}$$

(1) If $0 \leqslant x < \frac{3}{5}y$, then, $M_3(x, y) = \max\{\frac{2}{3}(y - x), \frac{4}{15}x, \frac{4}{15}y\} = \frac{2}{3}(y - x)$.

(2) If $\frac{3}{5}y \leqslant x < y$, then, $M_3(x, y) = \max\{\frac{2}{3}(y - x), \frac{4}{15}x, \frac{4}{15}y\} = \frac{4}{15}y$.

(3) If $y \leqslant x < \frac{5}{3}y$, then, $M_3(x, y) = \max\{\frac{2}{3}(y - x), \frac{4}{15}x, \frac{4}{15}y\} = \frac{4}{15}x$.

(4) If $\frac{5}{3}y \leqslant x$, then, $M_3(x, y) = \max\{\frac{2}{3}(y - x), \frac{4}{15}x, \frac{4}{15}y\} = \frac{2}{3}(x - y)$.

From the discussion above, we have that

(1) $0 \leqslant x < \frac{3}{5}y$, $\frac{8}{25}|x - y| = \frac{8}{25}(y - x) < \frac{3}{5} \cdot \frac{2}{3}(y - x)$;

(2) $\frac{3}{5}y \leqslant x < y$, $\frac{8}{25}|x - y| = \frac{8}{25}(y - x) < \frac{3}{5} \cdot \frac{4}{15}y = \frac{4}{25}y$;

(3) $y \leqslant x < \frac{5}{3}y$, $\frac{8}{25}|x - y| = \frac{8}{25}(x - y) < \frac{3}{5} \cdot \frac{4}{15}x = \frac{4}{25}x$;

(4) $\frac{5}{3}y \leqslant x$, $\frac{8}{25}|x - y| = \frac{8}{25}(x - y) < \frac{3}{5} \cdot \frac{2}{3}(x - y) = \frac{10}{25}(x - y)$.

It follows from above that

$$\psi(\gamma(fx, fy, fy)S(Tx, Ty, Ty)) \leqslant \psi(M_3(x, y)) - \varphi(M_3(x, y)), \quad \forall x, y \in X.$$

Moreover, there exists $x_0 \in X$ such that $\gamma(fx_0, Tx_0, Tx_0) \geqslant 1$. Indeed, for $x_0 = 1$, we have $\gamma(f1, T1, T1) = \gamma(\frac{1}{3}, \frac{1}{5}, \frac{1}{5}) = 1$.

Next, we will show that T is f-γ-admissible. For this, let $x, y, z \in X$ such that $\gamma(fx, fy, fz) \geqslant 1$. This implies that $fx, fy, fz \in [0, 1]$ and by the definition of f, we have $x, y, z \in [0, 3]$. Therefore, by the definition of T and f, we have

$$Tx = \frac{x}{5} \in [0, 1], \quad Ty = \frac{y}{5} \in [0, 1], \quad Tz = \frac{z}{5} \in [0, 1], \quad \gamma(Tx, Ty, Tz) = 1.$$

Thus, T is f-γ-admissible. Clearly $T(X) \subseteq f(X)$ and $f(X)$ is closed.

At last, let $\{fx_n\}$ be a sequence in X such that $\gamma(fx_n, fx_{n+1}, fx_{n+1}) \geqslant 1$ for all n and $fx_n \to fx \in f(X)$, as $n \to \infty$. Since $\gamma(fx_n, fx_{n+1}, fx_{n+1}) \geqslant 1$ for all $n \in \{0\} \cup \mathbb{N}$, by the definition of γ, we have $fx_n \in [0, 1]$ for all $n \in \{0\} \cup \mathbb{N}$ and $fx \in [0, 1]$. Then $\gamma(fx_n, fx, fx) \geqslant 1$.

Hence all the hypotheses of Theorem 3.32 are satisfied. Consequently, f and T have a unique coincidence point $x = 0$. Furthermore, here 0 is the unique common fixed point of f and T.

In the past several decades, there have been enormous results in the study of fixed point problems of contractive mappings in metric spaces endowed with a partial order. The first famous result in this direction was given by Turinici [27], where he extended the BCP in partially ordered set. After then, Ran and Reurings [20] generalized Turinici's theorem to matrix equations. Moreover, many interesting and useful results have been obtained relating to the existence of a fixed point for contraction type mappings in partially ordered metric spaces (see, e.g., [1, 9, 13, 16–19, 21, 22]). In what follows, we will present some fixed and common point results on an S-metric space endowed with a partial order. For this, we require the following concepts.

