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$\mathcal{F}$-sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity between dynamical systems and their induced hyperspace dynamical systems

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Abstract

The notions of F-sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity were introduced and studied by Wang et al. via Furstenberg families in [H.-Y. Wang, J.-C. Xiong, F. Tan, Discrete Dyn. Nat. Soc., **2010** (2010), 12 pages]. In this paper, the concepts of $\mathcal{F}$-collective sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity) and compact-type $\mathcal{F}$-collective sensitivity (resp. compact-type $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity) are introduced as stronger forms of the traditional sensitivity for dynamical systems and Hausdorff locally compact second countable (HLCSC) dynamical systems, respectively, where $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are Furstenberg families. It is proved that $\mathcal{F}$-sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity) of the induced hyperspace system defined on the space of non-empty compact subsets or non-empty finite subsets (Vietoris topology) is equivalent to the $\mathcal{F}$-collective sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity) of the original system; $\mathcal{F}$-sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity) of the induced hyperspace system defined on the space of all non-empty closed subsets (hit-or-miss topology) is equivalent to the compact-type $\mathcal{F}$-collective sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity) of the original HLCSC system. Moreover, it is shown that for a given dynamical system (E, d, f) and a given Furstenberg family $\mathcal{F}$, if (E, d, f) is $\mathcal{F}$-mixing, then it is $\mathcal{F}$-collectively sensitive. Additionally, we prove that for a given dynamical system (E, d, f) and a given Furstenberg family $\mathcal{F}$, (E, d, f) is $\mathcal{F}$-mixing if and only if $\underbrace{f \times f \times \cdots \times f}_{n}$ is $\mathcal{F}$-mixing for every $n \geqslant 2$.

Our results extend and improve some existing results.

Keywords: Furstenberg families, $\mathcal{F}$-collective sensitivity, $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity, compact-type $\mathcal{F}$-collective sensitivity, compact-type $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity, hyperspace dynamical systems, hit-or-miss topology.
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1. Introduction

It is well-known that sensitivity characterizes the unpredictability of chaotic phenomena. Therefore, it is very important to study what system is sensitive. This problem has gained much attention recently (see [1, 2, 6, 10, 12, 14, 15, 20, 29, 31]). There are several forms of sensitivity for dynamical systems (see [15]). In [29], the concepts of collective sensitivity and compact-type collective sensitivity were introduced as stronger conditions than the traditional sensitivity for dynamical systems and Hausdorff locally compact second countable (HLCSC) dynamical systems, respectively, and it was proved that sensitivity of

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the induced hyperspace system defined on the space of non-empty compact subsets or non-empty finite subsets (Vietoris topology) is equivalent to the collective sensitivity of the original system; sensitivity of the induced hyperspace system defined on the space of all non-empty closed subsets (hit-or-miss topology) is equivalent to the compact-type collective sensitivity of the original HLCSC system. Additionally, relations between these two concepts and other dynamics concepts that describe chaos are investigated [29].

Let (E, f) be a topological dynamical system, where E is a topological space and $f : E \to E$ is a continuous map, and let 2^E be the set of all nonempty closed subsets of E. If an appropriate hyperspace topology is chosen for 2^E and f is compatible, i.e., $f(F) \in 2^E$ for any $F \in 2^E$, then the induced hyperspace topological dynamical system $(2^E, 2^f)$ is well-defined, where $2^f : 2^E \to 2^E$, is defined by $2^f(F) = f(F), F \in 2^E$. It is well-known that (E, f) is topologically conjugated to the subsystem of $(2^E, 2^f)$ that consists of the singleton sets of E when E satisfies certain conditions and an appropriate hyperspace topology is selected, e.g., hit-or-miss topology (see [28–30]) or Vietoris topology (see [3, 16, 36]). Clearly, an invariant subset of the original system (E, f) becomes a fixed point of the hyperspace system $(2^E, 2^f)$. In general, the induced system $(2^E, 2^f)$ may inherit some dynamical properties of the system (E, f), but the dynamical properties of $(2^E, 2^f)$ are much more complex, as explored by recent studies on mixing, weak mixing, transitivity, dense set of periodic points, sensitive dependence on initial conditions, entropy, and chaos (see [8, 11, 13, 16–18, 22–24, 28, 29, 36]).

Recently, Wang et al. [28] first used the hit-or-miss topology to study some dynamical properties of the hyperspace dynamical system $(2^E, 2^f)$ induced by a HLCSC dynamical system (E, f). This hyperspace topology is metrizable when E is HLCSC and a concrete metric is available. Consequently, one can study sensitivity and other metric-related dynamical properties for locally compact hyperspace systems. However, for other hyperspace topologies currently employed in the literature of hyperspace dynamics, the Vietoris topology is non-metrizable (unless E is compact metrizable), thus limiting the scope to compact (hyperspace) systems when metric-related dynamical properties are concerned.

The Furstenberg family notion is a very useful tool in studying topologically dynamical systems and ergodic theory (see [25, 27, 31, 35]). In the past few years, some authors [25, 27, 31, 35] investigated proximity, mixing, chaos and sensitivity via Furstenberg family. Whang et al. introduced and investigated the concepts of $\mathcal{F}$-sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity via Furstenberg families, where $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_1$ are Furstenberg families. They provided some conditions under which a dynamical system is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive), and proved that the one-sided shift σ on $\Sigma_N = \prod_{i=1}^{\infty} E_i$ is $\mathcal{F}$-sensitive when $\mathcal{F}$ is Furstenberg family and $\mathcal{F}_1$ and $\mathcal{F}_1$ are full Furstenberg families are compatible with $(\Sigma_N \times \Sigma_N, \sigma \times \sigma)$, where $E_i = \{1, 2, \cdots, N\}$ for all $i \geqslant 1$. Recently, Wu et al. [34] studied the sensitivity of $(2^E, 2^f)$ in Furstenberg families. In particular, they proved that $\mathscr{F}$-sensitivity of $(2^E, 2^f)$ implies that of (E, f), and the converse is also true if the Furstenberg family $\mathscr{F}$ is a filter [34, Corollary 1, Theorem 4]. Wu [33] also proved that $(M(E), f_M)$ is **a**-transitive (resp., exact, uniformly rigid) if and only if (E, f) is weakly mixing and **a**-transitive (resp., exact, uniformly rigid), where $(M(E), f_M)$ is the induced dynamical system on the space of Borel probability measures with weak*-topology.

