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Solvability of a fractional functional equation arising in some epidemic models

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Abstract

We provide sufficient conditions for the existence of solutions to a fractional generalized Gripenberg equation, which arises in the study of the spread of an infectious disease that does not induce permanent immunity.

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1. Introduction

In [6], Gripenberg studied the qualitative behavior of solutions of the functional equation

$$x(t) = k\left[p(t) + \int_0^t A(t-s)x(s)\,ds\right]\left[f(t) + \int_0^t a(t-s)x(s)\,ds\right], \quad t \geqslant 0, \tag{1.1}$$

where $k > 0$ is a constant, p f, A and a are continuous functions. Equation (1.1) arises in the study of the spread of an infectious disease that does not induce permanent immunity. For more details about the meanings of the various parameters appearing in (1.1), we refer to [6]. In [3], using the Banach contraction principle, an existence result was proved for the following generalized Gripenberg's equation

$$x(t) = \left[g_1(t) + \int_0^t A_1(t-s)x(s)\,ds\right]\cdots\left[g_q(t) + \int_0^t A_q(t-s)x(s)\,ds\right], \quad t \geqslant 0.$$

In [7, 8], Olaru studied the solvability of the functional equation

$$x(t) = \prod_{i=1}^{q}(A_i x)(t), \quad t \in [a,b],$$

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where

$$(A_i x)(t) = g_i(t) + \int_a^t K_i(t, s, x(s))\, ds, \quad i = 1, \cdots, q.$$

The used techniques in [7, 8] are based on weakly Picard operators.

In this paper, we deal with the solvability of the fractional functional equation

$$x(t) = \prod_{i=1}^{q} \left(g_i(t) + \frac{(S_i x)(t)}{\Gamma(\alpha_i)} \int_a^t \frac{h_i'(s) K_i(t, s, x(s), (T_i x)(s))}{(h_i(t) - h_i(s))^{1-\alpha_i}}\, ds \right), \quad t \in [a, b], \tag{1.2}$$

where $q \geqslant 1$, $0 < \alpha_i < 1$, $g_i, h_i : [a, b] \to \mathbb{R}$, $S_i, T_i : C([a, b]; \mathbb{R}) \to C([a, b]; \mathbb{R})$, $1 \leqslant i \leqslant q$. Here, Γ is the gamma function.

2. Notations, definitions, and preliminaries

In this section, we present some auxiliary facts which will be used throughout this work. At first, let us fix some notations.

Let $(E, \cdot)$ be a Banach algebra over $\mathbb{R}$ with respect to a certain norm $\|\cdot\|_E$. We denote by 0_E the zero vector of E. For $x \in E$ and $r > 0$, we denote by $B(x, r)$ the open ball in E of center x and radius r, i.e.,

$$B(x, r) = \{y \in E : \|x - y\|_E < r\}.$$

We denote by $\mathcal{P}(E)$ the set of all nonempty subsets of E. If $M \in \mathcal{P}(E)$, then the symbol $\overline{M}$ denotes the closure of M. The symbol $\mathrm{Conv}(M)$ stands for the convex hull of M. For $(M, N) \in \mathcal{P}(E) \times \mathcal{P}(E)$ and $\alpha \in \mathbb{R}$, we denote

$$M + N = \{x + y : (x, y) \in M \times N\},$$

and

$$\alpha M = \{\alpha x : x \in M\}.$$

We denote by $\mathcal{P}_b(E)$ the set of all nonempty and bounded subsets of E. For $M \in \mathcal{P}_b(E)$, we denote by $\|M\|$ the norm of M, i.e.,

$$\|M\| = \sup\{\|x\|_E : x \in M\}.$$

We denote by $\mathcal{P}_{rc}(E)$ the set of all relatively compact subsets of E. For $M, N \in \mathcal{P}(E)$, we denote by MN the product set

$$MN = \{x \cdot y : (x, y) \in M \times N\}.$$

In what follows, we recall the axiomatic approach of a measure of noncompactness introduced by Banas and Goebel [1].

Definition 2.1. Let $\mu : \mathcal{P}_b(E) \to [0, \infty)$ be a given mapping. We say that μ is a measure of noncompactness in E if it satisfies the following axioms:

(A1) The family $\ker \mu = \mu^{-1}(\{0\})$ is a subset of $\mathcal{P}_{rc}(E)$.