Definition 3.34. Let $(X, \preceq)$ be a partially ordered set and $T : X \mapsto X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if $x, y \in X, x \preceq y$ then $Tx \preceq Ty$.

Definition 3.35. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$.

Definition 3.36. Let $(X, \preceq)$ be a partially ordered set and S be an S-metric on X. We say that $(X, \preceq, S)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$.

Definition 3.37. Let $(X, \preceq)$ be a partially ordered set and S be an S-metric on X. We say that $(X, \preceq, S)$ is f-regular where $f : X \mapsto X$, if for every nondecreasing sequence $\{fx_n\} \subset X$ such that $fx_n \to fx \in X$ as $n \to \infty$, there exists a subsequence $\{fx_{n_k}\}$ of $\{fx_n\}$ such that $fx_{n_k} \preceq fx$ for all $k \in \mathbb{N}$.

Definition 3.38. Let $(X, \preceq)$ be a partially ordered set and $T, f : X \mapsto X$ be two given mappings. We say that T is f-nondecreasing with respect to $\preceq$ if $x, y \in X, fx \preceq fy$ then $Tx \preceq Ty$. In particular, if we choose $f = I_x$, where I_x denotes the identity mapping, then we can get that f-nondecreasing reduces to nondecreasing.

Theorem 3.39. *Let $(X, \preceq)$ be a partially ordered set and S be an S-metric on X such that (X, S) is complete. Let $T, f : X \mapsto X$ be such that $T(X) \subseteq f(X)$ and T be a f-nondecreasing mapping w.r.t. $\preceq$. Suppose that there exist two functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$S(Tx, Ty, Ty) \leqslant \psi(M_2(x, y)) - \varphi(M_2(x, y)),$$

where $M_2(x, y) = \max\{S(fx, fy, fy), \frac{2S(fx,Tx,Tx)+S(fy,Ty,Ty)}{3}, \frac{2S(fy,Tx,Tx)+S(fx,Ty,Ty)}{3}\}$ for all $x, y \in X$ with $fx \preceq fy$. Suppose that the following assumptions hold:

(A1) *there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$;*

(A2) *$(X, \preceq, S)$ is f-regular.*

Also suppose that $f(X)$ is closed. Then, T and f have a coincidence point in X. Moreover, if every pair $(u, v) \in C(f, T) \times C(f, T)$, there exists $w \in X$ such that $fu \preceq fw$ and $fu \preceq fw$, and f and T commute at their coincidence points, then f and T have a unique common fixed point.

Proof. Define the mapping $\gamma : X^3 \mapsto [0, \infty)$ by

$$\gamma(x, y, y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily to check that the pair (T, f) is an S-γ-ψ-φ-contractive pair of mappings of type D', that is

$$\gamma(fx, fy, fy)S(Tx, Ty, Ty) \leqslant \psi(M_2(x, y)) - \varphi(M_2(x, y))$$

for all $x, y \in X$.

Notice that in view of (A1), we have $\gamma(fx_0, Tx_0, Tx_0) \geqslant 1$. Moreover, for all $x, y \in X$, from the f-monotone property of T, we have

$$\gamma(fx, fy, fy) \geqslant 1 \Rightarrow fx \preceq fy \Rightarrow Tx \preceq Ty \Rightarrow \gamma(Tx, Ty, Ty) \geqslant 1,$$

which implies that T is f-γ-admissible.

Now, let $\{fx_n\}$ be a sequence in X such that $\gamma(fx_n, fx_{n+1}, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $fx_n \to fx \in X$ as $n \to \infty$. From the f-regularity, there exists a subsequence $\{fx_{n_k}\}$ of $\{fx_n\}$ such that $fx_{n_k} \preceq fx$ for all $k \in \mathbb{N}$. Then, by the definition of γ, we have that $\gamma(fx_{n_k}, fx, fx) \geqslant 1$. Hence, all the hypotheses of Theorem 3.28 are satisfied. Therefore, we deduce that f and T have a coincidence point x^*, that is $fx^* = Tx^*$.