In the present paper, inspired by [31] and [29], we introduce the concepts of $\mathcal{F}$-collective sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity) and compact-type $\mathcal{F}$-collective sensitivity (resp. compact-type $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity) for dynamical systems and and Hausdorff locally compact second countable (HLCSC) dynamical systems, respectively, where $\mathcal{F}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ are Furstenberg families. It is shown that $\mathcal{F}$-sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity) of the induced hyperspace system defined on the space of non-empty compact subsets or non-empty finite subsets (Vietoris topology) is equivalent to the $\mathcal{F}$-collective sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity) of the original system; $\mathcal{F}$-sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity) of the induced hyperspace system defined on the space of all non-empty closed subsets (hit-or-miss topology) is equivalent to the compact-type $\mathcal{F}$-collective sensitivity (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity) of the original HLCSC system. Moreover, it is shown that for a given dynamical system (E, d, f) and a given Furstenberg family $\mathcal{F}$, if (E, d, f) is $\mathcal{F}$-mixing, then it is $\mathcal{F}$-collectively sensitive. Additionally, we prove that for a given dynamical system (E, d, f) and a given Furstenberg family $\mathcal{F}$,

(E, d, f) is $\mathcal{F}$-mixing if and only if $\underbrace{f \times f \times \cdots \times f}_{n}$ is $\mathcal{F}$-mixing for every $n \geqslant 2$. As applications, we present two examples. Our results extend and improve some existing ones.

The organization of this paper is as follows. In Section 2, we recall some notations and basic concepts. In Section 3, we introduce the concepts of $\mathcal{F}$-collective sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity for any dynamical system to characterize the $\mathcal{F}$-sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity of induced hyperspace dynamical systems defined on $\mathcal{C}$ and $\mathcal{F}_\infty$, equipped with the Vietoris topology, respectively, where $\mathcal{F}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ are Furstenberg families. Section 4 introduces the concepts of compact-type $\mathcal{F}$-collective sensitivity and compact-type $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity for HLCSC dynamical systems to characterize the $\mathcal{F}$-sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity of induced hyperspace dynamical systems defined on 2^E, equipped with the hit-or-miss topology, respectively, where $\mathcal{F}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ are Furstenberg families. Moreover, the relation between $\mathcal{F}$-collective sensitivity and $\mathcal{F}$-mixing for any dynamical systems is explored. In addition, as applications, we provide two examples to explore dynamical properties related to the concepts introduced and results established in Sections 3 and 4.

2. Preliminaries

For a fixed Hausdorff space E, let $\mathcal{F}(E) = \{F \text{ is a closed subset of } E\}$, $\mathcal{G}(E) = \{F : F \text{ is an open subset of } E\}$, and $\mathcal{K}(E) = \{F : F \text{ is a compact subset of } E\}$, abbreviated as $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{K}$.

The hit-or-miss topology τ_f (see [28, 29]) (also known as H-topology [9], Fell topology [4, 5, 21], Choquet-Matheron topology [26], or weak Vietoris topology [32]) on $\mathcal{F}$ is generated by the subbase $\mathcal{F}^K$, $K \in \mathcal{K}$; $\mathcal{F}_G$, $G \in \mathcal{G}$, where $\mathcal{F}^K = \{F \in \mathcal{F} : F \bigcap K = \emptyset\}$ and $\mathcal{F}_G = \{F \in \mathcal{F} : F \bigcap G \neq \emptyset\}$.

A topological base of τ_f (see [19, 28–30]) is $\mathcal{F}^K$, $K \in \mathcal{K}$; $\mathcal{F}^K_{G_1 G_2 \cdots G_n}$, $K \in \mathcal{K}$, $G_i \in \mathcal{G}$, $n \geqslant 1$, $1 \leqslant i \leqslant n$, where $\mathcal{F}^K_{G_1 G_2 \cdots G_n} = \mathcal{F}^K \bigcap \mathcal{F}_{G_1} \bigcap \mathcal{F}_{G_2} \bigcap \cdots \bigcap \mathcal{F}_{G_n}$.

For simplicity, we suppose that $\mathcal{F}^K_{G_1 G_2 \cdots G_n}$ means $\mathcal{F}^K$ when $n = 0$.

Clearly, $\mathcal{F}_0$ and 2^E are identical, where $\mathcal{F}_0 = \mathcal{F} \backslash \{\emptyset\}$. Throughout this paper, E is assumed to be a Hausdorff locally compact second countable space (HLCSC) and the hit-or-miss topology is equipped on $\mathcal{F}_0$ unless stated otherwise.

For any two topological spaces X and Y, a continuous map $f : X \to Y$ is perfect if f is a closed map and all fibers $f^{-1}(y)(y \in Y)$ are compact (see [7, 28, 29]).

Let E be Hausdorff and let $U, V \subset E$ be any two non-empty open sets. If $E \backslash U$ is compact, then U is called co-compact. If one of the U and V is (or both U and V are) co-compact, then U and V are said to be a pair of co-compact subsets of E, denoted by (U, V) (see [28, 29]).

In the following and here, by a dynamical system we mean a pair (E, f), where E is a metric space and $f : E \to E$ is a continuous map. For a given nonempty set A, let $\mathcal{P}(A)$ denote the collection of all subsets of A and let Z^+ be the set of all non-negative integers. For simplicity write $\mathcal{P} = \mathcal{P}(Z^+)$. A subset $\mathcal{F} \subset \mathcal{P}$ is a Furstenberg family if it is hereditary upwards, i.e., $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A Furstenberg family $\mathcal{F}$ is proper if it is a proper subset of $\mathcal{P}$, that is, $\mathcal{F}$ is proper if and only if $Z^+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. For a given subset $\mathcal{A} \subset \mathcal{P}$, it can generate a Furstenberg family $[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$. For a Furstenberg family $\mathcal{F}$, its dual family is defined by $\kappa\mathcal{F} = \{F \in \mathcal{P} : F \bigcap F' \neq \emptyset \text{ for any } F' \in \mathcal{F}\}$. Let $\mathcal{B}$ be the family of all infinite subsets of Z^+. Then its dual family $\kappa\mathcal{B}$ is the family of all cofinite subsets of Z^+. Let (E, f) be a dynamical system and $A, B \subset E$. Define the hitting time set $N_f(A, B) = \{n \in Z^+ : f^n(A) \bigcap B \neq \emptyset\}$. A dynamical system (E, f) is $\mathcal{F}$-transitive if for each pair of open and nonempty subsets $U, V \subset E$, $N_f(U, V) \in \mathcal{F}$. (E, f) is $\mathcal{F}$-mixing if $(E \times E, f \times f)$ is $\mathcal{F}$-transitive.

A Furstenberg family $\mathcal{F}$ is countably generated [25, 27, 31, 35] if there exists a countable subset $\mathcal{A} \subset \mathcal{P}$ such that $[\mathcal{A}] = \mathcal{F}$. It is clear that $\kappa\mathcal{B}$ is a countably generated proper family.