(A2) $(M, N) \in \mathcal{P}_b(E) \times \mathcal{P}_b(E),\ M \subset N \implies \mu(M) \leqslant \mu(N)$.

(A3) $\mu\left(\overline{M}\right) = \mu(M)$, $M \in \mathcal{P}_b(E)$.

(A4) $\mu\left(\mathrm{Conv}(M)\right) = \mu(M)$, $M \in \mathcal{P}_b(E)$.

(A5) $\mu\left(\lambda M + (1 - \lambda) N\right) \leqslant \lambda \mu(M) + (1 - \lambda)\, \mu(N)$, for $\lambda \in [0, 1]$, $(M, N) \in \mathcal{P}_b(E) \times \mathcal{P}_b(E)$.

(A6) If $\{M_n\}$ is a sequence of closed sets from $\mathcal{P}_b(E)$ such that $M_{n+1} \subset M_n$ for $n = 1, 2, \cdots$ and if $\lim_{n \to \infty} \mu(M_n) = 0$, then the intersection set $M_\infty = \bigcap_{n=1}^{\infty} M_n$ is nonempty.

Now, let us recall the following important result, which is called Drabo's fixed point theorem [1, 4].

Lemma 2.2. *Let* D *be a nonempty, bounded, closed and convex subset of* E *and let* $T : D \to D$ *be a continuous mapping. Suppose that there exists a constant* $k \in (0,1)$ *such that*

$$\mu(TM) \leqslant k\mu(M),$$

for any nonempty subset M *of* D*, where* μ *is a measure of noncompactness in* E*. Then* T *has at least one fixed point in* D*.*

The following concept introduced by Banas and Olszowy [2] will be useful later.

Definition 2.3. Let μ be a measure of noncompactness in E. We say that μ satisfies condition (m) if

$$\mu(MN) \leqslant \|M\|\mu(N) + \|N\|\mu(M), \quad (M,N) \in \mathcal{P}_b(E) \times \mathcal{P}_b(E).$$

3. A fixed point result via a measure of noncompactness satisfying condition (m)

In this section, we establish a fixed point theorem, which will be used for the proof of our main result. At first, we need the following lemma.

Lemma 3.1. *Let* μ *be a measure of noncompactness in* E *satisfying condition* (m)*. Let* $\{M_i\}_{i=1,\cdots,q}$ *be a finite sequence in* $\mathcal{P}_b(E)$*,* $q \geqslant 2$*. Then*

$$\mu\left(\prod_{i=1}^{q} M_i\right) \leqslant \sum_{i=1}^{q} \prod_{j=1, j\neq i}^{q} \|M_j\|\mu(M_i). \tag{3.1}$$

Proof. We shall use the induction principle. For $q = 2$, (3.1) follows immediately from Definition 2.3. Suppose now that (3.1) is satisfied for some $q \geqslant 2$. We have to prove that

$$\mu\left(\prod_{i=1}^{q+1} M_i\right) \leqslant \sum_{i=1}^{q+1} \prod_{j=1, j\neq i}^{q+1} \|M_j\|\mu(M_i). \tag{3.2}$$

Using (3.1) and Definition 2.3, we have

$$\begin{aligned}
\mu\left(\prod_{i=1}^{q+1} M_i\right) &= \mu\left(\left(\prod_{i=1}^{q} M_i\right) M_{q+1}\right)\\
&\leqslant \mu\left(\prod_{i=1}^{q} M_i\right)\|M_{q+1}\| + \mu(M_{q+1})\left\|\prod_{i=1}^{q} M_i\right\|\\
&\leqslant \mu\left(\prod_{i=1}^{q} M_i\right)\|M_{q+1}\| + \mu(M_{q+1})\prod_{j=1}^{q}\|M_j\|\\
&\leqslant \sum_{i=1}^{q} \prod_{j=1, j\neq i}^{q} \|M_j\|\mu(M_i)\|M_{q+1}\| + \mu(M_{q+1})\prod_{j=1}^{q}\|M_j\|\\
&= \sum_{i=1}^{q} \prod_{j=1, j\neq i}^{q+1} \|M_j\|\mu(M_i)\|M_{q+1}\| + \mu(M_{q+1})\prod_{j=1}^{q}\|M_j\|\\
&= \sum_{i=1}^{q+1} \prod_{j=1, j\neq i}^{q+1} \|M_j\|\mu(M_i).
\end{aligned}$$

Thus we proved (3.2). Finally, by the induction principle, (3.1) follows. □

Let $D \in \mathcal{P}(E)$. We are interested in the fixed point problem:

$$\begin{cases} \text{Find } x \in D \text{ such that} \\ x = \prod_{i=1}^{q} A_i x, \end{cases} \tag{3.3}$$

where $A_i : D \to E$, $i = 1, \cdots, q$, $q \geqslant 1$, are given operators.

