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Some approximate fixed point results and application on graph theory for partial (h-F)-generalized convex contraction mappings with special class of functions on complete metric space

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Abstract

In this paper, we introduce a new concept called partial (h-F)-generalized (and (h-F)-subgeneralized) convex contractions of order 3 (and with rank 3) using some auxiliary functions. Also we present some approximate fixed point results in metric space and approximate fixed point results in metric space endowed with a graph. Some examples are provided to illustrate the main results and to show the essentiality of the given hypotheses.

Keywords: ε-fixed point, α-admissible, partial (h-F)-generalized convex contractions of order 3, partial (h-F)-subgeneralized convex contractions of order 3, α-complete metric spaces, graph.
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1. Introduction

Over the past two decades the metric fixed point theory has played as a major tool for solving various problems in nonlinear functional analysis as well as useful tool for proving the existence theorems for nonlinear differential and integral equations. In many practical problems, the conditions in the fixed point theorems are too strong and so the existence of a fixed point does not need to be guaranteed. In this case, we can consider nearly fixed points or what so-called approximate fixed points.

In 2003, Tijs et al. [18], showed that under some weakenings the conditions of some well-known fixed point theorems the existence of approximate fixed points turns out to be still guaranteed. Inspired by the work of Tijs et al., Berinde [3] presented some fundamental approximate fixed point theorems in metric space. In 2013, Dey and Saha [7] studied the existence of approximate fixed point for the Reich operator

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[14] which in turn generalizes approximate fixed point theorems of Berinde [3]. There after many authors studied approximate fixed point results (see, for example, [6, 10, 13, 15] and the references therein). On the other hand, in 1982, Istratescu [9] introduced the concept of convex contractions and proved that each convex contraction mapping has a unique fixed point on a complete metric space. In 2013, Miandaragh et al. [12] extended the concept of convex contractions to generalized convex contractions and generalized convex contractions of order 2. They also established some approximate fixed point theorems for continuous mappings satisfying such contractive conditions in complete metric spaces. The results of Miandaragh et al. [12] was extended by Latif et al. [11].

In this paper, we introduce a new concept called partial (h-F)-generalized convex contractions of order 3 and partial (h-F)-subgeneralized convex contractions of order 3 with rank 3, to present some approximate fixed point results in metric space and approximate fixed point results in metric space endowed with a graph. Some examples are provided to illustrate the main results and to show the essentiality of the given hypotheses.

2. Preliminaries

The following definitions, examples and lemmas are needed in sequel.

Definition 2.1 ([15]). Let (X, d) be a metric space, $T : X \to X$ be a mapping and $\varepsilon > 0$ be a given real number. A point $x_0 \in X$ is said to be an ε-fixed point (approximate fixed point) of T, if

$$d(x_0, Tx_0) < \varepsilon.$$

For a metric space (X, d) and a given $\varepsilon > 0$, the set of all ε-fixed points of $T : X \to X$ is denoted by

$$F_\varepsilon(T) := \{x \in X : d(x, Tx) < \varepsilon\}.$$

Definition 2.2 ([10]). Let (X, d) be a metric space and $T : X \to X$ be a mapping. We say that T has the approximate fixed point property, if for all $\varepsilon > 0$, there exists an ε-fixed point of T, that is, for all $\varepsilon > 0$

$$F_\varepsilon(T) \neq \emptyset.$$

In 1996, Browder and Petryshyn [5] defined the following notions.

Definition 2.3 ([5]). A self-mapping T on a metric space (X, d) is said to be asymptotically regular at a point $x \in X$, if

$$d(T^n x, T^{n+1} x) \to 0, \quad \text{as } n \to \infty,$$

where $T^n x$ denotes the n-th iterate of T at x.

Lemma 2.4 ([11]). *Let (X, d) be a metric space and $T : X \to X$ be an asymptotically regular at a point $z \in X$, then T has the approximate fixed point property.*

In 2012, Samet et al. [17] introduced the concept of α-admissible mapping as follows:

Definition 2.5 ([17]). Let X be a non-empty set, $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$. We say that T is an α-admissible mapping, if for all $x, y \in X$, $\alpha(x, y) \geqslant 1$ implies $\alpha(Tx, Ty) \geqslant 1$.

Definition 2.6 ([2, 16]). Let $T : X \to X$, $\xi : X \times X \to \mathbb{R}^+$. We say that T is an ξ-subadmissible mapping, if $x, y \in X$, $\xi(x, y) \leqslant 1$ implies $\xi(Tx, Ty) \leqslant 1$.

Definition 2.7 ([8]). Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}^+$ be a mapping. The metric space X is said to be α-complete, if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, converges in X.

Definition 2.8. Let (X, d) be a metric space and $\mu : X \times X \to \mathbb{R}^+$ be a mapping. The metric space X is said to be μ-complete, if and only if every Cauchy sequence $\{x_n\}$ in X with $\mu(x_n, x_{n+1}) \leqslant 1$ for all $n \in \mathbb{N}$, converges in X.

Remark 2.9. If X is complete metric space, then X is α-complete (μ-complete) metric space. But the converse of above statement does not need to be true (see [11, example 2.9]).

Definition 2.10 ([11])**.** Let (X, d) be a metric space, $\alpha,\ \mu : X \times X \to [0, \infty)$ and $T : X \to X$ be mappings. We say that

- T is an α-continuous mapping on (X, d) if for each convergent sequence $\{x_n\}$ to x where $\alpha(x_n,x_{n+1}) \geqslant 1$ for all $n \in N$ implies $Tx_n \to Tx$.
- T is a μ-continuous mapping on (X, d), if for each convergent sequence $\{x_n\}$ to x where $\mu(x_n, x_{n+1}) \leqslant 1$ for all $n \in N$ implies $Tx_n \to Tx$.

Remark 2.11 ([11]). If T is a continuous mapping, then T is an α-continuous (μ-continuous) mapping, where $\alpha, \mu : X \times X \to [0, \infty)$ are arbitrary mappings.