For any two Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$, let $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \bigcap F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. A Furstenberg family $\mathcal{F}$ is full if it is proper and $\mathcal{F} \cdot \kappa\mathcal{F} \subset \mathcal{B}$. It is obvious that a Furstenberg family $\mathcal{F}$ is full if and only if $\kappa\mathcal{B} \cdot \mathcal{F} \subset \mathcal{F}$. One can easily see that $\kappa\mathcal{B}$ and $\mathcal{B}$ are full. Clearly, if $\mathcal{F}$ is full then $\kappa\mathcal{B} \subset \mathcal{F}$. A Furstenberg family $\mathcal{F}$ is a filterdual if it is proper and $\kappa\mathcal{F} \supset \kappa\mathcal{F} \cdot \kappa\mathcal{F}$.

Let (E, f) be a dynamical system and $\mathcal{F}$ be a Furstenberg family. For any $A \subset X$ and any $x \in X$, write $N_f(x, A) = N_f(\{x\}, A)$. A point $x \in X$ is called an $\mathcal{F}$-attaching point of the set A if $N_f(x, A) \in \mathcal{F}$. A Furstenberg family $\mathcal{F}$ is said to be compatible with the system (E, f) if the set of $\mathcal{F}$-attaching points of U is a G_δ set of E for each open and nonempty subset $U \subset E$ (see [27, 31, 35]).

Definition 2.1 ([31]). Let (E, d, f) be a topological dynamical system. (E, d, f) (or simply f) is said to be $\mathcal{F}$-sensitive if there exists a $\varepsilon > 0$ such that for every $x \in E$ and every open neighborhood U of x there exists $y \in U$ such that $\{n \in Z^+ : d(f^n(x), f^n(y)) > \varepsilon\}$ belongs to $\mathcal{F}$, where $\mathcal{F}$ is a Furstenberg family.

Definition 2.2 ([31]). Let (E, d, f) be a topological dynamical system. (E, d, f) (or simply f) is said to be $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive if there is a $\varepsilon > 0$ such that every $x \in E$ is a limit of points $y \in E$ such that $\{n \in Z^+ : d(f^n(x), f^n(y)) < \delta\}$ belongs to $\mathcal{F}_1$ for any $\delta > 0$ but $\{n \in Z^+ : d(f^n(x), f^n(y)) > \varepsilon\}$ belongs to $\mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are Furstenberg families.

Definition 2.3 ([31]). Let (E, d, f) be a topological dynamical system. (E, d, f) (or simply f) is said to be weakly $\mathcal{F}$-sensitive if there is a $\varepsilon > 0$–a weakly $\mathcal{F}$-sensitive constant–such that in every open and nonempty subset U of X there exist x and y of U such that the pair x, y is not $\mathcal{F}$-ε-asymptotic. That is, $\{n \in Z^+ : d(f^n(x), f^n(y)) > \varepsilon\} \in \mathcal{F}$.

3. $\mathcal{F}$-sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity of induced (sub) hyperspace dynamical systems equipped with the Vietoris topology

The following two lemmas are needed.

Lemma 3.1. *Let (X, d, f) be a subsystem of a given dynamical system (E, d, f) and $\mathcal{F}$ a Furstenberg family, where X is dense in E. If $f : E \to E$ is uniformly continuous, then (E, d, f) is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive) if and only if (X, d, f) is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive).*

Proof. It follows from the definitions. □

Lemma 3.2. *Let (X, d, f) be a subsystem of a given dynamical system (E, d, f) and $\mathcal{F}$ a Furstenberg family, where X is dense in E. If $2^f : (2^E, \tau_v) \to (2^E, \tau_v)$ or $2^f : (2^E, \tau_f) \to (2^E, \tau_f)$ is uniformly continuous, then $(\mathcal{C}, d_H, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive) if and only if $(\mathcal{F}_\infty, d_H, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive).*

Proof. It follows from Lemma 3.1. □

Definition 3.3. Let (E, d, f) be a given dynamical system, $\mathcal{F}$ a Furstenberg family and $\delta > 0$ a constant. (E, d, f) is said to be $\mathcal{F}$-collectively sensitive with the collective sensitivity constant δ if for any finitely many distinct points $x_1, x_2, \cdots, x_n \in E$ and any $\varepsilon > 0$, there exist the same number of distinct points $y_1, y_2, \cdots, y_n \in E$ satisfying the following two conditions:

(i) $d(x_i, y_i) < \varepsilon$ for all $1 \leqslant i \leqslant n$;

(ii) there exists an i_0 with $1 \leqslant i_0 \leqslant n$ such that $\{k \in Z^+ : d(f^k(x_i), f^k(y_{i_0})) > \delta$ or $d(f^k(x_{i_0}), f^k(y_i)) > \delta, 1 \leqslant i \leqslant n\} \in \mathcal{F}$.

Clearly, if $\mathcal{F}$ is a proper Furstenberg family, then $\mathcal{F}$-collective sensitivity is stronger than collective sensitivity.

Definition 3.4. Let (E, d, f) be a given dynamical system, $\mathcal{F}_1, \mathcal{F}_2$ two Furstenberg families and $\delta > 0$ a constant. (E, d, f) is said to be $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive with the collective sensitivity constant δ if for any finitely many distinct points $x_1, x_2, \cdots, x_n \in E$ and any $\varepsilon > 0$, there exist the same number of distinct points $y_1^j, y_2^j, \cdots, y_n^j \in E$ $(j = 1, 2, \cdots)$ satisfying the following three conditions:

(i) $d(x_i, y_i^j) < \varepsilon$ for all $1 \leqslant i \leqslant n$ $(j = 1, 2, \cdots)$;

(ii) for each $j \in \{1, 2, \cdots\}$, there exists an i_0 with $1 \leqslant i_0 \leqslant n$ such that

$$\{k \in Z^+ : d(f^k(x_i), f^k(y^j_{i_0})) > \delta \text{ or } d(f^k(x_{i_0}), f^k(y^j_i)) > \delta, 1 \leqslant i \leqslant n\} \in \mathcal{F}_1$$

and

$$\{k \in Z^+ : d(f^k(x_i), f^k(y^j_{i_0})) < \varepsilon \text{ or } d(f^k(x_{i_0}), f^k(y^j_i)) < \varepsilon, 1 \leqslant i \leqslant n\} \in \mathcal{F}_2$$

for any $j \geqslant 1$;

(iii) $\lim\limits_{j\to+\infty} y^j_i = x_i$, $i = 1, 2, \cdots, n$.

It is easily seen that $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity implies $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity by the above definitions.