We have the following fixed point result.

Theorem 3.2. *Assume that* D *is nonempty, bounded, closed, and convex subset of the Banach algebra* E. *Assume also that all the following conditions are satisfied:*

(i) A_i *is continuous,* $i = 1, \cdots, q$.

(ii) $A_i D$ *is bounded,* $i = 1, \cdots, q$.

(iii) $AD \subset D$, *where* $Ax = \prod_{i=1}^{q} A_i x$.

(iv) *There exist a finite sequence* $\{k_i\}_{i=1}^{q} \subset (0, \infty)$ *such that*

$$\mu(A_i M) \leqslant k_i \mu(M), \quad i = 1, \cdots, q,$$

for any nonempty subset M *of* D, *where* μ *is a measure of noncompactness in* E *satisfying condition* (m).

(v) $\sum_{i=1}^{q} k_i \prod_{j=1, j\neq i}^{q} \|A_j D\| < 1$.

Then Problem (3.3) *has at least one solution in* D.

Proof. Let M be nonempty subset of D. Using Lemma 3.1 and the considered assumptions, we obtain

$$\begin{aligned} \mu(AM) &= \mu\left(\prod_{i=1}^{q} A_i M\right) \\ &\leqslant \sum_{i=1}^{q} \prod_{j=1, j\neq i}^{q} \|A_j M\| \mu(A_i M) \\ &\leqslant \sum_{i=1}^{q} k_i \prod_{j=1, j\neq i}^{q} \|A_j M\| \mu(M) \\ &\leqslant \left(\sum_{i=1}^{q} k_i \prod_{j=1, j\neq i}^{q} \|A_j D\|\right) \mu(M). \end{aligned}$$

Thus we proved that

$$\mu(AM) \leqslant k\mu(M),$$

for every nonempty subset M of D, where

$$k = \sum_{i=1}^{q} k_i \prod_{j=1, j\neq i}^{q} \|A_j D\| < 1.$$

Applying Lemma 2.2, the result follows. □

Remark 3.3. For $q = 2$, Theorem 3.2 recduces to a fixed point theroem established in [2].

Remark 3.4. For $q = 1$, Theorem 3.2 recduces to Lemma 2.2.

4. A solvability result

In this section, we state and prove our main result concerning the existence of solutions to the fractional functional equation (1.2).

We take $E = C([a, b]; \mathbb{R})$ the Banach space of all real-valued and continuous functions in $[a, b]$ equipped with the norm

$$\|u\|_E = \max\{|u(t)| : t \in [a, b]\}, \quad u \in E.$$

Clearly E is a Banach algebra with respect to the operation "$\cdot$" defined by

$$(u \cdot v)(t) = u(t)v(t), \quad t \in [a, b], \ (u, v) \in E \times E.$$

Let $M \in \mathcal{P}_b(E)$. For $x \in M$ and $\varepsilon > 0$, set

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [a, b], |t - s| \leqslant \varepsilon\},$$
$$\omega(M, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in M\}.$$

It was proved in [1] that the mapping $\mu : \mathcal{P}_b(E) \to [0, \infty)$ defined by

$$\mu(M) = \lim_{\varepsilon \to 0^+} \omega(M, \varepsilon), \quad M \in \mathcal{P}_b(E), \tag{4.1}$$

is a measure of noncompactness in E. Moreover, μ satisfies condition (m) (see [2]).

Equation (1.2) is investigated under the following conditions:

(C1) $g_i \in E$, $i = 1, \cdots, q$.

(C2) $T_i : E \to E$, $i = 1, \cdots, q$, and
$$\|T_i x\|_E \leqslant c_i \|x\|_E^{\gamma_i}, \quad x \in E,$$
where $c_i > 0$ and $\gamma_i > 0$ are constants.

