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Some Coupled fixed point theorems in partially ordered A_b-metric space

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Abstract

In this paper, we use the concept of A_b-metric space which is obtained by generalizing the definitions of A-metric space and b-metrc space. Using this concept, we prove some coupled fixed point theorems in partially ordered A_b-metric space. Examples are also presented to verify the obtained results.

Keywords: Common fixed points, A-metric space, b-metric space, A_b-metric space.
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1. Introduction and preliminaries

In the study of fixed point, the generalization of metric space is one of the interesting topics for many researchers. Some of the generalizations of metric space are 2-metric space, D-metric space, D^*-metric space, G-metric space, S-metric space, rectangular metric or metric-like space, partial metric space, fuzzy metric space, probabilistic metric space, cone metric space, modular metric space, etc.

In the year 1989, Bhaktin [5] introduced the concept of b-metric space in which the triangle inequality is made by a more general form by introducing a scale factor which is a real number greater than or equal to one. By doing so the ordinary metric space becomes a special case of b-metric space when scale factor is equal to one.

Introduction of b-metric space opens a gate to further generalization of metric space. Due to the introduction of b-metric space, many generalizations of metric space came into existence. Mention can be made about G_b-metric space, S_b-metric space, b_2-metric space, cone b-metric space, partial b-metric space, fuzzy b-metric space, rectangular b-metric space etc. Some results about generalized metric space can be seen in [1, 5, 8, 12, 15–19, 21–23] and references there in.

Recently, Abbas et al. [1] introduced the concept of an n-tuple metric space and studied its topological properties. This new structure is named as A-metric space. The definition of A-metric is as follows.

Definition 1.1 ([1]). Let X be a nonempty set. A function $A : X^n \to [0, \infty)$ is called an A-metric on X if for any $x_i, a \in X, i = 1, 2, ..., n$, the following conditions hold:

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(A1) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geqslant 0$,
(A2) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = ... = x_{n-1} = x_n$,
(A3)

$$\begin{aligned} A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \leqslant [&A(x_1, x_1, x_1, ..., (x_1)_{n-1}, a) \\ &+ A(x_2, x_2, x_2, ..., (x_2)_{n-1}, a) \\ &+ A(x_3, x_3, x_3, ..., (x_3)_{n-1}, a) \\ &\vdots \\ &+ A(x_{n-1}, x_{n-1}, x_{n-1}, ..., (x_{n-1})_{n-1}, a) \\ &+ A(x_n, x_n, x_n, ..., (x_n)_{n-1}, a)]. \end{aligned}$$

The pair (X, A) is called an A-metric space.

Some examples of A-metric space are given in [1].

Example 1.2 ([1])**.** Let $X = [1, +\infty)$. Define $A : X^n \to [0, +\infty)$ by

$$A(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|$$

for all $x_i \in X, i = 1, 2, \ldots n$. Then (X, A) is an A-metric space.

Example 1.3 ([1])**.** Let $X = \mathbb{R}$. Define $A : X^n \to [1, +\infty)$ by

$$\begin{aligned} A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = |\sum_{i=2}^{n} x_i - (n-1)x_1| + |\sum_{i=3}^{n} x_i - (n-2)x_2| + \cdots \\ + |\sum_{i=n-3}^{n} x_i - 3x_{n-3}| + |\sum_{i=n-2}^{n} x_i - 2x_{n-2}| + |x_n - x_{n-1}| \end{aligned}$$

for all $x_i \in X, i = 1, 2, \ldots n$. Then (X, A) is an A-metric space.

The concept of b-metric space was introduced by Bakhtin [5]. The definition of b-metric space is as follows.

Definition 1.4 ([6])**.** Let X be a nonempty set. A b-metric on X is a function $d : X^2 \to [0, \infty)$ if there exists a real number $s \geqslant 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

Motivated by the concepts given by Abbas et al. [1] and Bakhtin [5], Ughade et al. [23] introduced a generalized form of n-tuple metric space. They named it as A_b-metric space and defined as follows.

Definition 1.5 ([23])**.** Let X be a nonempty set and $b \geqslant 1$ be a given real number. A function $A : X^n \to [0, \infty)$ is called an A_b-metric on X if for any $x_i, a \in X, i = 1, 2, ..., n$, the following conditions hold:

(A_b1) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geqslant 0$;
(A_b2) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = ... = x_{n-1} = x_n$;

(A_b3)

$$\begin{aligned}A(x_1,x_2,x_3,...,x_{n-1},x_n) \leqslant b[&A(x_1,x_1,x_1,...,(x_1)_{n-1},a)\\&+A(x_2,x_2,x_2,...,(x_2)_{n-1},a)\\&+A(x_3,x_3,x_3,...,(x_3)_{n-1},a)\\&\vdots\\&+A(x_{n-1},x_{n-1},x_{n-1},...,(x_{n-1})_{n-1},a)\\&+A(x_n,x_n,x_n,...,(x_n)_{n-1},a)].\end{aligned}$$

The pair (X, A_b) is called an A_b-metric space.

Definition 1.6. An A_b-metric space is said to be symmetric if $A(x_1, x_1, \ldots, (x_1)_{n-1}, x_2) = A(x_2, x_2, \ldots, (x_2)_{n-1}, x_1)$.

Note: A_b-metric space is more general than A-metric space. Moreover A-metric space is a special case of A_b-metric space with $b = 1$. b-metric space and S_b-metric space are also special cases of A_b-metric space with $n = 2$ and 3, respectively. Ordinary metric space and S-metric space are also special cases of A_b-metric space with $b = 1$ and respective values of n as 2 and 3.

Example 1.7 ([23]). Let $X = [1,+\infty)$. Define $A_b : X^n \to [1,+\infty)$ by

$$A_b(x_1,x_2,x_3,\ldots,x_{n-1},x_n) = \sum_{i=1}^{n}\sum_{i<j}|x_i - x_j|^2$$

for all $x_i \in X, i = 1, 2, \ldots n$. Then (X, A_b) is an A_b-metric space with $s = 2 > 1$.