Theorem 3.5. *Let (E, d, f) be a given dynamical system and $\mathcal{F}$ a Furstenberg family. Then $(\mathcal{F}_\infty, d_H, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive) if and only if (E, d, f) is $\mathcal{F}$-collectively sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive).*

Proof. Assume that $(\mathcal{F}_\infty, d_H, 2^f)$ is $\mathcal{F}$-sensitive with a sensitivity constant δ. For any finitely many distinct points $x_1, x_2, \cdots, x_n \in E$ and any $\varepsilon > 0$, without loss of generality, we can suppose $\varepsilon < \frac{1}{2}\min\{d(x_i, x_j) : 1 \leqslant i, j \leqslant n \text{ and } i \neq j\}$. By hypothesis, there exists $B \in \mathcal{F}_\infty$ with $d_H(A, B) < \varepsilon$ and $\{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B)) > \delta\} \in \mathcal{F}$ where $A = \{x_1, x_2, \cdots, x_n\}$. By $d_H(A, B) < \varepsilon$ and the assumption on ε, for any $y \in B$ there is only one $1 \leqslant i \leqslant n$ with $d(y, x_i) < \varepsilon$. Let $B_i = \{y \in B : d(y, x_i) < \varepsilon\}$ for every $1 \leqslant i \leqslant n$. Obviously, $B_i \neq \emptyset$ for any $i \in \{1, 2, \cdots, n\}$. For any $C \in \mathcal{F}_\infty$, Write $\overline{S}(C, \delta) = \{x \in E : d_H(x, C) \leqslant \delta\}$. For each $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B)) > \delta\}$, one of the following is not true:

(1) $\overline{S}((2^f)^k(A), \delta) \supseteq (2^f)^k(B)$;

(2) $\overline{S}((2^f)^k(B), \delta) \supseteq (2^f)^k(A)$.

If (1) does not hold for some $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B)) > \delta\}$, then there exists $\overline{y} \in B_{i_0}$ with $d(f^k(\overline{y}), f^k(x_i)) > \delta$ for all $1 \leqslant i \leqslant n$. Pick $y_i \in B_i$ for each i. In particular, take $y_{i_0} = \overline{y}$. Then we have (i) $d(x_i, y_i) < \varepsilon$ for any $i \in \{1, 2, \cdots, n\}$; and (ii) there exists an i_0 with $1 \leqslant i_0 \leqslant n$ and $d(f^k(x_i), f^k(y_{i_0})) > \delta$ for any $i \in \{1, 2, \cdots, n\}$.

If (2) does not hold for some $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B)) > \delta\}$, then there exists an i_0 with $1 \leqslant i_0 \leqslant n$ and $d(f^k(x_{i_0}), f^k(y)) > \delta$ for any $y \in B$. For each $1 \leqslant i \leqslant n$, pick $y_i \in B_i$. We have (i) $d(x_i, y_i) < \varepsilon$ for any $1 \leqslant i \leqslant n$; and (ii) $d(f^k(x_{i_0}), f^k(y_i)) > \delta$ for any $1 \leqslant i \leqslant n$.

From the above argument, it follows that (E, d, f) is $\mathcal{F}$-collectively sensitive.

Now we suppose that (E, d, f) is $\mathcal{F}$-collectively sensitive with a collective sensitivity constant δ. For any $A \in \mathcal{F}_\infty$ and $\varepsilon > 0$, write $A = \{x_1, x_2, \cdots, x_n\}$. By hypothesis, there exist n distinct points $y_1, y_2, \cdots, y_n \in E$ satisfying (i) and (ii) in Definition 3.3. Write $B = \{y_1, y_2, \cdots, y_n\}$. (i) means $d_H(A, B) < \varepsilon$; and (ii) means that for every $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B)) > \delta\}$, one of the following does not hold:

(3) $\overline{S}((2^f)^k(A), \delta) \supseteq (2^f)^k(B)$;

(4) $\overline{S}((2^f)^k(B), \delta) \supseteq (2^f)^k(A)$.

Therefore, for each $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B)) > \delta\}$, we have $d_H((2^f)^k(A), (2^f)^k(B)) > \delta$. Consequently, from the definition we know that $(\mathcal{F}_\infty, d_H, 2^f)$ is $\mathcal{F}$-sensitive.

Assume that $(\mathcal{F}_\infty, d_H, 2^f)$ is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive with a sensitivity constant δ. For any finitely many distinct points $x_1, x_2, \cdots, x_n \in E$ and any $\varepsilon > 0$, without loss of generality, we can assume $\varepsilon < \frac{1}{2}\min\{d(x_i, x_j) : 1 \leqslant i, j \leqslant n \text{ and } i \neq j\}$. By hypothesis, for each $j \in Z^+$, there exists $B_j \in \mathcal{F}_\infty$ with

$$\{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B_j)) < \varepsilon\} \in \mathcal{F}_1, \tag{3.1}$$
$$\{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B_j)) > \delta\} \in \mathcal{F}_2,$$

and

$$\lim_{j\to\infty} B_j = A, \tag{3.2}$$

where $A = \{x_1, x_2, \cdots, x_n\}$. By (3.2) and the assumption on ε, without loss of generality, we can suppose that for every $j \in Z^+$ and any $y \in B_j$, there is only one $1 \leqslant i_j \leqslant n$ with $d(y, x_i) < \varepsilon$. Let $B_{i_j} = \{y \in B_j : d(y, x_i) < \varepsilon\}$ for every $1 \leqslant i \leqslant n$ and any $j \in Z^+$. Obviously, $B_{i_j} \neq \emptyset$ for any $i \in \{1, 2, \cdots, n\}$ and any $j \in Z^+$. For any $C \in \mathcal{F}_\infty$, write $\overline{S}(C, \delta) = \{D \in \mathcal{F}_\infty : d_H(C, D) \leqslant \delta\}$. For every $j \in Z^+$ and every $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B_j)) > \delta\}$, one of the following is not true:

(5) $\overline{S}((2^f)^k(A), \delta) \supseteq (2^f)^k(B_j)$;

(6) $\overline{S}((2^f)^k(B_j), \delta) \supseteq (2^f)^k(A)$.

If (5) does not hold for some $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B_j)) > \delta\}$ and some $j \in Z^+$, then there exists $\overline{y}_j \in B_{i_{0_j}}$ with $d(f^k(\overline{y}_j), f^k(x_i)) > \delta$ for all $1 \leqslant i \leqslant n$. Pick $y_{i_j} \in B_{i_j}$ for each i and each $j \in Z^+$. In particular, take $y_{i_{0_j}} = \overline{y}_j$ for each $j \in Z^+$. Then for the above j, we have (i) $d(x_i, y_{i_j}) < \varepsilon$ for any $i \in \{1, 2, \cdots, n\}$; and (ii) there exists an i_{0_j} with $1 \leqslant i_0 \leqslant n$ and $d(f^k(x_i), f^k(y_{i_{0_j}})) > \delta$ for any $i \in \{1, 2, \cdots, n\}$.