Next, we need to show the existence and uniqueness of common fixed point. For this purpose, let $u, v \in C(f, T)$. By assumption, there exists $w \in X$ such that $fu \preceq fw$ and $fv \preceq fw$, which accounts to say from the definition of γ that $\gamma(fu, fw, fw) \geqslant 1$ and $\gamma(fv, fw, fw) \geqslant 1$. Thus, we deduce the existence and uniqueness of common fixed point by Theorem 3.29. □

Theorem 3.40. *Let $(X, \preceq)$ be a partially ordered set and S be an S-metric on X such that (X, S) is complete. Let $T, f : X \mapsto X$ be such that $T(X) \subseteq f(X)$ and T be a f-nondecreasing mapping w.r.t. $\preceq$. Suppose that there exist two functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$S(Tx, Ty, Ty) \leqslant \psi(M_3(x, y)) - \varphi(M_3(x, y)),$$

where $M_3(x, y) = \max\{S(fx, fy, fy), S(fx, Tx, Tx), S(fy, Ty, Ty)\}$ for all $x, y \in X$ with $fx \preceq fy$. Suppose that the following assumptions hold:

(A1) *there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$;*

(A2) *if a sequence $\{x_n\} \subset X$ such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;*

(A3) *for all $u, v \in C(f, T)$, then $fu \preceq fv$ or $fu \succeq fv$.*

Then, T and f have a unique coincidence point in X. Moreover, if f and T commute at their coincidence points, then f and T have a unique common fixed point.

Proof. Define the mapping $\gamma : X^3 \mapsto [0, \infty)$ by

$$\gamma(x, y, y) = \begin{cases} 1, & \text{if } x, y \in f(X) \text{ and } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the pair (T, f) is an S-γ-ψ-φ-contractive pair of mappings of type E', that is,

$$\psi(\gamma(fx, fy, fy)S(Tx, Ty, Ty)) \leqslant \psi(M_3(x, y)) - \varphi(M_3(x, y))$$

for all $x, y \in X$.

Notice that in view of (A1), we have $\gamma(fx_0, Tx_0, Tx_0) \geqslant 1$. Moreover, for all $x, y \in X$, from the f-monotone property of T, one can show easily that T is f-γ-admissible.

Now, let $\{x_n\}$ be a sequence in X such that $\gamma(x_n, x_{n+1}, x_{n+1}) \geqslant 1$ and $x_n \to x$ as $n \to \infty$. By the definition of γ, we have that

$$x_n, x_{n+1} \in f(X), \quad x_n \preceq x_{n+1}, \quad \forall n \in \mathbb{N}.$$

Since X is complete, we deduce that $x \in X$. By (A2), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$ and $\gamma(x_{n_k}, x, x) \geqslant 1$ for all $k \in \mathbb{N}$ and so X is γ-regular. Moreover, from the transitive property of partial order, we have that $\gamma(x_m, x_n, x_n) \geqslant 1$ for all $m, n \in \mathbb{N}$ with $m < n$. Hence, the hypotheses (A1)-(A4) of Theorem 3.32 are satisfied. Then, T and f have a unique common fixed point. □

From Theorem 3.39 and Theorem 3.40, if we set $f = I_x$ the identity mapping on X, we deduce the following corollaries on fixed point results on an S-metric space endowed with a partial order.

Corollary 3.41. *Let $(X, \preceq)$ be a partially ordered set and S be an S-metric on X such that (X, S) is complete. Let $T : X \mapsto X$ be a nondecreasing mapping w.r.t. $\preceq$. Suppose that there exist two functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$S(Tx, Ty, Ty) \leqslant \psi(M_2'(x, y)) - \varphi(M_2'(x, y)),$$

where $M_2'(x, y) = \max\{S(x, y, y), \frac{2S(x,Tx,Tx)+S(y,Ty,Ty)}{3}, \frac{2S(y,Tx,Tx)+S(x,Ty,Ty)}{3}\}$ for all $x, y \in X$ with $fx \preceq fy$. Suppose that the following assumptions hold:

(A1) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(A2) *$(X, \preceq, S)$ is regular.*

Then, T has a fixed point in X. Moreover, if $u \preceq v$ or $v \preceq u$, whenever $u = Tu$ and $v = Tv$, then T has a unique common fixed point in X.