Example 1.8 ([23]). Let $X = \mathbb{R}$. Define $A_b : X^n \to [1,+\infty)$ by

$$\begin{aligned}A_b(x_1,x_2,x_3,\ldots,x_{n-1},x_n) = |\sum_{i=2}^{n} x_i - (n-1)x_1|^2 + |\sum_{i=3}^{n} x_i - (n-2)x_2|^2 + \cdots\\ + |\sum_{i=n-3}^{n} x_i - 3x_{n-3}|^2 + |\sum_{i=n-2}^{n} x_i - 2x_{n-2}|^2 + |x_n - x_{n-1}|^2\end{aligned}$$

for all $x_i \in X, i = 1, 2, \ldots n$. Then (X, A_b) is an A_b-metric space with $s = 2 > 1$

Example 1.9. Let $X = [1,\infty)$. Define a function $A_b : X^n \to [0,\infty)$ by $A_b(x_1, x_2, \cdots, x_n) = |x_1 - \max\{x_2, \cdots, x_n\}|^2$ for all $x_1, x_2, \cdots, x_n \in X$. Then (X, A_b) is an A_b-metric on X with $b = 2$, and it is not difficult to see that (X, A_b) is not an A-metric space on X.

Lemma 1.10 ([23]). *Let (X, A) be A_b-metric space. Then $A(x, x, x, ..., x, y) \leqslant bA(y, y, y, ..., y, x)$ for all $x, y \in X$.*

Lemma 1.11. *Let (X, A) be A_b-metric space. Then for all $x, y, z \in X$ we have $A(x, x, x, ..., x, z) \leqslant (n-1)bA(x, x, x, ..., x, y) + b^2A(y, y, y, ..., y, z)$.*

Proof. By (A_b3), we have

$$\begin{aligned}A(x,x,x,...,x,y) &\leqslant b[A(x,x,x,...,x,y) + A(x,x,x,...,x,y) + \cdots + (n-1)\text{times} + A(z,z,z,...,z,y)]\\&\leqslant b[(n-1)A(x,x,x,...,x,y) + bA(y,y,y,...,y,z)]\\&= (n-1)bA(x,x,x,...,x,y) + b^2A(y,y,y,...,y,z).\end{aligned}$$

□

Lemma 1.12. *Let (X, A) be A_b-metric space. Then (X^2, D_A) is A_b-metric space on $X \times X$ with the metric D_A given by $D((x_1, y_1), (x_2, y_2), ..., (x_n, y_n)) = A(x_1, x_1, x_1, ..., x_n) + A(y_1, y_2, y_3, ..., y_n)$ for all $x_i, y_j \in X, i, j = 1, 2, ..., n$.*

Definition 1.13. The A_b-metric space (X, A) is said to be bounded if there exists a constant $r > 0$ such that $A(x, x, x, ..., x, y) \leqslant r$ for all $x, y \in X$. Otherwise, X is unbounded.

Definition 1.14. Given a point x_0 in A_b-metric space (X, A) and a positive real number r, the set $B(x_0, r) = \{y \in X : A(y, y, y, ..., y, x_0) < r\}$ is called an open ball centered at x_0 with radius r.

The set $\overline{B(x_0, r)} = \{y \in X : A(y, y, y, ..., y, x_0) \leqslant r\}$ is called a closed ball centered at x_0 with radius r.

Definition 1.15. A subset G in A_b-metric space (X, A) is said to be an open set if for each $x \in G$ there exists an $r > 0$ such that $B(x, r) \subset G$. A subset $F \subset X$ is called closed if $X \backslash F$ is open.

Lemma 1.16. *In any A_b-metric space (X, A), each open ball is an open set in X and each closed ball is also a closed set in X.*

Theorem 1.17. *Let (X, A) be A_b-metric space, then:*

(i) *An arbitrary union and finite intersection of open balls $B(x, r) \in X$ is open.*
(ii) *An arbitrary intersection and finite union of closed balls $B(x, r) \in X$ is closed.*

Theorem 1.18. *The collection $T = \{B(x, r) : x \in X, r > 0\}$ of all balls in A_b-metric space (X, A) is a basis for a topology τ on X.*

Definition 1.19. Let (X, A) be A_b-metric space. A sequence $\{x_k\}$ in X is said to converge to a point $x \in X$, if $A(x_k, x_k, x_k, ..., x_k, x) \to 0$ as $k \to \infty$. That is, for each $\varepsilon \geqslant 0$, there exists $N \in \mathbb{N}$ such that for all $k \geqslant N$ we have $A(x_k, x_k, x_k, ..., x_k, x) \leqslant \varepsilon$ and we write $\lim_{k\to\infty} x_k = x$.

Lemma 1.20. *Let (X, A) be A_b-metric space. If the sequence $\{x_k\}$ in X converges to a point x, then x is unique.*

Proof. Suppose $\{x_k\}$ converges to x and y. Then, for $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $k \geqslant N_1$ we have $A(x_k, x_k, \ldots, x_k, x) < \frac{\varepsilon}{2b^2(n-1)}$ and for every $k \geqslant N_2$, we get $A(x_k, x_k, \ldots, x_k, y) < \frac{\varepsilon}{2b^2}$. Choose $N = \max\{N_1, N_2\}$, therefore, for all $k \geqslant N$, we have

$$\begin{aligned} A(x, x, \ldots, x, y) &= (n-1)bA(x, x, \ldots, x, x_k) + b^2 A(x_k, x_k, \ldots, x_k, y) \\ &\leqslant (n-1)b^2 A(x_k, x_k, \ldots, x_k, x) + b^2 A(x_k, x_k, \ldots, x_k, y) \\ &\leqslant (n-1)b^2 \frac{\varepsilon}{2b^2(n-1)} + b^2 \times \frac{\varepsilon}{2b^2} \\ &= \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have $A(x, x, \ldots, x, y) = 0$ and so $x = y$. Establishing the uniqueness of $\{x_k\}$. □

Definition 1.21. Let (X, A) be A_b-metric space. A sequence $\{x_k\}$ in X is called a Cauchy sequence if $A(x_k, x_k, x_k, ..., x_k, x_m) \to 0$ as $k, m \to \infty$.