If (6) does not hold for some $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B)) > \delta\}$ and some $j \in Z^+$, then there exists an i_{0_j} with $1 \leqslant i_0 \leqslant n$ and $d(f^k(x_{i_{0_j}}), f^k(y)) > \delta$ for any $y \in B_j$. For each $1 \leqslant i \leqslant n$, pick $y_{i_j} \in B_{i_j}$. We have (i) $d(x_i, y_{i_j}) < \varepsilon$ for any $1 \leqslant i \leqslant n$; and (ii) $d(f^k(x_{i_0}), f^k(y_{i_j})) > \delta$ for any $1 \leqslant i \leqslant n$.

From the above argument, it follows that $\{k \in Z^+ : d(f^k(x_i), f^k(y^j_{i_0})) > \delta$ or $d(f^k(x_{i_0}), f^k(y^j_i)) > \delta, 1 \leqslant i \leqslant n\} \in \mathcal{F}_2$.

Clearly, for a fixed $j \in Z^+$, if $d_H((2^f)^k(A), (2^f)^k(B_j)) < \varepsilon$ for some $k \geqslant 0$ and some $\varepsilon > 0$, then we have $S((2^f)^k(A), \varepsilon) \supseteq (2^f)^k(B_j)$ and $S((2^f)^k(B_j), \varepsilon) \supseteq (2^f)^k(A)$.

If $S((2^f)^k(A), \varepsilon) \supseteq (2^f)^k(B_j)$ holds, there exists $y \in B_{i_{0_j}}$ with $d(f^k(\overline{y}_j), f^k(x_i)) < \varepsilon$ for all $1 \leqslant i \leqslant n$. Pick $y_{i_j} \in B_{i_j}$ for each i and each $j \in Z^+$. In particular, we can choose $y_{i_{0_j}} = \overline{y}_j$ for each $j \in Z^+$. Then for the above j, we have (i) $d(x_i, y_{i_j}) < \varepsilon$ for any $i \in \{1, 2, \cdots, n\}$; and (ii) there exists an i_{0_j} with $1 \leqslant i_0 \leqslant n$ and $d(f^k(x_i), f^k(y_{i_{0_j}})) < \varepsilon$ for any $i \in \{1, 2, \cdots, n\}$.

If $\overline{S}((2^f)^k(B_j), \varepsilon) \supseteq (2^f)^k(A)$ holds, then there exists an i_{0_j} with $1 \leqslant i_0 \leqslant n$ and $d(f^k(x_{i_{0_j}}), f^k(y)) < \varepsilon$ for any $y \in B_j$. For each $1 \leqslant i \leqslant n$, pick $y_{i_j} \in B_{i_j}$. We have (i) $d(x_i, y_{i_j}) < \varepsilon$ for any $1 \leqslant i \leqslant n$; and (ii) $d(f^k(x_{i_0}), f^k(y_{i_j})) < \varepsilon$ for any $1 \leqslant i \leqslant n$.

Since

$$\{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B_j)) < \varepsilon\} \in \mathcal{F}_1,$$

for each $j \in Z^+$, by the definition, the above argument, and (3.1), (E, d, f) is $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive.

Now we suppose that (E, d, f) is $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive with a collective sensitivity constant δ. For any $A \in \mathcal{F}_\infty$ and $\varepsilon > 0$, write $A = \{x_1, x_2, \cdots, x_n\}$. By hypothesis, for each $j \in Z^+$, there exist n distinct points $y_{1_j}, y_{2_j}, \cdots, y_{n_j} \in E$ satisfying (i), (ii), and (iii) in Definition 3.4. Write $B_j = \{y_{1_j}, y_{2_j}, \cdots, y_{n_j}\}$ for every $j \in Z^+$. By (iii) in Definition 3.4, without loss of generality, we can assume that $d_H(A, B_j) < \varepsilon$ for every $j \in Z^+$; and (ii) means that for a fixed $j \in Z^+$, one of the following does not hold:

(7) $\overline{S}((2^f)^k(A), \delta) \supseteq (2^f)^k(B_j)$;

(8) $\overline{S}((2^f)^k(B_j), \delta) \supseteq (2^f)^k(A)$.

Therefore, for the above j and each $k \in \{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B_j)) > \delta\}$, we have

$$d_H((2^f)^k(A), (2^f)^k(B_j)) > \delta.$$

By $d_H(A, B_j) < \varepsilon$ for every $j \in Z^+$, without loss of generality, one can suppose that $\lim_{j\to\infty} B_j = A$. By the above argument we obtain that $\{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B_j)) > \delta\} \in \mathcal{F}_2$ for every $j \in Z^+$. Similarly, (ii) means that for a fixed $j \in Z^+$ and a given $\varepsilon > 0$, one of the following does not hold:

(7) $\overline{S}((2^f)^k(A), \varepsilon) \supseteq (2^f)^k(B_j)$;

(8) $\overline{S}((2^f)^k(B_j), \varepsilon) \supseteq (2^f)^k(A)$.

Consequently, by the hypothesis and the above, $\{k \in Z^+ : d_H((2^f)^k(A), (2^f)^k(B_j)) < \varepsilon\} \in \mathcal{F}_1$ for every $j \in Z^+$. From the definition we know that $(\mathcal{F}_\infty, d_H, 2^f)$ is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive. Thus, the proof of Theorem 3.5 is finished. □

Theorem 3.6. *Let (E, d, f) be a given dynamical system and $\mathcal{F}$ a Furstenberg family. If $2^f : (2^E, \tau_v) \to (2^E, \tau_v)$ is uniformly continuous, then the following conditions are equivalent:*

(i) *$(\mathcal{C}, d_H, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive);*

(ii) *$(\mathcal{F}_\infty, d_H, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive);*

(iii) *(E, d, f) is $\mathcal{F}$-collectively sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive).*

Proof. It follows from Theorem 3.5 and Lemmas 3.1 and 3.2. □

In particular, for compact dynamical systems, we get the following result.