(C3) For all $i = 1, \cdots, q$,
$$\|T_i x - T_i y\|_E \leqslant L_i \|x - y\|_E^{\ell_i}, \quad (x, y) \in E \times E,$$
where $L_i > 0$ and $\ell_i > 0$ are constants.

(C4) $S_i : E \to E$, $i = 1, \cdots, q$, is continuous, and
$$\mu(S_i M) \leqslant \nu_i \mu(M), \quad M \in \mathcal{P}_b(E),$$
where $\nu_i > 0$ is a constant and μ is given by (4.1).

(C5) For all $i = 1, \cdots, q$, there exist constants $\theta_i > 0$, $\kappa_i > 0$ and $\phi_i > 0$ such that
$$\|S_i x\|_E \leqslant \theta_i + \kappa_i \|x\|_E^{\phi_i}, \quad x \in E.$$

(C6) $K_i : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, q$, is continuous.

(C7) There exists $\varphi_i : [0, \infty) \times [0, \infty) \to [0, \infty)$, $i = 1, \cdots, q$, a nondecreasing function with respect to each variable, such that
$$|K_i(t, s, u, v)| \leqslant \varphi_i(|u|, |v|), \quad (t, s, u, v) \in [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R}.$$

(C8) $h_i \in E \cap C^1((a, b]; \mathbb{R})$ is nondecreasing, $i = 1, \cdots, q$.

(C9) There exists $R > 0$ such that
$$\|g_i\|_E + \frac{(\theta_i + \kappa_i R^{\phi_i}) \varphi_i(R, c_i R_E^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \leqslant R^{1/q}, \quad i = 1, \cdots, q.$$

For $i = 1, \cdots, q$ and $x \in E$, let

$$(A_i x)(t) = g_i(t) + \frac{(S_i x)(t)}{\Gamma(\alpha_i)} \int_a^t \frac{h_i'(s) K_i(t, s, x(s), (T_i x)(s))}{(h_i(t) - h_i(s))^{1-\alpha_i}}\, ds, \quad t \in [a, b].$$

We have the following result.

Lemma 4.1. *For every* $i = 1, \cdots, q$*, the operator* $A_i : E \to E$ *is well-defined.*

Proof. Let $i \in \{1, \cdots, q\}$ be fixed, and let $x \in E$. We have to prove that $A_i x \in E$. In view of assumptions (C1) and (C4), it is sufficient to prove that $B_i x \in E$, where

$$(B_i x)(t) = \int_a^t \frac{h_i'(s) K_i(t, s, x(s), (T_i x)(s))}{(h_i(t) - h_i(s))^{1-\alpha_i}}\, ds, \quad t \in [a, b]. \tag{4.2}$$

At first, observe that $(B_i x)(t) < \infty$, for every $t \in [a, b]$. Indeed, using the considered assumptions, for all $t \in [a, b]$, we have

$$\begin{aligned} |(B_i x)(t)| &\leqslant \int_a^t \frac{h_i'(s) |K_i(t, s, x(s), (T_i x)(s))|}{(h_i(t) - h_i(s))^{1-\alpha_i}}\, ds \\ &\leqslant \int_a^t \frac{h_i'(s) \varphi_i(|x(s)|, |(T_i x)(s)|)}{(h_i(t) - h_i(s))^{1-\alpha_i}}\, ds \\ &\leqslant \int_a^t \frac{h_i'(s) \varphi_i(\|x\|_E, \|T_i x\|_E)}{(h_i(t) - h_i(s))^{1-\alpha_i}}\, ds \\ &\leqslant \varphi_i(\|x\|_E, c_i \|x\|_E^{\gamma_i}) \int_a^t \frac{h_i'(s)}{(h_i(t) - h_i(s))^{1-\alpha_i}}\, ds \\ &= \frac{\varphi_i(\|x\|_E, c_i \|x\|_E^{\gamma_i})(h_i(t) - h_i(a))^{\alpha_i}}{\alpha_i} < \infty. \end{aligned} \tag{4.3}$$

Now, we check that $B_i x$ is continuous at $t_0 = a$. Let $\{t_n\} \subset [a, b]$ be a sequence such that

$$t_n \to a, \text{ as } n \to \infty.$$

Using the estimate (4.3), we obtain

$$|(B_i x)(t_n)| \leqslant \frac{\varphi_i(\|x\|_E, c_i \|x\|_E^{\gamma_i})(h_i(t_n) - h_i(a))^{\alpha_i}}{\alpha_i}.$$

Since $h_i \in E$, we have

$$h_i(t_n) - h_i(a) \to 0, \text{ as } n \to \infty.$$

Therefore, from the above inequality,

$$(B_i x)(t_n) \to 0 = (B_i x)(a), \text{ as } n \to \infty,$$

which proves the continuity of $B_i x$ at $t_0 = a$.