That is, for each $\epsilon \geqslant 0$, there exists $N \in \mathbb{N}$ such that for all $k, m \in \mathbb{N}$ we have $A(x_k, x_k, x_k, ..., x_k, x_m) \leqslant \epsilon$ and we write $\lim_{k\to\infty} x_k = x$.

Lemma 1.22. *Every convergent sequence in A_b-metric space is a Cauchy sequence.*

Proof. Let $\{x_k\}$ be a convergent sequence in (X, A). Let $\lim_{k\to\infty} x_k = x$. Then, given $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $k \geqslant N_1$ we have $A(x_k, x_k, \ldots, x_k, x) < \frac{\varepsilon}{2b(n-1)}$ and for every $m \geqslant N_2$, we get $A(x_m, x_m, \ldots, x_m, x) < \frac{\varepsilon}{2b^2}$. Putting $N = \max\{N_1, N_2\}$. Therefore, for all $k, m \geqslant N$, we obtain

$$\begin{aligned} A(x_k, x_k, \ldots, x_k, x_m) &= (n-1)bA(x_k, x_k, \ldots, x_k, x) + b^2 A(x_m, x_m, \ldots, x_m, x) \\ &\leqslant (n-1)b \frac{\varepsilon}{2b(n-1)} + b^2 \times \frac{\varepsilon}{2b^2} \\ &= \varepsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. □

Remark 1.23. The converse of Lemma 1.22 does not hold in general. A Cauchy sequence in an A_b-metric space does not need to be convergent.

Definition 1.24. The A_b-metric space (X, A) is said to be complete if every Cauchy sequence in X is convergent.

Definition 1.25. Let (X, A_X) and (Z, A_Z) be A_b-metric spaces. A function $f : X \to Z$ is continuous at a point $x_0 \in X$, if $f^{-1}(G)$ is open in X for each open set G in Z. The function f is continuous on X if it is continuous at each points of X.

Theorem 1.26. *Let (X, A_X) and (Z, A_Z) be A_b-metric spaces. A function $f : X \to Z$ is continuous at a point $x_0 \in X$ iff it is sequentially continuous at x_0.*

Lemma 1.27. *Let (X, A) be A_b-metric space, then the function $A(x, x, x, ..., x, y)$ is continuous in all of its arguments. In other words, if there exist sequences $\{x_k\}$ and $\{y_k\}$ such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} y_k = y$, then $\lim_{k\to\infty} A(x_k, x_k, x_k, ..., x_k, y_k) = A(x, x, x, ..., x, y)$.*

Definition 1.28. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $S : X \times X \to X$ if

$$S(x, y) = x, \text{ and } S(y, x) = y.$$

In 1987 Guo and Lakshmikantham [7] introduced the concept of coupled fixed point. Later, Bhaskar and Lakshmikantham [6] proved a new fixed point theorem for a mixed monotone mapping in a metric space powered with partial ordered by using a weak contractivity type assumption. For more results on coupled fixed point results one can see research results in [2–5, 7, 9–11, 13, 14, 20] and references therein.

2. Main result

In this section, we obtain common coupled fixed point results of mappings satisfying more general contractive conditions in the framework of partially ordered A_b-metric spaces. We start with the following result.

Theorem 2.1. *Let $(X, \preceq, A)$ be a partially ordered complete A_b-metric space, and $f, g : X^2 \to X$ be two maps such that*

1. *the pair (f, g) has mixed weakly monotone property on X,*

$$x_0 \preceq f(x_0, y_0), f(y_0, x_0) \preceq y_0 \text{ or } x_0 \preceq g(x_0, y_0), g(y_0, x_0) \leqslant y_0 \text{ for some } x_0, y_0 \in X;$$

2. *there exist $a_i \succeq 0, i = 1, ..., 5$, satisfying $a_1 + a_2 + b^2(a_3 + a_4) + a_5 < 1$ and*

$$\begin{aligned} &A(f(x, y), f(x, y), \dots, f(x, y), g(u, v)) + A(f(y, x), f(y, x), \dots, f(y, x), g(v, u)) \\ &\leqslant a_1 D((x, y), (x, y), \dots, (x, y), (u, v)) + a_2 D((x, y), (x, y), \dots, (x, y), (f(x, y), f(y, x))) \\ &\quad + a_3 D((u, v), (u, v), \dots, (u, v), (g(u, v), g(v, u))) \\ &\quad + a_4 D((x, y), (x, y), \dots, (x, y), (g(u, v), g(v, u))) \\ &\quad + a_5 D((u, v), (u, v), \dots, (u, v), (f(x, y), f(y, x))) \end{aligned} \tag{2.1}$$

   for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$;

3. *either f or g is continuous or X has the following properties:*
   (a) *if $\{x_k\}$ is an increasing sequence with $x_k \to x$, then $x_k \preceq x$ for all $k \in \mathbb{N}$;*
   (b) *if $\{y_k\}$ is a decreasing sequence with $y_k \to y$, then $y \preceq y_k$ for all $k \in \mathbb{N}$.*

Then f and g have a coupled common fixed point in X.