Corollary 3.7. *Let (E, d, f) be a given compact dynamical system and $\mathcal{F}$ a Furstenberg family. Then the following conditions are equivalent:*

(i) *$(2^E, d_H, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive);*

(ii) *$(\mathcal{F}_\infty, d_H, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1; \mathcal{F}_2)$-sensitive);*

(iii) *(E, d, f) is $\mathcal{F}$-collectively sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive).*

4. $\mathcal{F}$-sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$sensitivity of induced hyperspace dynamical systems equipped with the hit-or-miss topology

Throughout this section, let E be a non-compact HLCSC space, d be a compact-type metric of E, and ρ be a metric of the hit-or-miss topology on $\mathcal{F} = 2^E \bigcup \{\emptyset\}$, and let $f : E \to E$ be a perfect mapping and $\mathcal{C}, \mathcal{F}_\infty$ be subspaces of $(\mathcal{F}, \tau_f)$, where τ_f is the hit-or-miss topology.

Definition 4.1 ([29]). A metric d of E is of compact-type if it can be extended to a metric $\overline{d}$ of the Alexandroff compactification ωE.

Definition 4.2. Let $\overline{d}$ be a metric of ωE and d be the restriction of $\overline{d}$ on E, and let (E, d, f) be a given dynamical system, $\mathcal{F}$ a Furstenberg family and $\delta > 0$ a constant. (E, d, f) is said to be compact-type $\mathcal{F}$-collectively sensitive with the collective sensitivity constant δ if for any finitely many distinct points $x_1, x_2, \cdots, x_n \in E$ and any $\varepsilon > 0$, there exist the same number of distinct points $y_1, y_2, \cdots, y_n \in E$ satisfying the following two conditions:

(i) $d(x_i, y_i) < \varepsilon$ for all $1 \leqslant i \leqslant n$;

(ii) there exists an i_0 with $1 \leqslant i_0 \leqslant n$ such that $K_1 \bigcup K_2 \in \mathcal{F}$, where

$$K_1 = \{k \in Z^+ : d(f^k(x_i), f^k(y_{i_0})) > \delta \text{ and } \overline{d}(f^k(y_{i_0}), \omega) > \delta, 1 \leqslant i \leqslant n\}$$

and

$$K_2 = \{d(f^k(x_{i_0}), f^k(y_i)) > \delta \text{ and } \overline{d}(f^k(x_{i_0}), \omega) > \delta, 1 \leqslant i \leqslant n\}.$$

Clearly, if $\mathcal{F}$ is a proper Furstenberg family, then compact-type $\mathcal{F}$-collective sensitivity is stronger than compact-type collective sensitivity.

Definition 4.3. Let (E, d, f) be a given dynamical system, $\mathcal{F}_1, \mathcal{F}_2$ two Furstenberg families and $\delta > 0$ a constant. (E, d, f) is said to be compact-type $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive with the collective sensitivity constant δ if for any finitely many distinct points $x_1, x_2, \cdots, x_n \in E$ and any $\varepsilon > 0$, there exist the same number of distinct points $y_1^j, y_2^j, \cdots, y_n^j \in E$ $(j = 1, 2, \cdots)$ satisfying the following three conditions:

(i) $d(x_i, y_i^j) < \varepsilon$ for all $1 \leqslant i \leqslant n$ $(j = 1, 2, \cdots)$;

(ii) for each $j \in \{1, 2, \cdots\}$, there exists an i_0 with $1 \leqslant i_0 \leqslant n$ such that $K_1 \bigcup K_2 \in \mathcal{F}_1$ and $K_3 \bigcup K_4 \in \mathcal{F}_2$ for any $\varepsilon > 0$, where

$$K_1 = \{k \in Z^+ : d(f^k(x_i), f^k(y_{i_0}^j)) > \delta \text{ and } \overline{d}(f^k(y_{i_0}^j), \omega) > \delta, 1 \leqslant i \leqslant n\},$$
$$K_2 = \{d(f^k(x_{i_0}), f^k(y_i^j)) > \delta \text{ and } \overline{d}(f^k(x_{i_0}), \omega) > \delta, 1 \leqslant i \leqslant n\},$$
$$K_3 = \{k \in Z^+ : d(f^k(x_i), f^k(y_{i_0}^j)) < \varepsilon \text{ and } \overline{d}(f^k(y_{i_0}^j), \omega) < \varepsilon, 1 \leqslant i \leqslant n\},$$

and

$$K_4 = \{d(f^k(x_{i_0}), f^k(y_i^j)) < \varepsilon \text{ and } \overline{d}(f^k(x_{i_0}), \omega) < \varepsilon, 1 \leqslant i \leqslant n\};$$

(iii) $\lim\limits_{j \to +\infty} y_i^j = x_i$, $i = 1, 2, \cdots, n$.

It is easy to see that compact-type $(\mathcal{F}_1, \mathcal{F}_2)$-collective sensitivity implies $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity by the definitions.

With the hit-or-miss topology, we obtain the following theorem:

Theorem 4.4. *If $(2^E, \rho, 2^f)$ is uniformly continuous, then the following conditions are equivalent:*

(i) *$(2^E, \rho, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive);*

(ii) *$(\mathcal{C}, \rho, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive);*

(iii) *$(\mathcal{F}_\infty, \rho, 2^f)$ is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive).*

Proof. Since $\mathcal{C}$ and $\mathcal{F}_\infty$ are dense subsets of 2^E under the hit-or-miss topology, it follows from Lemma 3.1 and Lemma 3.2. □

Theorem 4.5. *Let E be non-compact HLCSC, d_1 and d_2 be any two compact-type metrics of E, and $f : E \to E$ be a continuous map. Then (E, d_1, f) is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive) if and only if (E, d_2, f) is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$−sensitive).*

Proof. Clearly, It is enough to show that if (E, d_1, f) is $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive), then (E, d_2, f) is too. Let (E, d_1, f) be $\mathcal{F}$-sensitive (resp. $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive) with a sensitivity constant $\delta_1 > 0$. By hypothesis, there are two metrics $\overline{d}_1$ and $\overline{d}_2$ of ωE with $d_1 = \overline{d}_1|_{E \times E}$ and $d_2 = \overline{d}_2|_{E \times E}$. Since the identity mapping $i : (\omega E, \overline{d}_1) \to (\omega E, \overline{d}_2)$ is a homeomorphism, there is $\delta_2 > 0$ satisfying that $\overline{d}_2(x, y) \leqslant \delta_2$ $(x, y \in \omega E)$ implies that $\overline{d}_1(x, y) \leqslant \delta_1$. So, $d_2(x, y) \leqslant \delta_2$ $(x, y \in E)$ implies that $d_1(x, y) \leqslant \delta_1$. For any $x \in E$ and $\sigma > 0$, it is easy to see that $S_{d_2}(x, \sigma)$ is an open neighborhood of x under d_2. This means that $S_{d_2}(x, \sigma)$ is also an open neighborhood of x under d_1. By the $\mathcal{F}$-sensitivity of (E, d_1, f), there exists $y \in S_{d_2}(x, \sigma)$ with $\{n \in Z^+ : d_1(f^n(x), f^n(y))\} > \delta_1\} \in \mathcal{F}$. From the above relation of d_1 and d_2, we have $\{n \in Z^+ : d_2(f^n(x), f^n(y))\} > \delta_2\} \in \mathcal{F}$. Therefore, we obtain that if (E, d_1, f) is $\mathcal{F}$-sensitive, then so does (E, d_2, f). By a similar argument and the definition, one can easily prove that if (E, d_1, f) is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive, then so is (E, d_2, f). Thus, the proof is ended. □