Definition 2.12 ([1])**.** We say that $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I, if the following conditions are satisfied:

(i) h is continuous;

(ii) $y \leqslant h(1, y)$, for all $y \in \mathbb{R}^+$;

(iii) For $x \geqslant 1 \Rightarrow h(1, y) \leqslant h(x, y)$, for all $y \in \mathbb{R}^+$.

The following are the examples of function of subclass of type I for all $x, y \in \mathbb{R}^+$:

(1) $h(x, y) = (y + l)^x, l > 1$;

(2) $h(x, y) = (x + l)^y, l > 0$;

(3) $h(x, y) = xy$.

Definition 2.13 ([1])**.** Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. We say the pair (F, h) is a upper class of type I, if the following conditions are satisfied:

(i) F is a continuous function, h is a function of subclass of type I;

(ii) For $0 \leqslant s \leqslant 1 \Rightarrow F(s, t) \leqslant F(1, t)$;

(iii) $h(1, y) \leqslant F(s, t) \Rightarrow y \leqslant st$, for all $y, s, t \in \mathbb{R}^+$.

The following are the examples of function of upper class of type I for all $x, y, s, t \in \mathbb{R}^+$

(1) $h(x, y) = (y + l)^x,\ l > 1,\ F(s, t) = st + l$;

(2) $h(x, y) = (x + l)^y,\ l > 0,\ F(s, t) = (1 + l)^{st}$;

(3) $h(x, y) = xy,\ F(s, t) = st$;

(4) $h(x, y) = (y + l)^x,\ l > 1,\ F(s, t) = st + \frac{l}{k},\ k \geqslant 1$.

Definition 2.14. Let (X, d) be a metric space. The mapping $T : X \to X$ is called a partial (h-F)-generalized convex contraction of rank 3, if there exist a mapping $\alpha : X \times X \to \mathbb{R}^+$ and $a, b, c \geqslant 0$ with $a + b + c < 1$, satisfies the following condition: For all $x, y \in X$ with $\alpha(x, y) \geqslant 1$ implies

$$h(\alpha(x, y), d(T^3x, T^3y)) \leqslant F(1, ad(T^2x, T^2y) + bd(Tx, Ty) + cd(x, y)),$$

where pair (F, h) is a upper class of type I.

Definition 2.15. Let (X, d) be a metric space. The mapping $T : X \to X$ is called a partial (h-F)-subgeneralized convex contraction of rank 3, if there exist a mapping $\mu : X \times X \to \mathbb{R}^+$ and $a, b, c \in [0, \infty)$ with $a + b + c < 1$, satisfies the following condition:

For all $x, y \in X$ with $\mu(x, y) \leqslant 1$ implies

$$h(1, d(T^3x, T^3y)) \leqslant F(\mu(x, y), ad(T^2x, T^2y) + bd(Tx, Ty) + cd(x, y)),$$

where pair (F, h) is an upper class of type I .

3. Main results

In this section, we establish new approximate fixed point results for partial (h-F)-generalized (and (h-F)-subgeneralized) convex contraction mappings in α (and μ)-complete metric spaces.

Theorem 3.1. *Let (X, d) be a metric space and $T : X \to X$ be a partial (h-F)-generalized convex contraction of rank 3 with the mapping $\alpha : X \times X \to \mathbb{R}^+$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Then T has the approximate fixed point property. In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point. Moreover, if for each two fixed points x, y of T there exists $z \in X$ such that $\alpha(x, z) \geqslant 1$ and $\alpha(y, z) \geqslant 1$, then T has a unique fixed point.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = T^{n+1}(x_0) = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$. Then, it is clear that x_{n_0} is a fixed point of T and hence the proof is completed. From now on, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Since T is an α-admissible mapping and $\alpha(x_0, Tx_0) \geqslant 1$, we have $\alpha(x_1, Tx_1) = \alpha(Tx_0, T^2x_0) \geqslant 1$. Continuing in this manner, we get that $\alpha(x_n, Tx_n) = \alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that $\gamma = d(T^2x_0, T^3x_0) + d(Tx_0, T^2x_0) + d(x_0, Tx_0) = d(x_2, x_3) + d(x_1, x_2) + d(x_0, x_1)$, and $\beta = a + b + c$. Now since T is a partial (h-F)-generalized convex contraction of rank 3, we have

$$\begin{aligned} h(1, d(x_{n+3}, x_{n+4})) &\leqslant h(\alpha(x_n, x_{n+1}), d(x_{n+3}, x_{n+4})) \\ &\leqslant F(1, ad(x_{n+2}, x_{n+3}) + bd(x_{n+1}, x_{n+2}) + cd(x_n, x_{n+1})), \end{aligned}$$

which implies that for each $n \in \mathbb{N} \cup \{0\}$

$$d(x_{n+3}, x_{n+4}) \leqslant a\, d(x_{n+2}, x_{n+3}) + b\, d(x_{n+1}, x_{n+2}) + c\, d(x_n, x_{n+1}).$$

Therefore

$$d(x_3, x_4) \leqslant a\, d(x_2, x_3) + b\, d(x_1, x_2) + c\, d(x_0, x_1) \leqslant \beta\gamma. \tag{3.1}$$

