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On some new variations of Hardy type inequalities

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Abstract

The goal of this paper is to establish some new variations of the inequalities which originate from the well-known Hardy type inequalities. The method applied in this paper to achieve our results is related to the idea used by Levinson to obtain the generalizations of Hardy's integral inequality.

Keywords: Hardy type inequality, Hölder's inequality.
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1. Introduction

In [1], Hardy established the following inequality: if $p > 1$, $f(x) > 0$ for $0 < x < \infty$, and

$$R(x) = \frac{1}{x}\int_0^x F^p(x)dx,$$

then

$$\int_0^\infty R^p(x)dx < (\frac{p}{p-1})^p \int_0^\infty F^p(x)dx, \tag{1.1}$$

unless $f = 0$. The constant is the best possible.

The following variation of (1.1) has been proved by Izumi and Izumi [3]: if $p > 1$, and $s < -1$ and let f be a nonnegative and integrable function on $(0, \pi)$, if $x^s f^p(s)$ is integrable, then

$$\int_0^\pi x^s G^p(x)dx \leqslant (\frac{p}{-s-1})^p \int_0^\pi \mid f\frac{x}{2} - f(x) \mid^p dx, \tag{1.2}$$

where

$$G(x) = \int_{\frac{x}{2}}^x \frac{1}{t} f(t)dt.$$

The alternative proofs and extensions have been studied by many authors due to the importance of Hardy type inequalities, see [1–9]. Our results are the variations of Hardy inequality (1.1) given by Levinson [4] and the inequality (1.2) given by Izumi and Izumi [3].

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2. Main results

Throughout in this paper, we assume that all the integrals exist in their respective domains of their definitions.

Theorem 2.1. *Let* $p_r > 1 : 1 \leqslant r \leqslant n$, $m > 1$, $q > 0$ *be a constant and* f_r *be a nonnegative and integrable function on* $(0, b) : 0 < b < \infty$. *Also, let* w_r *be a positive and absolutely continuous function on* $(0, b)$ *and*

$$1 + \frac{2q_r x}{m-1}\frac{w'(x)}{w(x)} \geqslant \frac{1}{\alpha_r}; \quad 1 \leqslant r \leqslant n \tag{2.1}$$

for almost all $r \in (0, b)$ *and for some* $\alpha_r > 0$. *If* F_r *is defined by*

$$F_r(x) = \int_{\frac{x}{2}}^{x} \frac{1}{t}\Big(\int_{\frac{t}{2}}^{t} \frac{f_r(s)}{s} ds\Big) dt; \quad x \in (0, b),$$

then

$$\sum_{r=1}^{n} \int_0^b x^{-m} \frac{F_r^{p_r}(x)}{w_r^{q_r}(x)} \frac{F_r^{p_r+1}(x)}{w_r^{q_r+1}(x)} dx \leqslant \sum_{r=1}^{n} \frac{2p_r\alpha_r}{m-1})^{4p_r} \int_0^b x^{-m} w^{-2q_r}(x)\big[f_r(x) - f_r(\frac{x}{4})\big]^{2p_r} dx. \tag{2.2}$$

Proof. Using the inequality [7]

$$\sum_{r=1}^{n} C_r^{p_r} C_{r+1}^{p_{r+1}} \leqslant \sum_{r=1}^{n} C_r^{2p_r},$$

where $C_{n+1}^{p_{n+1}} = C_1^{p_1}$ (for C_r, $1 \leqslant r \leqslant n$ are real) and $p_r > 1$, we observe that

$$\sum_{r=1}^{n} \frac{F_r^{p_r}(x)}{w_r^{q_r}(x)} \frac{F_r^{p_r+1}(x)}{w_r^{q_r+1}(x)} \leqslant \sum_{r=1}^{n} \frac{F_r^{2p_r}(x)}{w_r^{2q_r}(x)}, \tag{2.3}$$

where $\frac{F_r^{p_r+1}(x)}{w_r^{q_r+1}(x)} = \frac{F_1^{p_1}(x)}{w_1^{q_1}(x)}$, $C_r^{p_r} = \frac{F_r^{p_r}(x)}{w_r^{q_r}(x)}$, and $C_{r+1}^{p_{r+1}} = \frac{F_r^{p_r+1}(x)}{w_r^{q_r+1}(x)}$.