Theorem 4.6. *Let (E, d, f) be a given dynamical system and $\mathcal{F}$ a Furstenberg family. Then $(F_\infty, \rho, 2^f)$ is $\mathcal{F}$-sensitive if and only if (E, d, f) is compact-type $\mathcal{F}$-collectively sensitive.*

Proof. Since the proof is similar to that of Theorem 3.5, it is omitted here. □

Theorem 4.7. *Let* (E, d, f) *be a given dynamical system and* $\mathcal{F}$ *a Furstenberg family. If* $(2^E, \rho, 2^f)$ *is uniformly continuous, then the following conditions are equivalent:*

(i) $(2^E, \rho, 2^f)$ *is* $\mathcal{F}$*-sensitive;*

(ii) $(\mathcal{C}, \rho, 2^f)$ *is* $\mathcal{F}$*-sensitive;*

(iii) $(F_\infty, \rho, 2^f)$ *is* $\mathcal{F}$*-sensitive;*

(iv) (E, d, f) *is compact-type* $\mathcal{F}$*-collectively sensitive.*

Proof. It follows from Theorems 4.4 and 4.6. □

The following lemma is needed to prove Theorem 4.10.

Lemma 4.8. *Let* $f : X \to X$ *be a continuous map of a metric space* X *and* $\mathcal{F}$ *a Furstenberg family. Then the following are equivalent:*

(1) f *is* $\mathcal{F}$*-mixing;*

(2) $\underbrace{f \times f \times \cdots \times f}_{n}$ *is* $\mathcal{F}$*-transitive for every* $n \geqslant 2$*;*

(3) $\underbrace{f \times f \times \cdots \times f}_{n}$ *is* $\mathcal{F}$*-mixing for every* $n \geqslant 2$.

Proof. (1)⇒ (2). By hypothesis and the definition, $f \times f$ is $\mathcal{F}$-transitive. Assume that $\underbrace{f \times f \times \cdots \times f}_{n}$ is $\mathcal{F}$-transitive for some $n \geqslant 2$. Let $U_i, V_i \subset X$ be nonempty and open for all $1 \leqslant i \leqslant n+1$. By hypothesis and the definition, there exists $s \in Z^+$ satisfying that U and V are nonempty and open subsets of X, where $U = U_n \bigcap f^{-s}(U_{n+1})$ and $V = V_n \bigcap f^{-s}(V_{n+1})$. By the above assumption, we get that

$$\{m \in Z+ : (\underbrace{f \times f \times \cdots \times f}_{n})^m (U_1 \times U_2 \times \cdots \times U_{n-1} \times U) \bigcap (V_1 \times V_2 \times \cdots \times V_{n-1} \times V) \neq \emptyset\} \in \mathcal{F}.$$

This implies that

$$\{m \in Z+ : (\underbrace{f \times f \times \cdots \times f}_{n+1})^m (U_1 \times U_2 \times \cdots \times U_{n+1}) \bigcap (V_1 \times V_2 \times \cdots \times V_{n+1}) \neq \emptyset\} \in \mathcal{F}.$$

By induction, $\underbrace{f \times f \times \cdots \times f}_{n}$ is $\mathcal{F}$-transitive for every $n \geqslant 2$.

(2)⇒ (3). By hypothesis, $\underbrace{f \times f \times \cdots \times f}_{n} \times \underbrace{f \times f \times \cdots \times f}_{n}$ is $\mathcal{F}$-transitive for every $n \geqslant 2$. So, $\underbrace{f \times f \times \cdots \times f}_{n}$ is $\mathcal{F}$-mixing for every $n \geqslant 2$.

(3)⇒ (1). The proof is easy by the definition and is omitted. □

Remark 4.9. In Lemma 4.8 it is not required that Furstenberg families are full. So, this lemma extends and improves some existing results.

Theorem 4.10. *Let* (E, d, f) *be a given dynamical system and* $\mathcal{F}$ *a Furstenberg family. If* $(2^E, \rho, 2^f)$ *is uniformly continuous, then the following conditions are equivalent:*

(i) $(2^E, \rho, 2^f)$ *is* $\mathcal{F}$*-sensitive;*

(ii) $(\mathcal{C}, \rho, 2^f)$ *is* $\mathcal{F}$*-sensitive;*

(iii) $(F_\infty, \rho, 2^f)$ *is $\mathcal{F}$-sensitive;*

(iv) (E, d, f) *is compact-type $\mathcal{F}$-collectively sensitive.*

Proof. Let $e_1, e_2 \in E$ be any two points with $d(e_1, e_2) \geqslant 10\delta$. Write $B_i = \{x \in E : d(e_i, x) < \delta\}$ for $i \in \{1, 2\}$. Then B_1 and B_2 are two disjoint open balls. Given a finite set of distinct points $x_i \in E$, $i = 1, 2, \cdots, n$ and $\varepsilon > 0$, let O_i denote the open ball centered at x_i with radius ε for each $1 \leqslant i \leqslant n$. By Lemma 4.8, any n-product of f is $\mathcal{F}$-transitive $n \geqslant 2$. This implies, for $n = 2m$, that $\{k \in Z^+ : f^k(O_i) \bigcap B_j \neq \emptyset, i = 1, 2, \cdots, m, j = 1, 2\} \in \mathcal{F}$. Then, for any given $k \in \{k \in Z^+ : f^k(O_i) \bigcap B_j \neq \emptyset, i = 1, 2, \cdots, m, j = 1, 2\}$, there are two points $z_i, z_i' \in O_i$ with $f^k(z_i) \in G_1$ and $f^k(z_i') \in G_2$ for each $i = 1, 2, \cdots, m$. Therefore, we have $d(f^k(z_i), f^k(z_i')) > 8\delta$ for each $i = 1, 2, \cdots, m$. This means that for the above k, either $d(f^k(x_1), f^k(z_1)) > 4\delta$ or $d(f^k(x_1), f^k(z_1')) > 4\delta$. Let $y_i = z_i$ or $y_i = z_i'$ for $i = 1, 2, \cdots, m$. This implies $d(f^k(x_1), f^k(y_i)) > \delta$ for the above k and each $i = 1, 2, \cdots, m$. Consequently, by the definition, (E, d, f) is $\mathcal{F}$-collectively sensitive with constant δ. □

Remark 4.11. It is known that if (E, d, f) is weakly mixing, then it is $\mathcal{B}$-mixing. By the definition one can easily prove that f is weakly mixing if and only if $f \times f$ is weakly mixing. We also know that if (E, d, f) is weakly mixing, then it is $\tau\mathcal{B}$-mixing. Since $\mathcal{B}$-collective sensitivity and $\tau\mathcal{B}$-collective sensitivity imply collective sensitivity, this theorem extends and improves some existing results.