Also,

$$\begin{aligned} d(x_4, x_5) &\leqslant a\, d(x_3, x_4) + b\, d(x_2, x_3) + c\, d(x_1, x_2) \\ &\leqslant a^2\, d(x_2, x_3) + ab\, d(x_1, x_2) + ac\, d(x_0, x_1) + b\, d(x_2, x_3) + c\, d(x_1, x_2) \\ &\leqslant a\, d(x_2, x_3) + a\, d(x_1, x_2) + a\, d(x_0, x_1) + b\, d(x_2, x_3) + c\, d(x_1, x_2) \\ &= (a + b)\, d(x_2, x_3) + (a + c)\, d(x_1, x_2) + a\, d(x_0, x_1) \\ &\leqslant \beta d(x_2, x_3) + \beta d(x_1, x_2) + \beta d(x_0, x_1) \leqslant \beta\gamma, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} d(x_5, x_6) &\leqslant a\, d(x_4, x_5) + b\, d(x_3, x_4) + c\, d(x_2, x_3) \\ &\leqslant a[(a + b)\, d(x_2, x_3) + (a + c)\, d(x_1, x_2) + a\, d(x_0, x_1)] \\ &\quad + b[a\, d(x_2, x_3) + b\, d(x_1, x_2) + c\, d(x_0, x_1)] + c\, d(x_2, x_3) \\ &\leqslant a\, d(x_2, x_3) + a\, d(x_1, x_2) + a\, d(x_0, x_1) + b\, d(x_2, x_3) + b\, d(x_1, x_2) + b\, d(x_0, x_1) + c\, d(x_2, x_3) \\ &= (a + b + c)\, d(x_2, x_3) + (a + b)\, d(x_1, x_2) + (a + c)\, d(x_0, x_1) \\ &\leqslant \beta d(x_2, x_3) + \beta d(x_1, x_2) + \beta d(x_0, x_1) \leqslant \beta\gamma. \end{aligned} \tag{3.3}$$

From (3.1), (3.2), (3.3) and $a, b, c \leqslant \beta$, we have

$$d(x_6, x_7) \leqslant ad(x_5, x_6) + bd(x_4, x_5) + cd(x_3, x_4) \leqslant a\beta\gamma + b\beta\gamma + c\,\beta\gamma \leqslant \beta^2\gamma. \tag{3.4}$$

Also from (3.2), (3.3), (3.4) and $a, b, c \leqslant \beta, \beta^2 \leqslant \beta$, we have

$$d(x_7, x_8) \leqslant ad(x_6, x_7) + bd(x_5, x_6) + cd(x_4, x_5) \leqslant a\beta\gamma + b\beta\gamma + c\,\beta\gamma \leqslant \beta^2\gamma. \tag{3.5}$$

Now using (3.3), (3.4), (3.5) and $a, b, c \leqslant \beta, \beta^2 \leqslant \beta$, we have

$$d(x_8, x_9) \leqslant ad(x_7, x_8) + bd(x_6, x_7) + cd(x_5, x_6) \leqslant a\beta\gamma + b\beta\gamma + c\,\beta\gamma \leqslant \beta^2\gamma.$$

By continuing in this manner, it is easy to see that

$$d(x_n, x_{n+1}) \leqslant \beta^l\gamma, \quad n \in \mathbb{N}, \tag{3.6}$$

where $n = 3l + r$, $r \in \{0, 1, 2\}$ for all $l \in \mathbb{N}$. This implies that

$$d(x_n, x_{n+1}) = d(T^n x_0, T^{n+1} x_0) \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore, T is an asymptotically regular at a point $x_0 \in X$.

By using Lemma 2.4, we conclude that T has the approximate fixed point property.

Next, we show that T has a fixed point provides that T is α-continuous and (X, d) is an α-complete metric space.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N} \cup \{0\}$ such that $n > m$, then by (3.6)

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leqslant \sum_{j=0}^{s} \beta^l\gamma + \beta^{l+1}\gamma + \beta^{l+1}\gamma + \beta^{l+1}\gamma + \cdots \quad (\text{where } s \in \{1, 2, 3\}) \\ &\leqslant \beta^l\gamma + \beta^l\gamma + \beta^l\gamma + \beta^{l+1}\gamma + \beta^{l+1}\gamma + \beta^{l+1}\gamma + \cdots \\ &\leqslant \sum_{i=l}^{\infty} 3\beta^i\gamma = \frac{3\beta^l\gamma}{1-\beta}. \end{aligned}$$

Therefore, for $n > m$ and when $m = 3l$ or $m = 3l + 1$ or $m = 3l + 2$ for all $l \in N$

$$d(x_m, x_n) \leqslant \frac{3\beta^l\gamma}{1-\beta} \to 0, \quad \text{as} \quad m \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. Now as $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and using α-completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Also, as a result of T is α-continuous, then $Tx_n \to Tx^*$ as $n \to \infty$. Therefore

$$Tx^* = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus T has a fixed point. To show the uniqueness, let x^* and y^* be fixed points of T, then by hypothesis there is $z \in X$ such that $\alpha(x^*, z) \geqslant 1$ and $\alpha(y^*, z) \geqslant 1$. Since T is α-admissible, we get $\alpha(x^*, T^m z) \geqslant 1$ and $\alpha(y^*, T^m z) \geqslant 1$ for all $m \in N$. Assume that $\nu = d(x^*, T^3 z) + d(x^*, T^2 z) + d(x^*, Tz)$, and $\beta = a + b + c$

$$\begin{aligned} h(1, d(x^*, T^{m+3} z)) &\leqslant h(\alpha(x^*, T^m z), d(x^*, T^{m+3} z)) \\ &\leqslant F(1, ad(x^*, T^{m+2} z) + bd(x^*, T^{m+1} z) + cd(x^*, T^m z)), \end{aligned}$$

which implies

$$d(x^*, T^{m+3} z) \leqslant ad(x^*, T^{m+2} z) + bd(x^*, T^{m+1} z) + cd(x^*, T^m z).$$

Therefore,

$$d(x^*, T^4 z) \leqslant ad(x^*, T^3 z) + bd(x^*, T^2 z) + cd(x^*, Tz) \leqslant \beta\nu. \tag{3.7}$$

Using (3.7) we have

$$d(x^*,T^5z) \leqslant ad(x^*,T^4z)+bd(x^*,T^3z)+cd(x^*,T^2z) \leqslant \beta\nu. \tag{3.8}$$

Using (3.7), (3.8) we have

$$d(x^*,T^6z) \leqslant ad(x^*,T^5z)+bd(x^*,T^4z)+cd(x^*,T^3z) \leqslant \beta\nu.$$