Multiplying both sides of (2.3) by x^{-m} and integrating from 0 to b, we obtain

$$\sum_{r=1}^{n} \int_0^b x^{-m} \frac{F_r^{p_r}(x)}{w_r^{q_r}(x)} \frac{F_r^{p_r+1}(x)}{w_r^{q_r+1}(x)} dx \leqslant \sum_{r=1}^{n} \int_0^b x^{-m} \frac{F_r^{2p_r}(x)}{w_r^{2q_r}(x)} dx. \tag{2.4}$$

Integrating $\int_0^b x^{-m} \frac{F_r^{2p_r}(x)}{w_r^{2q_r}(x)} dx$ by parts, we observe that

$$\Big[1 + \frac{2q_r x}{m-1}\frac{w_r'(x)}{w_r(x)} dx\Big] \int_0^b x^{-m} \frac{F_r^{2p_r}(x)}{w_r^{2q_r}(x)} dx \leqslant \frac{2p_r}{m-1} \int_0^b x^{-m} \frac{F_r^{2p_r-1}(x)}{w_r^{2q_r}(x)} \Big(\int_{\frac{x}{4}}^{x} \frac{f_r(s)}{s} ds\Big) dx. \tag{2.5}$$

Since $m > 1$, from (2.1) and applying Hölder's inequality [2] on (2.5) we have

$$\int_0^b x^{-m} \frac{F_r^{2p_r}(x)}{w_r^{2q_r}(x)} dx \leqslant \Big(\frac{2p_r\alpha_r}{m-1}\Big)^{2p_r} \int_0^b x^{-m} \frac{1}{w_r^{2q_r}(x)} R_r^{2p_r}(x) dx, \tag{2.6}$$

where

$$R_r(x) = \int_{\frac{x}{4}}^{x} \frac{f_r(s)}{s} ds \tag{2.7}$$

with $1 \leqslant r \leqslant n$.

Again integrating $\int_0^b x^{-m}\frac{R_r^{2p_r}(x)}{w_r^{2q_r}(x)}dx$ by parts, we observe that

$$\begin{aligned}\int_0^b x^{-m}\frac{R_r^{2p_r}(x)}{w_r^{2q_r}(x)}dx &= \frac{b^{-m+1}}{-m+1}\frac{R_r^{2p_r}(b)}{w_r^{2q_r}(b)}\\ &-\frac{1}{-m+1}\int_0^b x^{-m+1}\big[(2p_r)R_r^{2p_r-1}(x)w_r^{-2q_r}(x)\frac{1}{x}\big[f_r(x)-f_r(\frac{x}{4})\big]\\ &-(2q_r)Rr^{2p_r}(x)w_r^{-2q_r-1}(x)w_r'(x)\big]dx.\end{aligned} \tag{2.8}$$

Following the same steps from (2.5)-(2.7) with suitable modifications on (2.8), we get

$$\int_0^b x^{-m}\frac{R_r^{2p_r}(x)}{w_r^{2q_r}(x)}dx \leqslant (\frac{2p_r\alpha_r}{m-1})^{2p_r}\int_0^b x^{-m}\frac{1}{w_r^{2q_r}(x)}\big[f_r(x)-f_r(\frac{x}{4})\big]^{2p_r}dx.$$