Now, suppose that $t \in (a, b]$, and let $\{t_n\} \subset [a, b]$ be a sequence such that $t_n \to t$, as $n \to \infty$. Without restriction of the generality, we may assume that $t_n \geqslant t$, for every n. Under the considered assumptions, we have

$$\begin{aligned} |(B_i x)(t_n) - (B_i x)(t)| &\leqslant \left| \int_a^{t_n} \frac{h_i'(s) K_i(t_n, s, x(s), (T_i x)(s))}{(h_i(t_n) - h_i(s))^{1-\alpha_i}}\, ds - \int_a^t \frac{h_i'(s) K_i(t, s, x(s), (T_i x)(s))}{(h_i(t) - h_i(s))^{1-\alpha_i}}\, ds \right| \\ &\leqslant \left| \int_a^{t_n} \frac{h_i'(s) K_i(t_n, s, x(s), (T_i x)(s))}{(h_i(t_n) - h_i(s))^{1-\alpha_i}}\, ds - \int_a^{t_n} \frac{h_i'(s) K_i(t, s, x(s), (T_i x)(s))}{(h_i(t_n) - h_i(s))^{1-\alpha_i}}\, ds \right| \end{aligned}$$

$$
\begin{aligned}
&+\left|\int_a^{t_n} \frac{h_i'(s)K_i(t,s,x(s),(T_ix)(s))}{(h_i(t_n)-h_i(s))^{1-\alpha_i}}\,ds-\int_a^{t} \frac{h_i'(s)K_i(t,s,x(s),(T_ix)(s))}{(h_i(t_n)-h_i(s))^{1-\alpha_i}}\,ds\right|\\
&+\left|\int_a^{t} \frac{h_i'(s)K_i(t,s,x(s),(T_ix)(s))}{(h_i(t_n)-h_i(s))^{1-\alpha_i}}\,ds-\int_a^{t} \frac{h_i'(s)K_i(t,s,x(s),(T_ix)(s))}{(h_i(t)-h_i(s))^{1-\alpha_i}}\,ds\right|\\
\leqslant &\int_a^{t_n} \frac{h_i'(s)\,|K_i(t_n,s,x(s),(T_ix)(s))-K_i(t,s,x(s),(T_ix)(s))|}{(h_i(t_n)-h_i(s))^{1-\alpha_i}}\,ds\\
&+\int_t^{t_n} \frac{h_i'(s)|K_i(t,s,x(s),(T_ix)(s))|}{(h_i(t_n)-h_i(s))^{1-\alpha_i}}\,ds\\
&+\int_a^{t}\left(\frac{h_i'(s)}{(h_i(t)-h_i(s))^{1-\alpha_i}}-\frac{h_i'(s)}{(h_i(t_n)-h_i(s))^{1-\alpha_i}}\right)|K_i(t,s,x(s),(T_ix)(s))|\,ds\\
\leqslant &\frac{\omega_n}{\alpha_i}(h_i(t_n)-h_i(a))^{\alpha_i}+\frac{\varphi_i(\|x\|_E,c_i\|x\|_E^{\gamma_i})}{\alpha_i}(h_i(t_n)-h_i(t))^{\alpha_i}\\
&+\frac{\varphi_i(\|x\|_E,c_i\|x\|_E^{\gamma_i})}{\alpha_i}\left[(h_i(t)-h_i(a))^{\alpha_i}+(h_i(t_n)-h_i(t))^{\alpha_i}-(h_i(t_n)-h_i(a))^{\alpha_i}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_n = \sup\Big\{ &|K_i(t_1,s,y_1,y_2)-K_i(t_2,s,y_1,y_2)| :\ t_1,t_2,s\in[a,b],\ |t_1-t_2|\leqslant|t_n-t|,\\
&|y_1|\leqslant\|x\|_E,\ |y_2|\leqslant c_i\|x\|_E^{\gamma_i}\Big\}.
\end{aligned}
$$