By continuing the same process as in proof of Theorem 3.1, we get

$$d(x^*,T^nz) \leqslant \beta^l\nu,$$

where $n = 3l + r$, $r \in \{0,1,2\}$ for all $l \in \mathbb{N}$. This implies that $d(x^*,T^nz) \to 0$ as $n \to \infty$. Similarly, we can prove that $d(y^*,T^nz) \to 0$ as $n \to \infty$. By the uniqueness of limit, we have $x^* = y^*$ and then T has a unique fixed point. This completes the proof. □

Corollary 3.2. *Let (X,d) be a metric space, $\alpha : X\times X \to \mathbb{R}^+$ and $T : X \to X$ satisfied the following contraction:*

For all $x,y \in X$ with $\alpha(x,y) \geqslant 1$ implies

$$d(T^3x,T^3y) \leqslant ad(T^2x,T^2y)+bd(Tx,Ty)+cd(x,y),$$

where $a+b+c < 1$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geqslant 1$. Then T has the approximate fixed point property. In addition, if T is α-continuous and (X,d) is an α-complete metric space, then T has a fixed point.

Moreover, if for each two fixed points x,y of T there exists $z \in X$ such that $\alpha(x,z) \geqslant 1$ and $\alpha(y,z) \geqslant 1$, then T has a unique fixed point.

Proof. By taking $h(x,y) = y$ and $F(s,t) = st$, the mapping T is partial (h-F)-generalized convex contraction of rank 3, so the result follows from Theorem 3.1. □

Corollary 3.3. *Let (X,d) be a metric space, $\alpha : X\times X \to \mathbb{R}^+$ and $T : X \to X$ satisfies the following contraction:*

For all $x,y \in X$ with $\alpha(x,y) \geqslant 1$ implies

$$(\alpha(x,y)+\tau)^{d(T^3x,T^3y)} \leqslant (1+\tau)^{ad(T^2x,T^2y)+bd(Tx,Ty)+cd(x,y)},$$

where $a+b+c < 1$ and $\tau > 0$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geqslant 1$. Then T has the approximate fixed point property. In addition, if T is α-continuous and (X,d) is an α-complete metric space, then T has a fixed point.

Moreover, if for each two fixed points x,y of T there exists $z \in X$ such that $\alpha(x,z) \geqslant 1$ and $\alpha(y,z) \geqslant 1$, then T has a unique fixed point.

Proof. By taking $h(x,y) = (x+\tau)^y$ and $F(s,t) = (1+\tau)^{st}$, where $\tau > 0$, the mapping T is partial (h-F)-generalized convex contraction of rank 3, so the result follows from Theorem 3.1. □

The following example illustrates Theorem 3.1 and shows that the existence of $z \in X$ for any two fixed points $x,y \in X$ such that $\alpha(x,z) \geqslant 1$ and $\alpha(y,z) \geqslant 1$ is essential to guarantee the uniqueness of the fixed point.

Example 3.4. Let $X = (0,\infty)$ and $d : X\times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define $T : X \to X$ and $\alpha : X\times X \to [0,\infty)$ by

$$Tx = \begin{cases} \frac{x+4}{5}, & x \in [1,3], \\ \frac{3x-4}{2}, & x \in (3,5), \\ 5x+20, & \text{otherwise}, \end{cases}$$

$h(x,y) = xy,\ F(s,t) = st$, and

$$\alpha(x,y) = \begin{cases} 1, & x,y \in [1,3], \\ 0, & \text{otherwise}. \end{cases}$$

For $\alpha(x,y) \geqslant 1$, we have $x, y \in [1,3]$ and so,

$$\begin{aligned}
h(\alpha(x,y), d(T^3x, T^3y)) = d(T^3x, T^3y) &\leqslant d\left(T^2\frac{x+4}{5}, T^2\frac{y+4}{5}\right) \\
&= d\left(T\frac{x+24}{25}, T\frac{y+24}{25}\right) \\
&= d\left(\frac{x+124}{125}, \frac{y+124}{125}\right) \\
&= \frac{1}{125}|x-y| \\
&\leqslant \frac{3}{4}\left|\frac{x+24}{25} - \frac{y+24}{25}\right| + \frac{2}{16}\left|\frac{x+4}{5} - \frac{y+4}{5}\right| + \frac{1}{64}|x-y| \\
&= ad(T^2x, T^2y) + bd(Tx, Ty) + cd(x,y) \\
&= F(1, ad(T^2x, T^2y) + bd(Tx, Ty) + cd(x,y)).
\end{aligned}$$

Therefore, T is (h-F)-partial generalized convex contraction with $a = \frac{3}{4}$, $b = \frac{1}{8}$ and $c = \frac{1}{64}$.

Moreover, it is easy to see that T is an α-admissible and there exists $x_0 = 2 \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(2, T(2)) = \alpha(2, \frac{6}{5}) \geqslant 1.$$

It is easy to see that all conditions of Theorem 3.1 are satisfied, $x = 1$, $x = 4$ are two fixed points of T and there is no $z \in X$ such that $\alpha(1, z) \geqslant 1$ and $\alpha(4, z) \geqslant 1$.

Theorem 3.5. *Let (X, d) be a metric space and $T : X \to X$ be a partial (h-F)-subgeneralized convex contraction of rank 3 with the mapping $\mu : X \times X \to \mathbb{R}^+$. Assume that T is μ-subadmissible and there exists $x_0 \in X$ such that $\mu(x_0, Tx_0) \leqslant 1$. Then T has the approximate fixed point property.*

In addition, if T is μ-continuous and (X, d) is μ-complete metric space, then T has a fixed point. Moreover, if for each two fixed points x, y of T, there exists $z \in X$ such that $\mu(x, z) \leqslant 1$ and $\mu(y, z) \leqslant 1$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ such that $\mu(x_0, Tx_0) \leqslant 1$. Define a sequence $\{x_n\} \subset X$ as in the proof of Theorem 3.1. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then it is clear that x_{n_0} is a fixed point of T and hence the proof is completed.

Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a μ-subadmissible mapping and $\mu(x_0, Tx_0) \leqslant 1$, we deduce that $\mu(x_1, Tx_1) = \mu(Tx_0, T^2x_0) \leqslant 1$, continuing this process, we deduce that $\mu(x_n, Tx_n) = \mu(x_n, x_{n+1}) \leqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$.

The rest of the proof follows directly from the same argument as in the proof of Theorem 3.1, by replacing $\alpha(x, y)$ by $\mu(x, y)$.

□

Definition 3.6. Let (X, d) be a metric space. The mapping $T : X \to X$ is called a partial (h-F)-generalized convex contraction of rank 3 and order 3, if there exist a mapping $\alpha : X \times X \to \mathbb{R}^+$ and constants $a_1, a_2, a_3, b_1, b_2, b_3 \in [0, 1)$, with $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 < 1$, satisfies the following condition:

For all $x, y \in X$, with $\alpha(x, y) \geqslant 1$ implies

$$\begin{aligned}
h(\alpha(x,y), d(T^3x, T^3y)) \leqslant F(1, &a_3 d(T^2x, T^3x) + a_2 d(Tx, T^2x) + a_1 d(x, Tx) \\
&+ b_3 d(T^2y, T^3y) + b_2 d(Ty, T^2y) + b_1 d(y, Ty)),
\end{aligned} \tag{3.9}$$

where the pair (F, h) is an upper class of type I.

Definition 3.7. Let (X, d) be a metric space. The mapping $T : X \to X$ is called a partial (h-F)-subgeneralized convex contraction of rank 3 and order 3, if there exist a mapping $\mu : X \times X \to \mathbb{R}^+$ and $a_1, b_1, a_2, b_2, a_3, b_3 \in [0,1)$, with $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 < 1$, satisfies the following condition:

For all $x, y \in X$, with $\mu(x,y) \leqslant 1$ implies

$$\begin{aligned} h(1, d(T^3x, T^3y)) \leqslant F(\mu(x,y), &a_3 d(T^2x, T^3x) + a_2 d(Tx, T^2x) + a_1 d(x, Tx) \\ &+ b_3 d(T^2y, T^3y) + b_2 d(Ty, T^2y) + b_1 d(y, Ty)), \end{aligned}$$

where the pair (F, h) is an upper class of type I.

Theorem 3.8. *Let (X, d) be a metric space and $T : X \to X$ be a partial $(h\text{-}F)$-generalized convex contraction of rank 3 and order 3 with the based mapping $\alpha : X \times X \to \mathbb{R}^+$. Assume that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Then T has the approximate fixed point property.*

In addition, if T is α-continuous and (X, d) is an α-complete metric space, then T has a fixed point. Moreover, if $\alpha(x, y) \geqslant 1$ for each two fixed points x, y of T then T has a unique fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Define a sequence $\{x_n\} \subseteq X$ as in Theorem 3.1 by $x_{n+1} = T^{n+1}(x_0) = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Since the result is clear if $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, we consider $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By the assumption of the mapping T is an α-admissible mapping and $\alpha(x_0, Tx_0) \geqslant 1$, we deduce that $\alpha(x_n, Tx_n) = \alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Assume that $s = d(T^2x_0, T^3x_0) + d(Tx_0, T^2x_0) + d(x_0, Tx_0) = d(x_2, x_3) + d(x_1, x_2) + d(x_0, x_1)$, $\xi = 1 - b_3$ and $\sigma = a_1 + a_2 + a_3 + b_1 + b_2$. Using that T is partial h-F-generalized convex contraction, we have

$$\begin{aligned} h(1, d(x_{n+3}, x_{n+4})) &\leqslant h(\alpha(x_n, x_{n+1}), d(x_{n+3}, x_{n+4})) \\ &\leqslant F(1, a_3 d(T^{n+2}x_0, T^{n+3}x_0) + a_2 d(T^{n+1}x_0, T^{n+2}x_0) + a_1 d(T^n x_0, T^{n+1}x_0) \\ &\quad + b_3 d(T^{n+3}x_0, T^{n+4}x_0) + b_2 d(T^{n+2}x_0, T^{n+3}x_0) + b_1 d(T^{n+1}x_0, T^{n+2}x_0)), \end{aligned}$$

which implies that,

$$\begin{aligned} d(x_{n+3}, x_{n+4}) \leqslant\ &a_3 d(x_{n+2}, x_{n+3}) + a_2 d(x_{n+1}, x_{n+2}) + a_1 d(x_n, x_{n+1}) \\ &+ b_3 d(x_{n+3}, x_{n+4}) + b_2 d(x_{n+2}, x_{n+3}) + b_1 d(x_{n+1}, x_{n+2}). \end{aligned} \tag{3.10}$$

Using above inequality for $n = 0$, we have

$$\begin{aligned} d(x_3, x_4) &\leqslant a_3 d(x_2, x_3) + a_2 d(x_1, x_2) + a_1 d(x_0, x_1) \\ &\quad + b_3 d(x_3, x_4) + b_2 d(x_2, x_3) + b_1 d(x_1, x_2) \\ &\leqslant a_1 s + (b_2 + a_3)s + (b_1 + a_2)s + b_3 d(x_3, x_4). \end{aligned}$$

Thus,

$$d(x_3, x_4) \leqslant \frac{\sigma}{\xi} s. \tag{3.11}$$

From inequality (3.10) (with $n = 1$) and (3.11) we get

$$\begin{aligned} d(x_4, x_5) &\leqslant a_3 d(x_3, x_4) + a_2 d(x_2, x_3) + a_1 d(x_1, x_2) \\ &\quad + b_3 d(x_4, x_5) + b_2 d(x_3, x_4) + b_1 d(x_2, x_3) \\ &\leqslant a_1 s + (b_2 + a_3)\frac{\sigma}{\xi} s + (b_1 + a_2)s + b_3 d(x_4, x_5) \\ &\leqslant a_1 s + (b_2 + a_3)s + (b_1 + a_2)s + b_3 d(x_4, x_5). \end{aligned}$$