Proof. Let (x_0, y_0) be a given point in $X \times X$. Choose $x_1 = f(x_0, y_0), y_1 = f(y_0, x_0), x_2 = g(x_1, y_1)$ and $y_2 = g(y_1, x_1)$. From the condition $x_0 \preceq f(x_0, y_0), y_0 \succeq f(y_0, x_0)$, and the fact that (f, g) has mixed weakly monotone property we have

$$x_1 = f(x_0, y_0) \preceq g(f(x_0, y_0), f(y_0, x_0)) = g(x_1, y_1) \Rightarrow x_1 \preceq x_2$$

and

$$x_2 = g(x_1, y_1) \preceq f(g(x_1, y_1), g(y_1, x_1)) = f(x_2, y_2) \Rightarrow x_2 \preceq x_3.$$

Thus,

$$y_1 = f(x_0, y_0) \succeq g(f(y_0, x_0), f(x_0, y_0)) = g(y_1, x_1) \Rightarrow y_1 \succeq y_2$$

and

$$y_2 = g(y_1, x_1) \succeq f(g(y_1, x_1), g(x_1, y_1)) = f(y_2, x_2) \Rightarrow y_2 \succeq y_3.$$

Continuing this way, we obtain

$$\begin{aligned} &x_{2k+1} = f(x_{2k}, y_{2k}), y_{2k+1} = f(y_{2k}, x_{2k}), \\ &x_{2k+2} = g(x_{2k+1}, y_{2k+1}), y_{2k+2} = g(y_{2k+1}, x_{2k+1}) \text{ for all } k \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Therefore the sequences $\{x_k\}$ and $\{y_k\}$ are monotone:

$$\begin{aligned} &x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_k \preceq x_{2k+1} \preceq x_{2k+2} \preceq \cdots, \\ &y_0 \succeq y_1 \succeq y_2 \succeq y_3 \succeq \cdots \succeq y_k \succeq y_{2k+1} \succeq y_{2k+2} \succeq \cdots. \end{aligned} \tag{2.3}$$

Now we show that these sequences are Cauchy. From the contractive condition (2.1) we have for all $k \in \mathbb{N}$

$$\begin{aligned} &A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \\ &\quad = A(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), \ldots, f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1})) \\ &\qquad + A(f(y_{2k}, x_{2k}), f(y_{2k}, x_{2k}), \ldots, f(y_{2k}, x_{2k}), g(y_{2k+1}, x_{2k+1})) \\ &\quad \leqslant a_1 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1})) \\ &\qquad + a_2 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (f(x_{2k}, y_{2k}), f(y_{2k}, x_{2k}))) \\ &\qquad + a_3 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \ldots, (x_{2k+1}, y_{2k+1}), (g(x_{2k+1}, y_{2k+1}), g(y_{2k+1}, x_{2k+1}))) \\ &\qquad + a_4 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (g(x_{2k+1}, y_{2k+1}), g(y_{2k+1}, x_{2k+1}))) \\ &\qquad + a_5 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \ldots, (x_{2k+1}, y_{2k+1}), (f(x_{2k}, y_{2k}), f(y_{2k}, x_{2k}))). \end{aligned}$$

Applying (2.2), we get

$$\begin{aligned} &A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \\ &\quad \leqslant a_1 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1})) \\ &\qquad + a_2 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1})) \\ &\qquad + a_3 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \ldots, (x_{2k+1}, y_{2k+1}), (x_{2k+2}, y_{2k+2})) \\ &\qquad + a_4 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+2}, y_{2k+2})) \\ &\qquad + a_5 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \ldots, (x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1})) \\ &\quad = a_1 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1})) \\ &\qquad + a_2 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1}))) \\ &\qquad + a_3 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \ldots, (x_{2k+1}, y_{2k+1})(x_{2k+2}, y_{2k+2}))) \\ &\qquad + a_4 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+2}, y_{2k+2}))) \\ &\quad = a_1 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1})) \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&+ a_2 D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1}))) \\
&+ a_3 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \ldots, (x_{2k+1}, y_{2k+1}), (x_{2k+2}, y_{2k+2}))) \\
&+ (n-1) b a_4 [D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1})))] \\
&+ b^2 a_4 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \ldots, (x_{2k+1}, y_{2k+1}), (x_{2k+2}, y_{2k+2}))) \\
= {}& (a_1 + a_2 + (n-1) b a_4) D((x_{2k}, y_{2k}), (x_{2k}, y_{2k}), \ldots, (x_{2k}, y_{2k}), (x_{2k+1}, y_{2k+1})) \\
&\times (a_3 + b^2 a_4) D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \ldots, (x_{2k+1}, y_{2k+1}), (x_{2k+2}, y_{2k+2}))) \\
= {}& (a_1 + a_2 + (n-1) b a_4)[A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, y_{2k}, \ldots, y_{2k}, y_{2k+1})] \\
&\times (a_3 + b^2 a_4)[A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) \\
&+ A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})].
\end{aligned} \tag{2.5}
$$

Similarly we obtain

$$
\begin{aligned}
& A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) + A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) \\
&\quad \leqslant (a_1 + a_2 + (n-1) b a_4)[A(y_{2k}, y_{2k}, \ldots, y_{2k}, y_{2k+1}) + A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1})] \\
&\quad \times (a_3 + b^2 a_4)[A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \\
&\quad + A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})].
\end{aligned} \tag{2.6}
$$

From (2.4) and (2.5), we have

$$
\begin{aligned}
& 2[A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})] \\
&\quad \leqslant 2[(a_1 + a_2 + (n-1) b a_4)[A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, y_{2k}, \ldots, y_{2k}, y_{2k+1})]] \\
&\quad + 2[(a_3 + b^2 a_4)[A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) \\
&\quad + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})]].
\end{aligned} \tag{2.7}
$$

This implies that

$$
\begin{aligned}
& [A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})][1 - (a_3 + b^2 a_4)] \\
&\quad \leqslant [(a_1 + a_2 + (n-1) b a_4)[A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, y_{2k}, \ldots, y_{2k}, y_{2k+1})]].
\end{aligned}
$$

Thus

$$
\begin{aligned}
& A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \\
&\quad \leqslant \frac{(a_1 + a_2 + (n-1) b a_4)}{1 - (a_3 + b^2 a_4)} [A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, y_{2k}, \ldots, y_{2k}, y_{2k+1})].
\end{aligned}
$$

Let $\delta = [\frac{(a_1 + a_2 + (n-1) b a_4)}{1 - (a_3 + b^2 a_4)}]$, then $0 \leqslant \delta < 1$ and

$$
\begin{aligned}
& A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \\
&\quad \leqslant \delta(A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, y_{2k}, \ldots, y_{2k}, y_{2k+1})).
\end{aligned}
$$