Since K_i is uniformly continuous on the compact set

$$[a,b]\times[a,b]\times[-\|x\|_E,\|x\|_E]\times[-c_i\|x\|_E^{\gamma_i},c_i\|x\|_E^{\gamma_i}],$$

we have

$$\lim_{n\to\infty}\omega_n=0. \tag{4.4}$$

Now, we have

$$|(B_ix)(t_n)-(B_ix)(t)|\leqslant\chi_n, \tag{4.5}$$

where

$$
\begin{aligned}
\chi_n = &\frac{\omega_n}{\alpha_i}(h_i(t_n)-h_i(a))^{\alpha_i}+\frac{\varphi_i(\|x\|_E,c_i\|x\|_E^{\gamma_i})}{\alpha_i}(h_i(t_n)-h_i(t))^{\alpha_i}\\
&+\frac{\varphi_i(\|x\|_E,c_i\|x\|_E^{\gamma_i})}{\alpha_i}\left[(h_i(t)-h_i(a))^{\alpha_i}+(h_i(t_n)-h_i(t))^{\alpha_i}-(h_i(t_n)-h_i(a))^{\alpha_i}\right].
\end{aligned}
$$

Passing to the limit as $n\to\infty$, using the continuity of h_i and (4.4), we infer that

$$\lim_{n\to\infty}\chi_n=0,$$

which implies from (4.5) that

$$\lim_{n\to\infty}|(B_ix)(t_n)-(B_ix)(t)|=0.$$

This proves the continuity of B_ix on $[a,b]$. Therefore, $A_i: E\to E$ is well-defined. □

Lemma 4.2. *For every $i=1,\cdots,q$, $A_i\overline{B(0_E,R)}$ is bounded.*

Proof. Let $i\in\{1,\cdots,q\}$ be fixed, and let $x\in\overline{B(0_E,R)}$. Under the considered assumptions, for all $t\in[a,b]$, we have

$$|(A_ix)(t)|\leqslant\|g_i\|_E+\frac{|(S_ix)(t)|}{\Gamma(\alpha_i)}|(B_ix)(t)|.$$

From (4.3), we have

$$|(B_i x)(t)| \leqslant \frac{\varphi_i(R, c_i R_E^{\gamma_i})(h_i(b)-h_i(a))^{\alpha_i}}{\alpha_i}.$$

On the other hand, by (C5), we have

$$|(S_i x)(t)| \leqslant \theta_i + \kappa_i R^{\phi_i}, \quad t \in [a,b].$$

Therefore, we obtain

$$\|A_i x\|_E \leqslant \|g_i\|_E + \frac{(\theta_i + \kappa_i R^{\phi_i})\varphi_i(R, c_i R_E^{\gamma_i})(h_i(b)-h_i(a))^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

Using assumption (C9), we get

$$\|A_i x\|_E \leqslant R^{1/q}.$$

Therefore, we have

$$A_i \overline{B(0_E, R)} \subset \overline{B(0_E, R^{1/q})}, \tag{4.6}$$

which yields the desired result.

□

Lemma 4.3. *For every* $i = 1, \cdots, q$, $A_i : \overline{B(0_E, R)} \to E$ *is continuous.*

Proof. Let $i \in \{1, \cdots, q\}$ be fixed. We have just to prove that $B_i : \overline{B(0_E, R)} \to E$ is continuous, where B_i is defined by (4.2). Fix $\varepsilon > 0$ and take $x, y \in \overline{B(0_E, R)}$ such that $\|x-y\|_E < \varepsilon$. Then, for any $t \in [a,b]$, we have

$$\begin{aligned}
|(B_i x)(t) - (B_i y)(t)| &= \left| \int_a^t \frac{h_i'(s) K_i(t,s,x(s),(T_i x)(s))}{(h_i(t)-h_i(s))^{1-\alpha_i}}\, ds - \int_a^t \frac{h_i'(s) K_i(t,s,y(s),(T_i y)(s))}{(h_i(t)-h_i(s))^{1-\alpha_i}}\, ds \right| \\
&\leqslant \int_a^t \frac{h_i'(s)}{(h_i(t)-h_i(s))^{1-\alpha_i}} |K_i(t,s,x(s),(T_i x)(s)) - K_i(t,s,y(s),(T_i y)(s))|\, ds \\
&\leqslant \chi_\varepsilon \int_a^t \frac{h_i'(s)}{(h_i(t)-h_i(s))^{1-\alpha_i}}\, ds \\
&\leqslant \frac{\chi_\varepsilon}{\alpha_i}(h_i(b)-h_i(a))^{\alpha_i},
\end{aligned}$$