Corollary 3.42. *Let $(X, \preceq)$ be a partially ordered set and S be an S-metric on X such that (X, S) is complete. Let $T : X \mapsto X$ be a nondecreasing mapping w.r.t. $\preceq$. Suppose that there exist two functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$S(Tx, Ty, Ty) \leqslant \psi(M_3'(x, y)) - \varphi(M_3'(x, y)),$$

where $M_3'(x, y) = \max\{S(x, y, y), S(x, Tx, Tx), S(y, Ty, Ty)\}$ for all $x, y \in X$ with $x \preceq y$. Suppose that the following assumptions hold:

(A1) *there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$;*

(A2) *if a sequence $\{x_n\} \subset X$ such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;*

(A3) *if $u \preceq v$ or $v \preceq u$, whenever $u = Tu$ and $v = Tv$.*

Then, T has a unique fixed point in X.

Acknowledgment

The first author was supported in part by Natural Science Foundation of Hainan Province (Grant No.114014), Scientific Research Fund of Hainan Province Education Department (Grant No.Hnjg2016ZD-20). The second author was partially supported by National Natural Science Foundation of China (Grant No.61573010), Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (No.2015QZJ01), Artificial Intelligence of Key Laboratory of Sichuan Province (No.2015RZJ01), Scientific Research Fund of Sichuan Provincial Education Department (No.16ZA0256), Science Research Fund of Science and Technology Department of Sichuan Province (N0.2017JY0125), Scientific Research Fund of Sichuan University of Science and Engineering (No.2014RC01 No.2014RC03).

References

[1] R. P. Agarwal, M. A. El-Gebeily, D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal., **87** (2008), 109–116. 3

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1

[3] V. Berinde, *Iterative approximation of fixed points*, Second edition. Lecture Notes in Mathematics, Springer, Berlin, (2007). 2.2

[4] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **29** (2002), 531–536. 1

[5] M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, *Partial metric spaces*, Amer. Math. Monthly, **116** (2009), 708–718. 1

[6] R. Caccioppoli, *Un teorema generale sullesistenza di elementi uniti in una trasformazione funzionale*, Rend. Accad. Lincei, **11** (1930), 794–799. 1

[7] N. V. Dung, *On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces*, Fixed Point Theory Appl., **2013** (2013), 17 pages. 1

[8] S. Gähler, *Über die Uniformisierbarkeit 2-metrischer Räume*, (German) Math. Nachr., **28** (1964/1965), 235–244. 1

[9] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393. 1, 3

[10] L.-G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332** (2007), 1468–1476. 1

[11] R. Kannan, *Some results on fixed points*, II, Amer. Math. Monthly, **76** (1969), 405–408. 1

[12] E. Karapınar, B. Samet, *Generalized α-ψ contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal., **2012** (2012), 17 pages. 1, 2

[13] V. Lakshmikantham, L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70** (2009), 4341–4349. 1, 3

[14] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297. 1

[15] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–239. 1

[16] J. J. Nieto, R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin. (Engl. Ser.), **23** (2007), 2205–2212. 3

[17] S. Radenović, *Bhaskar-Lakshmikantham type results for monotone mappings in partially ordered metric spaces*, Int. J. Nonlinear Anal. Appl., **5** (2014), 96–103.

[18] S. Radenović, *Coupled fixed point theorems for monotone mappings in partially ordered metric spaces*, Kragujevac J. Math., **38** (2014), 249–257.

[19] S. Radenović, *Some coupled coincidence points results of monotone mappings in partially ordered metric spaces*, Int. J. Nonlinear Anal. Appl., **5** (2014), 174–184. 3

[20] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443. 3

[21] R. Saadati, S. M. Vaezpour, *Monotone generalized weak contractions in partially ordered metric spaces*, Fixed Point Theory, **11** (2010), 375–382. 1, 3

[22] B. Samet, *Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal., **2010** (2010), 4508–4517. 3

[23] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for $\alpha\psi$-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1, 2.1, 2, 2, 2

[24] S. Sedghi, N. V. Dung, *Fixed point theorems on S-metric spaces*, Mat. Vesnik, **66** (2014), 113–124. 1

[25] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik, **64** (2012), 258–266. 1, 2, 2.9, 2.10, 2.12, 2.13

[26] S. Sedghi, N. Shobe, H.-Y. Zhou, *A common fixed point theorem in D^*-metric spaces*, Fixed Point Theory Appl., **2007** (2007), 13 pages. 1

[27] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl., **117** (1986), 100–127. 3

[28] M. Zhou, X.-L. Liu, *On coupled common fixed point theorems for nonlinear contractions with the mixed weakly monotone property in partially ordered S-metric spaces*, J. Funct. Spaces, **2016** (2016), 9 pages. 1