It gives,

$$d(x_4, x_5) \leqslant \frac{\sigma}{\xi} s. \tag{3.12}$$

Again using (3.10) (with $n = 2$), (3.11) and (3.12) we have

$$\begin{aligned}
d(x_5,x_6) &\leqslant a_3 d(x_4,x_5) + a_2 d(x_3,x_4) + a_1 d(x_2,x_3) \\
&\quad + b_3 d(x_5,x_6) + b_2 d(x_4,x_5) + b_1 d(x_3,x_4) \\
&= a_1 d(x_2,x_3) + (a_2+b_1) d(x_3,x_4) + (a_3+b_2) d(x_4,x_5) + b_3 d(x_5,x_6) \\
&\leqslant a_1 s + (a_2+b_1)\frac{\sigma}{\xi} s + (a_3+b_2)\frac{\sigma}{\xi} s + b_3 d(x_5,x_6) \\
&\leqslant a_1 s + (a_2+b_1) s + (a_3+b_2) s + b_3 d(x_5,x_6).
\end{aligned}$$

Hence,

$$d(x_5,x_6) \leqslant \frac{\sigma}{\xi} s. \tag{3.13}$$

Moreover, by (3.10) (with $n = 3$), (3.11), (3.12) and (3.13) we have

$$\begin{aligned}
d(x_6,x_7) &\leqslant a_3 d(x_5,x_6) + a_2 d(x_4,x_5) + a_1 d(x_3,x_4) \\
&\quad + b_3 d(x_6,x_7) + b_2 d(x_5,x_6) + b_1 d(x_4,x_5) \\
&= a_1 d(x_3,x_4) + (a_2+b_1) d(x_4,x_5) + (a_3+b_2) d(x_5,x_6) + b_3 d(x_6,x_7) \\
&\leqslant a_1 \frac{\sigma}{\xi} s + (a_2+b_1)\frac{\sigma}{\xi} s + (a_3+b_2)\frac{\sigma}{\xi} s + b_3 d(x_5,x_6),
\end{aligned}$$

and so,

$$d(x_6,x_7) \leqslant (\frac{\sigma}{\xi})^2 s.$$

By continuing in the same process as in proof of Theorem 3.1 we get

$$d(x_n,x_{n+1}) \leqslant (\frac{\sigma}{\xi})^l s,$$

where $n = 3l + r$, $r \in \{0,1,2\}$. This implies that

$$d(x_n,x_{n+1}) = d(T^n x_0, T^{n+1} x_0) \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore, T is an asymptotically regular at a point $x_0 \in X$, then by Lemma 2.4, we conclude that T has the approximate fixed point property.

The proof of existence of fixed point of T can be done by replacing β by $\frac{\sigma}{\xi}$ in proof of Theorem 3.1.

Now, we will show the uniqueness of fixed point under our assumption. Assume that x, y are two fixed points of T. By the assumption $\alpha(x,y) \geqslant 1$, by condition (3.9) and the properties of h and F we have

$$\begin{aligned}
h(1, d(x,y)) &\leqslant h(\alpha(x,y), d(T^3x, T^3y)) \\
&= h(\alpha(x,y), d(x,y)) \\
&\leqslant F(1, a_3 d(T^2x, T^3x) + a_2 d(Tx, T^2x) + a_1 d(x, Tx) \\
&\quad + b_3 d(T^2y, T^3y) + b_2 d(Ty, T^2y) + b_1 d(y, Ty)),
\end{aligned}$$

therefore,

$$\begin{aligned}
d(x,y) &\leqslant a_3 d(T^2x, T^3x) + a_2 d(Tx, T^2x) + a_1 d(x, Tx) \\
&\quad + b_3 d(T^2y, T^3y) + b_2 d(Ty, T^2y) + b_1 d(y, Ty) \\
&= 0,
\end{aligned}$$

which implies that $x = y$ □

Corollary 3.9. *Let* (X, d) *be a metric space,* $\alpha : X \times X \to \mathbb{R}^+$ *and* $T : X \to X$ *be mappings satisfy the following contraction:*

For all $x, y \in X$ *with* $\alpha(x, y) \geqslant 1$ *implies*

$$\begin{aligned} d(T^3x, T^3y) \leqslant\ & a_3 d(T^2x, T^3x) + a_2 d(Tx, T^2x) + a_1 d(x, Tx) \\ & + b_3 d(T^2y, T^3y) + b_2 d(Ty, T^2y) + b_1 d(y, Ty), \end{aligned}$$

where $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 < 1$. *Assume that* T *is* α*-admissible and there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$. *Then* T *has the approximate fixed point property. In addition, if* T *is* α*-continuous and* (X, d) *is an* α*-complete metric space, then* T *has a fixed point.*

Moreover, if $\alpha(x, y) \geqslant 1$ *for each two fixed points* x, y *of* T *then* T *has a unique fixed point.*

Proof. By taking $h(x, y) = y$ and $F(s, t) = st$, the mapping T is partial (h-F)-generalized convex contraction of rank 3, so the result follows from Theorem 3.8. □

Corollary 3.10. *Let* (X, d) *be a metric space,* $\alpha : X \times X \to \mathbb{R}^+$ *and* $T : X \to X$ *be mappings satisfy the following contraction:*