For all $k \in \mathbb{N}$, applying (2.1) again and by interchanging the roles of f and g, we obtain

$$
\begin{aligned}
& A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2k+3}) + A(y_{2k+2}, y_{2k+2}, \ldots, y_{2k+2}, y_{2k+3}) \\
&\quad \leqslant \delta(A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})).
\end{aligned}
$$

It follows from (2.6) that

$$
\begin{aligned}
& A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \\
&\quad \leqslant \delta(A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, y_{2k}, \ldots, y_{2k}, y_{2k+1}))
\end{aligned}
$$

$$\leqslant \delta^2(A(x_{2k-1}, x_{2k-1}, \ldots, x_{2k-1}, x_{2k}) + A(y_{2k-1}, y_{2k-1}, \ldots, y_{2k-1}, y_{2k}))$$
$$\leqslant \delta^3(A(x_{2k-2}, x_{2k-2}, \ldots, x_{2k-2}, x_{2k-1}) + A(y_{2k-2}, y_{2k-2}, \ldots, y_{2k-2}, y_{2k-1})).$$

This implies

$$\begin{aligned} &A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \\ &\quad \leqslant \delta^{2k+1}(A(x_0, x_0, \ldots, x_0, x_1) + A(y_0, y_0, \ldots, y_0, y_1)). \end{aligned} \tag{2.8}$$

Similarly, by (2.7), we get

$$\begin{aligned} &A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2k+3}) + A(y_{2k+2}, y_{2k+2}, \ldots, y_{2k+2}, y_{2k+3}) \\ &\quad \leqslant \delta^{2k+2}(A(x_0, x_0, \ldots, x_0, x_1) + A(y_0, y_0, \ldots, y_0, y_1)). \end{aligned} \tag{2.9}$$

By Lemma 1.11, we have for all $k, m \in \mathbb{N}$ with $k \leqslant m$

$$\begin{aligned} A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2m+1}) \leqslant\ & b(n-1)A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) \\ & + b^2A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2m+1}) \end{aligned}$$

and

$$\begin{aligned} A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2m+1}) \leqslant\ & b(n-1)A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \\ & + b^2A(y_{2k+2}, y_{2k+2}, \ldots, y_{2k+2}, y_{2m+1}). \end{aligned}$$

So we have

$$\begin{aligned} &A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2m+1}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2m+1}) \\ &\leqslant b[(n-1)(A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + (n-1)A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}))] \\ &\quad + b^2[A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2m+1}) + A(y_{2k+2}, y_{2k+2}, \ldots, y_{2k+2}, y_{2m+1})] \\ &= b(n-1)[A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})] \\ &\quad + b^3(n-1)[A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2k+3}) + A(y_{2k+2}, y_{2k+2}, \ldots, y_{2k+2}, y_{2k+3})] \\ &\quad \vdots \\ &\quad + b^{2(2m-2k)-1}(n-1)[A(x_{2m-1}, x_{2m-1}, \ldots, x_{2m-1}, x_{2m}) + A(y_{2m-1}, y_{2m-1}, \ldots, y_{2m-1}, y_{2m})] \\ &\quad + b^{2(2m-2k)}(n-1)[A(x_{2m}, x_{2m}, \ldots, x_{2m}, x_{2m+1}) + A(y_{2m}, y_{2m}, \ldots, y_{2m}, y_{2m+1})] \\ &\Rightarrow A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2m+1}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2m+1}) \\ &\leqslant b(n-1)[A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})] \\ &\quad + b^3(n-1)[A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2k+3}) + A(y_{2k+2}, y_{2k+2}, \ldots, y_{2k+2}, y_{2k+3})] \\ &\quad \vdots \\ &\quad + b^{2(2m-2k)}(n-1)[A(x_{2m}, x_{2m}, \ldots, x_{2m}, x_{2m+1}) + A(y_{2m}, y_{2m}, \ldots, y_{2m}, y_{2m+1})] \\ &= b(n-1)[\delta^{2k+1} + b^2\delta^{2k+2} + \cdots + b^{2(2m-2k)-1}\delta^{2m}] \times [A(x_0, x_0, \ldots, x_0, x_1) + A(y_0, y_0, \ldots, y_0, y_1)] \\ &\Rightarrow A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2m+1}) + A(y_{2k+1}, y_{2k+1}, \ldots, y_{2k+1}, y_{2m+1}) \\ &\leqslant b(n-1)\frac{\delta^{2k+1}}{1-\delta b^2}[(A(x_0, x_0, \ldots, x_0, x_1) + A(y_0, y_0, \ldots, y_0, y_1))]. \end{aligned}$$

Similarly, we have

$$A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2m+1}) + A(y_{2k}, y_{2k}, \ldots, y_{2k}, y_{2m+1})$$

$$\leqslant (n-1)\frac{b\delta^{2k}}{1-\delta b^2}(A(x_0,x_0,\ldots,x_0,x_1)+A(y_0,y_0,\ldots,y_0,y_1))$$

and

$$\begin{aligned}&A(x_{2k},x_{2k},\ldots,x_{2k},x_{2m})+A(y_{2k},y_{2k},\ldots,y_{2k},y_{2m})\\&=(n-1)\frac{b\delta^{2k}}{1-\delta b^2}(A(x_0,x_0,\ldots,x_0,x_1)+A(y_0,y_0,\ldots,y_0,y_1)).\end{aligned}$$

Hence, for all $k,m\in\mathbb{N}$ with $k\leqslant m$, we have

$$A(x_k,x_k,\ldots,x_k,x_m)+A(y_k,y_k,\ldots,y_k,y_m)\leqslant (n-1)\frac{b\delta^{k}}{1-\delta b^2}(A(x_0,x_0,\ldots,x_0,x_1)+A(y_0,y_0,\ldots,y_0,y_1)).$$

Since $0\leqslant\delta=[\frac{(\alpha_1+\alpha_2+(n-1)b\alpha_4)}{1-(\alpha_3+b^2\alpha_4)}]<1$, we have

$$\lim_{k,m\to\infty}(A(x_k,x_k,\ldots,x_k,x_m)+A(y_k,y_k,\ldots,y_k,y_m))=0.$$

That is,

$$\lim_{k,m\to\infty}A(x_k,x_k,\ldots,x_k,x_m)=\lim_{k,m\to\infty}A(y_k,y_k,\ldots,y_k,y_m)=0.$$

Therefore, $\{x_k\}$ and $\{y_k\}$ are both Cauchy sequences in X. By the completeness of X, there exist $x,y\in X$ such that $x_k\to x$ and $y_k\to y$ as $k\to\infty$.