Theorem 4.12. *Let (E, d, f) be a given dynamical system and $\mathcal{F}_1, \mathcal{F}_2$ two Furstenberg families. Then $(F_\infty, \rho, 2^f)$ is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive if and only if (E, d, f) is compact-type $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive.*

Proof. Since the proof is similar to that of Theorem 3.5, it is omitted here. □

Theorem 4.13. *Let (E, d, f) be a given dynamical system and $\mathcal{F}_1, \mathcal{F}_2$ two Furstenberg families. If $(2^E, \rho, 2^f)$ is uniformly continuous, then the following conditions are equivalent:*

(i) $(2^E, \rho, 2^f)$ *is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive;*

(ii) $(\mathcal{C}, \rho, 2^f)$ *is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive;*

(iii) $(F_\infty, \rho, 2^f)$ *is $(\mathcal{F}_1, \mathcal{F}_2)$-sensitive;*

(iv) (E, d, f) *is compact-type $(\mathcal{F}_1, \mathcal{F}_2)$-collectively sensitive.*

Proof. It follows from Theorems 4.4 and 4.12. □

The dynamical system given in the following example has the compact-type $\mathcal{F}$-collective sensitivity, $\mathcal{F}$ is the family of all cofinite subsets of Z^+.

Example 4.14 ([29, Example 6.1])**.** For a given integer $p \geqslant 2$. Let $\Sigma(p) = \{s = (\cdots, s_{-1}, s_0, s_1, \cdots) : s_n \in \{1, 2, \cdots, p\}, n \in Z\}$ and $(\Sigma(p), \sigma)$ be the full two-sided p-shift, where Z is the set of all integers and $\sigma : \Sigma(p) \to \Sigma(p)$ is defined by $\sigma(s) = t$, where $t_n = s_{n+1}$ for any $n \in Z$. The metric $\overline{d}$ of $(\Sigma(p), \sigma)$ is defined as $d(s, t) = \sum\limits_{n \in Z} \frac{\delta(s_n, t_n)}{2^{|n|}}$, where $\delta(i, j) = 1$ if $i \neq j$ and $\delta(i, j) = 0$ if $i = j$. Let $E = \Sigma(p) \setminus \{(\cdots, 1, 1, 1, \cdots)\}$ and $f = \sigma|E$. Then (E, f^k) is compact-type $\mathcal{F}$-collective sensitive and compact-type $(\mathcal{B}, \mathcal{B})$-collective sensitive for any integer $k \geqslant 1$, where $\mathcal{F}$ is the family of all cofinite subsets of Z^+ and $\mathcal{B}$ is the family of all infinite subsets of Z^+.

Proof. It is enough to show that (E, f) is compact-type $\mathcal{F}$-collective sensitive and compact-type $(\mathcal{B}, \mathcal{B})$-collective sensitive.

Clearly, E is an open and dense subset of $\Sigma(p)$. Since $(\Sigma(p), \sigma)$ is topologically mixing, (E, f) is topologically mixing by the definition, which implies (E, f) is c-mixing. By Theorem 3.4 in [28], $(2^E, \rho, 2^f)$ is topologically mixing. So, $(2^E, \rho, 2^f)$ is c-mixing. By Theorem 4.10, $(2^E, \rho, 2^f)$ is $\mathcal{F}$-sensitive. Since

$(2^{\Sigma(p)}, \rho, 2^{\sigma})$ is uniformly continuous, $(2^E, \rho, 2^f)$ is uniformly continuous. So, it follows from Theorem 4.7 that (E, f) is compact-type $\mathcal{F}$-collective sensitive.

By the above argument, $(2^E, \rho, 2^f)$ is topologically mixing. Hence, $(2^E, \rho, 2^f)$ is $\mathcal{B}$-mixing. It is easily seen that $\mathcal{B}$ is a full filterdual, and $\kappa\mathcal{B}$ is countably generated. From Theorem 4.5 in [31], we know that $(2^E, \rho, 2^f)$ is $(\mathcal{B}, \mathcal{B})$-sensitive. By Theorem 4.13, (E, f) is compact-type $(\mathcal{B}, \mathcal{B})$-collective sensitive. □

Remark 4.15. The result in the above example improves that of Examples 6.1 from [29]. It follows from the example that $(\Sigma(p), \sigma)$ is $\kappa\mathcal{B}-$collective sensitive and $(\mathcal{B}, \mathcal{B})$-collective sensitive.

Example 4.16 ([29, Example 6.2])**.** Let $E = (0, +\infty)$, equipped with the subspace topology of $R = (-\infty, +\infty)$. Let $f : E \to E$ be a continuous map defined as $f(x) = \frac{1}{x^2}$ for any $x \in E$. Now we consider two metrics d_1 and d_2 on E, where d_1 is the restriction of the usual metric of R (i.e., $d_1(x, y) = |x - y|$) and d_2 is defined as follows. Let $h(x) = 1 - \frac{1}{x}$ for $x \in (0, 1)$, and $h(x) = x - 1$ for $x \in [1, +\infty)$. Let S^1 be the circle $x^2 + (y - 1)^2 = 1$ (north pole $P = (0, 2)$ removed) equipped with the metric d_S induced by the usual metric of R^2, and let $g : R \to S^1$ be the stereographical projection with $g(0) = (0, 0)$. Clearly, h and g are homeomorphisms, which implies that $g \circ h : E \to S^1$ is a homeomorphism. Now, we define d_2 by $d_2(x, y) = d_S(g \circ h(x), g \circ h(y))$ for any $(x, y) \in E \times E$. Then (E, d_1, f) is $\mathcal{F}$-collectively sensitive and any $\delta > 0$ is a $\mathcal{F}$-collective sensitivity constant, and (E, d_2, f) is not $\mathcal{F}$-collectively sensitive, where $\mathcal{F}$ is the family of all cofinite subsets of Z^+.

Proof. It follows from Example 6.2 in [29] and its proof. □

Remark 4.17. The result of Example 4.16 improves that of Examples 6.2 in [29].

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