For all $x, y \in X$ *with* $\alpha(x, y) \geqslant 1$ *implies*

$$\begin{aligned} (d(T^3x, T^3y) + \tau)^{\alpha(x,y)} \leqslant\ & a_3 d(T^2x, T^3x) + a_2 d(Tx, T^2x) + a_1 d(x, Tx) \\ & + b_3 d(T^2y, T^3y) + b_2 d(Ty, T^2y) + b_1 d(y, Ty) + \frac{\tau}{\zeta}, \end{aligned}$$

where $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 < 1$, $\tau > 1$ *and* $\zeta \geqslant 1$. *Assume that* T *is* α*-admissible and there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$. *Then* T *has the approximate fixed point property. In addition, if* T *is* α*-continuous and* (X, d) *is an* α*-complete metric space, then* T *has a fixed point.*

Moreover, if $\alpha(x, y) \geqslant 1$ *for each two fixed points* x, y *of* T *then* T *has a unique fixed point.*

Proof. By taking $h(x, y) = (y + \tau)^x$ and $F(s, t) = st + \frac{\tau}{\zeta}$, where $\tau > 1$, $\zeta \geqslant 1$, the mapping T is partial (h-F)-generalized convex contraction of rank 3, so the result follows from Theorem 3.8. □

Theorem 3.11. *Let* (X, d) *be a metric space and* $T : X \to X$ *be a partial* (h-F)*-subgeneralized convex contraction of rank 3 and order 3 with the mapping* $\mu : X \times X \to \mathbb{R}^+$. *Assume that* T *is* μ*-subadmissible and there exists* $x_0 \in X$ *such that* $\mu(x_0, Tx_0) \leqslant 1$. *Then* T *has the approximate fixed point property.*

In addition, if T *is* μ*-continuous and* (X, d) *is a* μ*-complete metric space, then* T *has a fixed point. Moreover, if* $\mu(x, y) \leqslant 1$ *for each two fixed points* x, y *of* T *then* T *has a unique fixed point.*

Proof. The proof follows from the same argument as in proof of Theorem 3.8. □

4. Approximate fixed point with graph

In this section, we give the existence of approximate fixed point property on a metric space endowed with graph. Before presenting our results, we give the following notions and definitions.

Let (X, d) be a metric space. We assume that G is a reflexive graph with the vertex set $V(G) = X$ and the edge set E(G) contains no multiple edges. So we can identify G with the pair $(V(G), E(G))$. Wherever it appears Δ means the set of all edges $\{(x, x); x \in X\}$.

Our graph theory notations and terminology are standard and can be found in any graph theory books, such as ([4]).

Definition 4.1. Given a graph G. Then

1. An infinite path is a sequence of vertices $(x_n)_{n=0}^{\infty}$ such that $(x_{n-1}, x_n) \in E(G)$ for $n = 1, 2, \cdots$.
2. An infinite path $(x_n)_{n=0}^{\infty}$ is a Cauchy path if and only if $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence in (X, d).

3. An infinite path $(x_n)_{n=0}^{\infty}$ is called convergent path if and only if $(x_n)_{n=0}^{\infty}$ is convergent sequence in (X, d).
4. The metric space X is said to be $E(G)$-complete if and only if every Cauchy path $\{x_n\}$ in G converges in (X, d).

Definition 4.2. Let X be a nonempty set endowed with a graph G. and $T : X \to X$ be a mapping. Then

1. T preserves edge, if for every $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$.
2. T is an $E(G)$-continuous mapping on (X, d), if for each convergent infinite path $\{x_n\}$ to $x \in X$ we have $Tx_n \to Tx$.

Theorem 4.3. *Let (X, d) be a metric space endowed with a graph G and a mapping $T : X \to X$ satisfies the following contraction:*

For all $(x, y) \in E(G)$ we have

$$d(T^3x, T^3y) \leqslant a d(T^2x, T^2y) + b d(Tx, Ty) + c d(x, y), \tag{4.1}$$

where $a + b + c < 1$. Assume that T preserves edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. Then T has the approximate fixed point property. In addition, if T is $E(G)$-continuous and (X, d) is an $E(G)$-complete metric space, then T has a fixed point. Moreover, if for any two fixed points x and y, there exists $z \in X$ such that $(x, z) \in E(G)$ and $(z, y) \in E(G)$, then T has a unique fixed point.

Proof. Consider a mapping $\alpha : X \times X \to [0, \infty)$ to be defined by:

$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in E(G), \\ 0, & \text{otherwise}. \end{cases}$$

For any $x, y \in X$ such that $\alpha(x, y) \geqslant 1$ implies $(x, y) \in E(G)$ and T satisfies condition (4.1).

By assumption, there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, so by definition of α we have $\alpha(x_0, Tx_0) = 1$. Since T preserves edges, then clearly T is α-admissible mapping.

Thus, by Corollary 3.2, T has approximate fixed point property. One can easily see that the rest of hypotheses of Corollary 3.2 are satisfied, so T has a fixed point which is unique. □

Theorem 4.4. *Let (X, d) be a metric space endowed with a graph G and a mapping $T : X \to X$ satisfies the following contraction:*

For all $(x, y) \in E(G)$ we have

$$\begin{aligned} d(T^3x, T^3y) \leqslant\ & a_1 d(T^2x, T^3x) + a_2 d(Tx, T^2x) + a_3 d(x, Tx) + b_1 d(T^2y, T^3y) + b_2 d(Ty, T^2y) \\ & + b_3 d(y, Ty), \end{aligned}$$

where $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 < 1$. Assume that T preserves edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. Then T has the approximate fixed point property. In addition, if T is $E(G)$-continuous and (X, d) is an $E(G)$-complete metric space, then T has a fixed point. Moreover, if for any two fixed points x and y we have $(x, y) \in E(G)$, then T has a unique fixed point.

Proof. The proof follows from Corollary 3.9 by similar argument to proof of Theorem 4.3. □

The following example shows that the assumption " there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$" is essential.