We next show that the pair (x,y) is a coupled common fixed point of f and g.

Now, suppose f is continuous, then we have

$$x=\lim_{k\to\infty}x_{2k+1}=\lim_{k\to\infty}f(x_{2k},y_{2k})=f(\lim_{k\to\infty}x_{2k},\lim_{k\to\infty}y_{2k})=f(x,y)$$

and

$$y=\lim_{k\to\infty}y_{2k+1}=\lim_{k\to\infty}f(y_{2k},x_{2k})=f(\lim_{k\to\infty}y_{2k},\lim_{k\to\infty}x_{2k})=f(y,x).$$

Applying (2.1), we have

$$\begin{aligned}&A(f(x,y),f(x,y),\ldots,f(x,y),g(x,y))+A(f(y,x),f(y,x),\ldots,f(y,x),g(y,x))\\&\quad\leqslant a_1D((x,y),(x,y),\ldots,(x,y),(x,y))\\&\qquad+a_2D((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x)))+a_3D((x,y),(x,y),\ldots,(x,y),(g(x,y),g(y,x)))\\&\qquad+a_4D((x,y),(x,y),\ldots,(x,y),(g(x,y),g(y,x)))+a_5D((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x)))\\&\quad=a_2D((x,y),(x,y),\ldots,(x,y),(x,y))+a_3D((x,y),(x,y),\ldots,(x,y),(g(x,y),g(y,x)))\\&\qquad+a_4D((x,y),(x,y),\ldots,(x,y),(g(x,y),g(y,x)))+a_5D((x,y),(x,y),\ldots,(x,y),(x,y))\\&\quad=(a_3+a_4)D((x,y),(x,y),\ldots,(x,y),(g(x,y),g(y,x)))\\&\quad=(a_3+a_4)(A(x,x,\ldots,x,g(x,y))+(A(y,y,\ldots,y,g(y,x))).\end{aligned}$$

Therefore

$$A(x,x,\ldots,x,g(x,y))+A(y,y,\ldots,y,g(y,x))\leqslant(a_3+a_4)(A(x,x,\ldots,x,g(x,y))+A(y,y,\ldots,y,g(y,x))).$$

Since $0\leqslant(a_3+a_4)<1$, $(A(x,x,x,\ldots,x,g(x,y))=A(y,y,y,\ldots,y,g(y,x))=0$. That is, $g(x,y)=x$ and $g(y,x)=y$. This implies (x,y) is a coupled fixed point of g.

In a similar fashion, suppose g is continuous, then we have

$$x=\lim_{k\to\infty}x_{2k+2}=\lim_{k\to\infty}g(x_{2k+1},y_{2k+1})=g(\lim_{k\to\infty}x_{2k+1},\lim_{k\to\infty}y_{2k+1})=g(x,y)$$

and

$$y = \lim_{k\to\infty} y_{2k+2} = \lim_{k\to\infty} g(y_{2k+1}, x_{2k+1}) = g(\lim_{k\to\infty} y_{2k+1}, \lim_{k\to\infty} x_{2k+1}) = g(y, x).$$

Applying (2.1) again, we also get

$$A(x, x, \dots, x, f(x, y)) + A(y, y, \dots, y, f(y, x)) \leqslant (a_2 + a_5)(A(x, x, \dots, x, f(x, y)) + A(x, x, \dots, x, f(y, x))).$$

This implies $f(x, y) = x$ and $f(y, x) = y$ and so (x, y) is as well a coupled fixed point of f. Therefore, (x, y) is a coupled common fixed point of f and g.

Finally, suppose X satisfies hypotheses 3 (a) and 3 (b). Then by (2.3) we get $x_k \preceq x$ and $y \preceq y_k$ for all $k \in \mathbb{N}$. Applying Lemmas 1.11 and 1.12, we obtain

$$\begin{aligned}
&D((x, y), (x, y), \dots, (f(x, y), f(y, x)))\\
&\quad\leqslant b(n-1)D((x, y), (x, y), \dots, (x, y), (x_{2k+2}, y_{2k+2}))\\
&\qquad + b^2 D((x_{2k+2}, y_{2k+2}), (x_{2k+2}, y_{2k+2}), \dots, (x_{2k+2}, y_{2k+2}), (f(x, y), f(y, x)))\\
&\quad= b(n-1)D((x, y), (x, y), \dots, (x, y), (x_{2k+2}, y_{2k+2}))\\
&\qquad + b^2 D(g(x_{2k+1}, y_{2k+1}), g(y_{2k+1}, x_{2k+1})), (g(x_{2k+1}, y_{2k+1}), g(y_{2k+1}, x_{2k+1})), \dots\\
&\qquad\ \dots, (g(x_{2k+1}, y_{2k+1}), g(y_{2k+1}, x_{2k+1})), (f(x, y), f(y, x))\\
&\quad\leqslant b(n-1)D((x, y), (x, y), \dots, (x, y), (x_{2k+2}, y_{2k+2}))\\
&\qquad + b^2 A(g(x_{2k+1}, y_{2k+1}), g(x_{2k+1}, y_{2k+1}), \dots, g(x_{2k+1}, y_{2k+1}), f(x, y))\\
&\qquad + b^2 A(g(y_{2k+1}, x_{2k+1}), g(y_{2k+1}, x_{2k+1}), \dots, g(y_{2k+1}, x_{2k+1}), f(y, x)).
\end{aligned}$$