where

$$\begin{aligned}
\chi_\varepsilon = \sup \Big\{ & |K_i(t,s,u_1,v_1) - K_i(t,s,u_2,v_2)| : t,s \in [a,b],\ u_1,u_2 \in [-R,R], \\
& v_1, v_2 \in [-c_i R^{\gamma_i}, c_i R^{\gamma_i}],\ |u_1-u_2| \leqslant \varepsilon,\ |v_1-v_2| \leqslant L_i \varepsilon^{\ell_i} \Big\}.
\end{aligned}$$

Therefore, we have

$$\|B_i x - B_i y\|_E \leqslant \frac{\chi_\varepsilon}{\alpha_i}(h_i(b)-h_i(a))^{\alpha_i}.$$

By the uniform continuity of K_i on the compact set

$$[a,b] \times [a,b] \times [-R,R] \times [-c_i R^{\gamma_i}, c_i R^{\gamma_i}],$$

it follows that

$$\chi_\varepsilon \to 0, \text{ as } \varepsilon \to 0.$$

This proves that B_i is continuous on $\overline{B(0_E, R)}$. □

Now, for $i = 1, \cdots, q$, we estimate $\mu(A_i M)$, where M is a nonempty subset of $\overline{B(0_E, R)}$ and μ is the measure of noncompactness in E given by (4.1). Fix $\varepsilon > 0$, $x \in M$, and take $t_1, t_2 \in [a, b]$ such that $|t_1 - t_2| \leqslant \varepsilon$. Without loss of the generality, we may assume that $t_2 \geqslant t_1$. Then, taking into account our assumptions, we get

$$\begin{aligned} |(A_i x)(t_2) - (A_i x)(t_1)| &= \left| g_i(t_2) + \frac{(S_i x)(t_2)}{\Gamma(\alpha_i)}(B_i x)(t_2) - g_i(t_1) - \frac{(S_i x)(t_1)}{\Gamma(\alpha_i)}(B_i x)(t_1) \right| \\ &\leqslant |g_i(t_2) - g_i(t_1)| + \frac{1}{\Gamma(\alpha_i)} |(S_i x)(t_2)(B_i x)(t_2) - (S_i x)(t_1)(B_i x)(t_1)| \\ &\leqslant \omega(g_i, \varepsilon) + \frac{1}{\Gamma(\alpha_i)} |(S_i x)(t_2)(B_i x)(t_2) - (S_i x)(t_1)(B_i x)(t_1)|, \end{aligned} \tag{4.7}$$

where

$$\omega(g_i, \varepsilon) = \sup\{|g_i(\tau_2) - g_i(\tau_1)| : \tau_1, \tau_2 \in [a, b], |\tau_1 - \tau_2| \leqslant \varepsilon\}.$$

But using (4.3), we have

$$\begin{aligned} &|(S_i x)(t_2)(B_i x)(t_2) - (S_i x)(t_1)(B_i x)(t_1)| \\ &\leqslant |(S_i x)(t_2)(B_i x)(t_2) - (S_i x)(t_2)(B_i x)(t_1)| \\ &\quad + |(S_i x)(t_2)(B_i x)(t_1) - (S_i x)(t_1)(B_i x)(t_1)| \\ &\leqslant |(S_i x)(t_2)||(B_i x)(t_2) - (B_i x)(t_1)| + |(B_i x)(t_1)||(S_i x)(t_2) - (S_i x)(t_1)| \\ &\leqslant (\theta_i + \kappa_i R^{\phi_i})|(B_i x)(t_2) - (B_i x)(t_1)| + |(B_i x)(t_1)|\omega(S_i x, \varepsilon) \\ &\leqslant (\theta_i + \kappa_i R^{\phi_i})|(B_i x)(t_2) - (B_i x)(t_1)| \\ &\quad + \frac{\varphi_i(R, c_i R^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i} \omega(S_i x, \varepsilon)}{\alpha_i}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |(S_i x)(t_2)(B_i x)(t_2) - (S_i x)(t_1)(B_i x)(t_1)| &\leqslant (\theta_i + \kappa_i R^{\phi_i})|(B_i x)(t_2) - (B_i x)(t_1)| \\ &\quad + \frac{\varphi_i(R, c_i R^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i} \omega(S_i x, \varepsilon)}{\alpha_i}. \end{aligned} \tag{4.8}$$