Example 4.5. Let $X = [0, 1]$ and $d : X \times X \to R$ be defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \frac{1}{3}, & x \in [0, \frac{1}{4}], \\ \frac{x}{9}, & (\frac{1}{4}, 1], \end{cases}$$

and the edge set of the graph G

$$E(G) = \{(0, y) : y \in [0, \frac{1}{4}]\} \cup \Delta.$$

Now, it is easy to see that for any $(x, y) \in E(G)$, we have either $y = x$ (which implies condition (4.1) is satisfied), or $x = 0,\ y \in [0, \frac{1}{4}]$, and so

$$d(0, y) = y, \quad d(T0, Ty) = d(\frac{1}{3}, \frac{1}{3}) = 0, \quad d(T^2 0, T^2 y) = d(\frac{1}{27}, \frac{1}{27}) = 0,$$
$$d(T^3 0, T^3 y) = d(\frac{1}{3}, \frac{1}{3}) = 0.$$

Hence,

$$\begin{aligned} d(T^3 0, T^3 y) &\leqslant a(0) + b(0) + c\,y \\ &= a\,d(T^2 0, T^2 y) + b\,d(T0, Ty) + c\,d(0, y). \end{aligned}$$

Therefore, T satisfies condition (4.1) with $a = b = c = \frac{1}{16}$. To show that T is an $E(G)$-continuous, the tail of any convergent path x_n to x, should be either

(i) a constant path; or

(ii) subset of $[0, \frac{1}{4}]$ such that the non-zero term of the path converges to 0.

If (i) is the case, then $x_n = (c) \to c$. If $c \in [0, \frac{1}{4}]$, then $T(x_n) = \frac{1}{3} \to \frac{1}{3} = T(c)$, if $c \in (\frac{1}{4}, 1]$ then $T(x_n) = \frac{c}{9} \to \frac{c}{9} = T(c)$. If (ii) is the case, then the tail of $T(x_n) = \frac{1}{3} \to \frac{1}{3} = T(0)$. Therefore, T is $E(G)$-continuous.

By using argument as above one can show that $([0, 1], d)$ is an $E(G)$-complete. Moreover, it is clear that there is no $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.

One can note that the mapping T has no fixed point. Furthermore, T has no approximate fixed point property. For instance assume there is an $x \in F_{\frac{1}{100}}(T)$. Then $d(x, Tx) < \frac{1}{100}$. Thus, if $x \in [0, \frac{1}{4}]$, then $|x - \frac{1}{3}| < \frac{1}{100}$ which implies $x > \frac{97}{300}$ which is a contradiction. If $x \in (\frac{1}{4}, 1]$, then $|x - \frac{x}{9}| < \frac{1}{100}$ which implies $x < \frac{9}{800}$ which is a contradiction.

Example 4.6. Let $X = (0, \infty)$ and $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to X$ by

$$Tx = \begin{cases} \frac{x+4}{3}, & x \in [2, 5], \\ 5x + 10, & \text{otherwise}, \end{cases}$$

and the edge set of the graph G,

$$E(G) = \{(2, 2 + \frac{1}{n}) : n \in \mathbb{N}\} \cup \Delta.$$

For $(x, y) \in E(G)$, either $(x, y) = (2, 2 + \frac{1}{n})$ or $(x, y) = (x, x)$. If $(x, y) = (x, x)$, then $(Tx, Tx) \in E(G)$ and it is clear that condition (4.1) is satisfied. If $(x, y) = (2, 2 + \frac{1}{n})$, then $(T2, T(2 + \frac{1}{n})) = (2, 2 + \frac{1}{3n}) \in E(G)$. Moreover,

$$\begin{aligned} d(2, 2 + \frac{1}{n}) &= \frac{1}{n}, \\ d(T2, T(2 + \frac{1}{n})) &= \frac{1}{3n}, \\ d(T^2 2, T^2(2 + \frac{1}{n})) &= \frac{1}{9n}, \\ d(T^3 2, T^3(2 + \frac{1}{n})) &= \frac{1}{27n}. \end{aligned}$$

Hence,

$$\begin{aligned} d(T^3 2, T^3(2+\frac{1}{n})) &= \frac{1}{27n} \\ &\leqslant \frac{1}{40}\frac{1}{9n} + \frac{1}{40}\frac{1}{3n} + \frac{1}{40}\frac{1}{n} \\ &= ad(T^2 2, T^2(2+\frac{1}{n})) + bd(T2, T(2+\frac{1}{n})) + cd(2, 2+\frac{1}{n}). \end{aligned}$$

Thus, T satisfies condition (4.1) with $a = b = c = \frac{1}{40}$. Moreover, $(x_0, Tx_0) = (2, T(2)) = (2, 2) \in E(G)$. To show that the mapping T is $E(G)$-continuous follow the same argument as in previous example, also it is clear that (X, d) is $E(G)$-complete. Therefore, all conditions of Theorem 4.3 are satisfied and $x = 2$ is the unique fixed point of T.

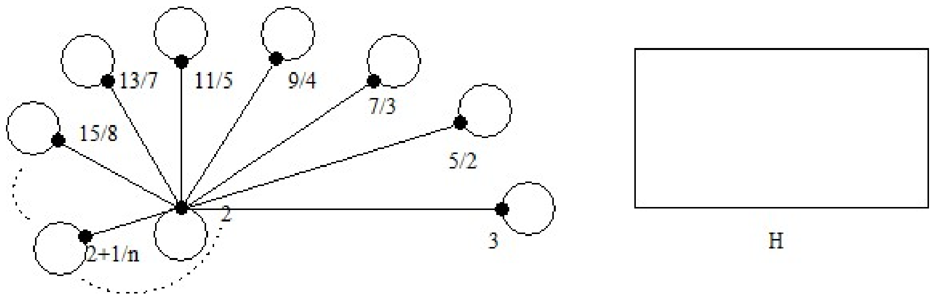

Figure 1: This figure depicts graph G in Example 4.6 where H is the subgraph of G with vertex set $V(H) = (0, \infty) - \{2, 2 + \frac{1}{n} : n \in \mathbb{N}\}$ and edge set $E(H) = \{(x, x); x \in V(H)\}$.

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