From (2.4), (2.6), and (2.8), we get the required inequality (2.2). □

Theorem 2.2. *Let* $p_r>1: 1\leqslant r\leqslant 2$, $m>1$ *and* $q>0$ *be a constant,* f_r *be a nonnegative and integrable function on* $(0,b), 0<b<\infty$, *and* w_r *be a positive and absolutely continuous function on* $(0,b)$ *and*

$$1+\frac{q_rM_1x}{m-1}\frac{w'(x)}{w(x)} \geqslant \frac{1}{\beta_r};\ \ 1\leqslant r\leqslant 2 \tag{2.9}$$

for almost all $r\in(0,b)$ *and for some* $\beta_r>0$. *If* F_r *is defined by*

$$F_r(x)=\int_0^x\frac{1}{t}\Big(\int_{\frac{t}{2}}^t\frac{f_r(s)}{s}ds\Big)dt;\ \ x\in(0,b),$$

then

$$\begin{aligned}&\int_0^b x^{-m}\Big[\frac{F_1^{p_1}(x)}{w_1^{q_1}(x)}\frac{F_2^{p_2}(x)}{w_2^{q_2}(x)}\Big]^{m_2}\Big[\frac{F_1^{p_1}(x)}{w_1^{q_1}(x)}\frac{F_2^{p_2}(x)}{w_2^{q_2}(x)}\Big]^{m_1}dx\\ &\leqslant 2^{m_1-1}\sum_{r=1}^{2}(\frac{p_rM_1\beta_r}{m-1})^{2p_rM_1}\int_0^b x^{-m}w^{-q_rM_1}(x)\big[f_r(x)-f_r(\frac{x}{2})\big]^{p_rM_1}dx,\end{aligned} \tag{2.10}$$

where $M_1=m_1+2m_2$.

Proof. Using the inequality [6]

$$[C_1C_2]^{m_2}[C_1C_2]^{m_1}\leqslant 2^{m_1-1}[C_1^{M_1}+C_2^{M_1}],$$

where C_1 and C_2 are positive reals, m_1 and m_2 are positive integers, and $M_1=m_1+2m_2$, we observe that

$$\Big[\frac{F_1^{p_1}(x)}{w_1^{q_1}(x)}\frac{F_2^{p_2}(x)}{w_2^{q_2}(x)}\Big]^{m_2}\Big[\frac{F_1^{p_1}(x)}{w_1^{q_1}(x)}\frac{F_2^{p_2}(x)}{w_2^{q_2}(x)}\Big]^{m_1}\leqslant 2^{m_1-1}\Big[\frac{F_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)}+\frac{F_2^{p_2M_1}(x)}{w_2^{q_2M_1}(x)}\Big], \tag{2.11}$$

where $C_1=\frac{F_1^{p_1}(x)}{w_1^{q_1}(x)}$ and $C_2=\frac{F_2^{p_2}(x)}{w_2^{q_2}(x)}$.

Multiplying both sides of (2.11) by x^{-m} and integrating from 0 to b, we obtain

$$\begin{aligned}&\int_0^b x^{-m}\Big[\frac{F_1^{p_1}(x)}{w_1^{q_1}(x)}\frac{F_2^{p_2}(x)}{w_2^{q_2}(x)}\Big]^{m_2}\Big[\frac{F_1^{p_1}(x)}{w_1^{q_1}(x)}\frac{F_2^{p_2}(x)}{w_2^{q_2}(x)}\Big]^{m_1}dx\\ &\leqslant 2^{m_1-1}\Big[\int_0^b x^{-m}\frac{F_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)}dx+\int_0^b x^{-m}\frac{F_2^{p_2M_1}(x)}{w_2^{q_2M_1}(x)}dx\Big],\end{aligned} \tag{2.12}$$

where $M_1 = m_1 + 2m_2$. Let us consider both the integrals on the right side of (2.12) simultaneously. Integrating $\int_0^b x^{-m} \frac{F_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)} dx$ by parts, we observe that

$$\left[1 + \frac{q_1M_1 x}{m-1} \frac{w_1'(x)}{w_1(x)} dx\right] \int_0^b x^{-m} \frac{F_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)} dx \leqslant \frac{p_1M_1}{m-1} \int_0^b x^{-m} \frac{F_1^{p_1M_1-1}(x)}{w_1^{q_1M_1}(x)} \Big(\int_{\frac{x}{2}}^x \frac{f_1(s)}{s} ds \Big) dx. \tag{2.13}$$