This implies

$$\begin{aligned}
&D((x, y), (x, y), \dots, (f(x, y), f(y, x)))\\
&\quad\leqslant b(n-1)A(x, x, \dots, x, x_{2k+2}) + b(n-1)A(y, y, \dots, y, y_{2k+2}))\\
&\qquad + b^2 A(g(x_{2k+1}, y_{2k+1}), g(x_{2k+1}, y_{2k+1}), \dots, g(x_{2k+1}, y_{2k+1}), f(x, y))\\
&\qquad + b^2 A(g(y_{2k+1}, x_{2k+1}), g(y_{2k+1}, x_{2k+1}), \dots, g(y_{2k+1}, x_{2k+1}), f(y, x)).
\end{aligned} \tag{2.10}$$

By using (2.1) and interchanging the roles of f with g we obtain

$$\begin{aligned}
&A(g(x_{2k+1}, y_{2k+1}), g(x_{2k+1}, y_{2k+1}), \dots, g(x_{2k+1}, y_{2k+1}), f(x, y))\\
&\qquad + A(g(y_{2k+1}, x_{2k+1}), g(y_{2k+1}, x_{2k+1}), \dots, g(y_{2k+1}, x_{2k+1}), f(y, x))\\
&\quad\leqslant a_1 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}) \dots, (x_{2k+1}, y_{2k+1}), (x, y))\\
&\qquad + a_2 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}) \dots, (x_{2k+1}, y_{2k+1}), (g(x_{2k+1}, y_{2k+1}), g(y_{2k+1}, x_{2k+1})))\\
&\qquad + a_3 D((x, y), (x, y), \dots, (x, y), (f(x, y), f(y, x)))\\
&\qquad + a_4 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \dots, (x_{2k+1}, y_{2k+1}), (f(x, y), f(y, x)))\\
&\qquad + a_5 D((x, y), (x, y), \dots, (x, y), (g(x_{2k+1}, y_{2k+1}), g(y_{2k+1}, x_{2k+1})))\\
&\quad= a_1 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}) \dots, (x_{2k+1}, y_{2k+1}), (x, y))\\
&\qquad + a_2 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}) \dots, (x_{2k+1}, y_{2k+1}), (x_{2k+2}, y_{2k+2}))\\
&\qquad + a_3 D((x, y), (x, y), \dots, (x, y), (f(x, y), f(y, x)))\\
&\qquad + a_4 D((x_{2k+1}, y_{2k+1}), (x_{2k+1}, y_{2k+1}), \dots, (x_{2k+1}, y_{2k+1}), (f(x, y), f(y, x)))\\
&\qquad + a_5 D((x, y), (x, y), \dots, (x, y), (x_{2k+2}, y_{2k+2})).
\end{aligned}$$

It follows from (2.8) and (2.9) that

$$D((x, y), (x, y), \dots, (f(x, y), f(y, x)))$$

$$\begin{aligned}
&\leqslant b[(n-1)A(x,x,\dots,x,x_{2k+2})+(n-1)A(y,y,\dots,y,y_{2k+2})]\\
&\quad + b^2[a_1D((x_{2k+1},y_{2k+1}),(x_{2k+1},y_{2k+1})\dots,(x_{2k+1},y_{2k+1}),(x,y))\\
&\quad + a_2D((x_{2k+1},y_{2k+1}),(x_{2k+1},y_{2k+1}),\dots,(x_{2k+1},y_{2k+1}),(x_{2k+2},y_{2k+2}))\\
&\quad + a_3D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x)))\\
&\quad + a_4D((x_{2k+1},y_{2k+1}),(x_{2k+1},y_{2k+1}),\dots,(x_{2k+1},y_{2k+1}),(f(x,y),f(y,x)))\\
&\quad + a_5D((x,y),(x,y),\dots,(x,y),(x_{2k+2},y_{2k+2}))].
\end{aligned}$$

Taking the limit as $k\to\infty$ in (2.10), we get

$$\begin{aligned}
&D((x,y),(x,y),\dots,(f(x,y),f(y,x)))\\
&\leqslant b[(n-1)A(x,x,\dots,x,x)+(n-1)A(y,y,\dots,y,y)]b^2[a_1D((x,y),(x,y),\dots,(x,y),(x,y))\\
&\quad + a_2D((x,y),(x,y)\dots,(x,y),(x,y))+a_3D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x)))\\
&\quad + a_4D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x)))+a_5D((x,y),(x,y),\dots,(x,y),(x,y))]\\
&= b^2[a_3D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x)))+a_4D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x)))].
\end{aligned}$$

Therefore,

$$D((x,y),(x,y),\dots,(f(x,y),f(y,x)))\leqslant b^2(a_3+a_4)D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))).$$

Since $b^2(a_3+a_4)<1$, we have $D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x)))=0$. That is $f(x,y)=x$ and $f(y,x)=y$. This implies (x,y) is a coupled fixed point of f.

Similarly, we can show that $g(x,y)=x$ and $g(y,x)=y$.

Hence, $f(x,y)=x=g(x,y)$ and $f(y,x)=y=g(y,x)$. Thus (x,y) is a coupled common fixed point of f and g. This completes the proof. □

Theorem 2.2. *In addition to the hypotheses of Theorem 2.1, if X is a totally ordered set, then f and g have a unique coupled common fixed point. Furthermore, any fixed point of f is a fixed point of g, and conversely.*

Proof. Let X be a totally ordered set. Suppose (x,y), (x^*,y^*) are coupled common fixed points of f and g. That is, $f(x,y)=x, f(y,x)=y$, and $g(x^*,y^*)=x^*$, $g(y^*,x^*)=y^*$. We show that $x=x^*, y=y^*$ and subsequently $x=y$.

Observe that if X is a totally ordered set, then, for every $(x,y),(x^*,y^*)\in X\times X$, there exists $(u,v)\in X\times X$ that is comparable to (x,y) and (x^*,y^*).

So we let $(x,y)\preceq(x^*,y^*)$ without loss of generality, then it follows from Lemma 1.12 and Theorem 2.1 that

$$\begin{aligned}
&D((x,y),(x,y),\dots,(x,y),(x^*,y^*))\\
&= A(x,x,\dots,x,x^*)+A(y,y,\dots,y,y^*)\\
&= A(f(x,y),f(x,y),\dots,f(x,y),g(x^*,y^*))+A(f(y,x),f(y,x),\dots,f(y,x),g(y^*,x^*))\\
&\leqslant a_1D((x,y),(x,y),\dots,(x,y),(x^*,y^*))+a_2D((x,y),(x,y)\dots,(x,y),(f(x,y),f(y,x))\\
&\quad + a_3D((x^*,y^*),(x^*,y^*),\dots,(x^*,y^*),(g(x^*,y^*),g(y^*,x^*)))\\
&\quad + a_4D((x,y),(x,y),\dots,(x,y),(g(x^*,y^*),g(y^*,x^*)))+a_5D((x^*,y^*),(x^*,y^*),\dots,(x^*,y^*),(f(x,y),f(y,x)))\\
&= a_1D((x,y),(x,y),\dots,(x,y),(x^*,y^*))+a_2D((x,y),(x,y)\dots,(x,y),(x,y))\\
&\quad + a_3D((x^*,y^*),(x^*,y^*),\dots,(x^*,y^*),(x^*,y^*))\\
&\quad + a_4D((x,y),(x,y),\dots,(x,y),(x^*,y^*))+a_5D((x^*,y^*),(x^*,y^*),\dots,(x^*,y^*),(x,y))\\
&= a_1D((x,y),(x,y),\dots,(x,y),(x^*,y^*))+a_4D((x,y),(x,y),\dots,(x,y),(x^*,y^*))\\
&\quad + a_5D((x^*,y^*),(x^*,y^*),\dots,(x^*,y^*),(x,y))\\
&= (a_1+a_4+a_5)D((x,y),(x,y),\dots,(x,y),(x^*,y^*)).
\end{aligned}$$

Since $0 \leqslant (a_1 + b^2 a_4 + a_5) < 1$, we have $D((x, y), (x, y), \ldots, (x, y), (x^*, y^*)) = 0$, which implies $x = x^*$ and $y = y^*$.

Proceeding, we show that any fixed point of f is a fixed point of g, and conversely. Applying Lemma 1.12 and Theorem 2.1 we have

$$
\begin{aligned}
&A(x, x, \ldots, x, y) + A(y, y, \ldots, y, x)\\
&= A(f(x, y), f(x, y), \ldots, f(x, y), g(y, x)) + A(f(y, x), f(y, x), \ldots, f(y, x), g(x, y))\\
&\leqslant a_1 D((x, y), (x, y), \ldots, (x, y), (y, x)) + a_2 D((x, y), (x, y) \ldots, (x, y), (f(x, y), f(y, x))\\
&\quad + a_3 D((y, x), (y, x), \ldots, (y, x), (g(y, x), g(x, y)))\\
&\quad + a_4 D((x, y), (x, y), \ldots, (x, y), (g(y, x), g(x, y))) + a_5 D((y, x), (y, x), \ldots, (y, x), (f(x, y), f(y, x)))\\
&= a_1 D((x, y), (x, y), \ldots, (x, y), (y, x)) + a_2 D((x, y), (x, y) \ldots, (x, y), (x, y))\\
&\quad + a_3 D((y, x), (y, x), \ldots, (y, x), (y, x))\\
&\quad + a_4 D((x, y), (x, y), \ldots, (x, y), (y, x)) + a_5 D((y, x), (y, x), \ldots, (y, x), (x, y))\\
&= a_1 D((x, y), (x, y), \ldots, (x, y), (y, x)) + a_4 D((x, y), (x, y), \ldots, (x, y), (y, x))\\
&\quad + a_5 D((x, y), (x, y), \ldots, (x, y), (y, x)).
\end{aligned}
$$

Therefore

$$
A(x, x, \ldots, x, y) + A(y, y, \ldots, y, x) \leqslant (a_1 + a_4 + a_5)(A(x, x, \ldots, x, y) + A(y, y, \ldots, y, x)).
$$

Since $(a_1 + b^4 a_4 + a_5) < 1$, $A(x, x, \ldots, x, y) + A(y, y, \ldots, y, x) = 0$. That is, $x = y$. The coupled common fixed point of f and g is unique. □

Example 2.3. Let $(\mathbb{R}, \preceq, A)$ be totally ordered complete A_b-metric space with A_b-metric defined as in Example 1.7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two maps defined by $f(x, y) = \frac{6x-3y+24n-3}{24n}$ and $g(x, y) = \frac{8x-4y+32n-4}{32n}$ for all $n \geqslant 1$. The pair (f, g) has the mixed weakly monotone property on R and

$$
\begin{aligned}
&A(f(x, y), f(x, y), \ldots, f(x, y), g(u, v)) + A(f(y, x), f(y, x), \ldots, f(y, x), g(v, u))\\
&= (n-1) \mid f(x, y) - g(u, v) \mid + (n-1) \mid f(y, x) - g(v, u) \mid\\
&= (n-1) \mid \frac{6x - 3y + 24n - 3}{24n} - \frac{8u - 4v + 32n - 4}{32n} \mid + (n-1) \mid \frac{6y - 3x + 24n - 3}{24n} - \frac{8v - 4u + 32n - 4}{32n} \mid\\
&= \frac{(n-1)}{768n} \mid 192(x-u) + 96(y-v) \mid + \frac{(n-1)}{768n} \mid 192(y-v) + 96(x-u) \mid\\
&\leqslant \frac{192(n-1)}{768n}(\mid |x-u| + |y-v| + |y-v| + |x-u| \mid)\\
&= \frac{(n-1)}{2n}(\mid |x-u| + |y-v| \mid).
\end{aligned}
$$

Then the contractive condition (2.2) is satisfied with $a_1 = \frac{n-1}{n}$, $a_2 = a_3 = a_4 = a_5 = 0$. Moreover, (1.1) is the unique coupled common fixed point of f and g.

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