Since $m > 1$, from (2.9) and applying Hölder's inequality [2] on (2.13), we have

$$\int_0^b x^{-m} \frac{F_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)} dx \leqslant \Big(\frac{p_1M_1\beta_1}{m-1}\Big)^{p_1M_1} \int_0^b x^{-m} \frac{1}{w_r^{q_1M_1}(x)} R_1^{p_1M_1}(x) dx, \tag{2.14}$$

where

$$R_1(x) = \int_{\frac{x}{2}}^x \frac{f_1(s)}{s} ds. \tag{2.15}$$

Again integrating $\int_0^b x^{-m} \frac{R_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)} dx$ by parts, we observe that

$$\begin{aligned} \int_0^b x^{-m} \frac{R_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)} dx &= \frac{b^{-m+1}}{-m+1} \frac{R_1^{p_1M_1}(b)}{w_1^{q_1M_1}(b)} \\ &\quad - \frac{1}{-m+1} \int_0^b x^{-m+1} \Big[(p_1M_1) R_1^{p_1M_1-1}(x) w_1^{-q_1M_1}(x) \frac{1}{x} \big[f_1(x) - f_1\big(\frac{x}{2}\big) \big] \\ &\quad - (q_1M_1) R_1^{p_1M_1}(x) w_1^{-q_1M_1-1}(x) w_1'(x) \Big] dx. \end{aligned} \tag{2.16}$$

Following the same steps from (2.13)-(2.14) with suitable modifications on (2.16), we get

$$\int_0^b x^{-m} \frac{R_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)} dx \leqslant \Big(\frac{p_1M_1\beta_1}{m-1}\Big)^{p_1M_1} \int_0^b x^{-m} \frac{1}{w_1^{q_1M_1}(x)} \big[f_1(x) - f_1\big(\frac{x}{2}\big) \big]^{p_1M_1} dx. \tag{2.17}$$

From (2.14) and (2.17), we have

$$\int_0^b x^{-m} \frac{F_1^{p_1M_1}(x)}{w_1^{q_1M_1}(x)} dx \leqslant \Big(\frac{p_1M_1\beta_1}{m-1}\Big)^{2p_1M_1} \int_0^b x^{-m} w^{-q_1M_1}(x) \big[f_1(x) - f_1\big(\frac{x}{2}\big) \big]^{p_1M_1} dx. \tag{2.18}$$

Similarly integrating $\int_0^b x^{-m} \frac{F_2^{p_2M_1}(x)}{w_2^{q_2M_1}(x)} dx$ by parts, we get

$$\begin{aligned} \int_0^b x^{-m} \frac{F_2^{p_2M_1}(x)}{w_2^{q_2M_1}(x)} dx = \frac{b^{-m+1}}{-m+1} \frac{F_2^{p_2M_1}(b)}{w_2^{q_2M_1}(b)} - \frac{1}{-m+1} \int_0^b x^{-m+1} \Big[(p_2M_1) F_2^{p_{2M_1}-1}(x) w_2^{-q_2M_1}(x) F_2'(x) \\ - (q_2M_1) F_2^{p_2M_1}(x) w_2^{-q_2M_1-1}(x) w_2'(x) \Big] dx. \end{aligned}$$

The rest of the proof follows from (2.13)-(2.15) with suitable changes, therefore

$$\int_0^b x^{-m} \frac{F_2^{p_2M_1}(x)}{w_2^{q_2M_1}(x)} dx \leqslant \Big(\frac{p_2M_1\beta_1}{m-1}\Big)^{2p_2M_1} \int_0^b x^{-m} w^{-q_2M_1}(x) \big[f_2(x) - f_2\big(\frac{x}{2}\big) \big]^{p_2M_1} dx. \tag{2.19}$$

From (2.12), (2.18), and (2.19) we get the desired inequality (2.10). □

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