Let us estimate now

$$|(B_i x)(t_2) - (B_i x)(t_1)|.$$

Under the considered assumptions, we have

$$\begin{aligned} |(B_i x)(t_2) - (B_i x)(t_1)| &\leqslant \left| \int_a^{t_2} \frac{h_i'(s) K_i(t_2, s, x(s), (T_i x)(s))}{(h_i(t_2) - h_i(s))^{1-\alpha_i}} \, ds - \int_a^{t_1} \frac{h_i'(s) K_i(t_1, s, x(s), (T_i x)(s))}{(h_i(t_1) - h_i(s))^{1-\alpha_i}} \, ds \right| \\ &\leqslant \left| \int_a^{t_2} \frac{h_i'(s) K_i(t_2, s, x(s), (T_i x)(s))}{(h_i(t_2) - h_i(s))^{1-\alpha_i}} \, ds - \int_a^{t_2} \frac{h_i'(s) K_i(t_1, s, x(s), (T_i x)(s))}{(h_i(t_2) - h_i(s))^{1-\alpha_i}} \, ds \right| \\ &\quad + \left| \int_a^{t_2} \frac{h_i'(s) K_i(t_1, s, x(s), (T_i x)(s))}{(h_i(t_2) - h_i(s))^{1-\alpha_i}} \, ds - \int_a^{t_1} \frac{h_i'(s) K_i(t_1, s, x(s), (T_i x)(s))}{(h_i(t_2) - h_i(s))^{1-\alpha_i}} \, ds \right| \\ &\quad + \left| \int_a^{t_1} \frac{h_i'(s) K_i(t_1, s, x(s), (T_i x)(s))}{(h_i(t_2) - h_i(s))^{1-\alpha_i}} \, ds - \int_a^{t_1} \frac{h_i'(s) K_i(t_1, s, x(s), (T_i x)(s))}{(h_i(t_1) - h_i(s))^{1-\alpha_i}} \, ds \right| \\ &\leqslant \int_a^{t_2} \frac{h_i'(s) \, |K_i(t_2, s, x(s), (T_i x)(s)) - K_i(t_1, s, x(s), (T_i x)(s))|}{(h_i(t_2) - h_i(s))^{1-\alpha_i}} \, ds \\ &\quad + \int_t^{t_2} \frac{h_i'(s)|K_i(t_1, s, x(s), (T_i x)(s))|}{(h_i(t_2) - h_i(s))^{1-\alpha_i}} \, ds \end{aligned}$$

$$
\begin{aligned}
&+\int_a^{t_1}\left(\frac{h_i'(s)}{(h_i(t_1)-h_i(s))^{1-\alpha_i}}-\frac{h_i'(s)}{(h_i(t_2)-h_i(s))^{1-\alpha_i}}\right)|K_i(t_1,s,x(s),(T_ix)(s))|\,ds\\
&\leqslant \frac{U_\varepsilon}{\alpha_i}(h_i(t_2)-h_i(a))^{\alpha_i}+\frac{\varphi_i(\|x\|_E,c_i\|x\|_E^{\gamma_i})}{\alpha_i}(h_i(t_2)-h_i(t_1))^{\alpha_i}\\
&\quad+\frac{\varphi_i(\|x\|_E,c_i\|x\|_E^{\gamma_i})}{\alpha_i}\left[(h_i(t_1)-h_i(a))^{\alpha_i}+(h_i(t_2)-h_i(t_1))^{\alpha_i}-(h_i(t_2)-h_i(a))^{\alpha_i}\right]\\
&\leqslant \frac{U_\varepsilon}{\alpha_i}(h_i(t_2)-h_i(a))^{\alpha_i}+\frac{2\varphi_i(\|x\|_E,c_i\|x\|_E^{\gamma_i})}{\alpha_i}(h_i(t_2)-h_i(t_1))^{\alpha_i}\\
&\leqslant \frac{U_\varepsilon}{\alpha_i}(h_i(b)-h_i(a))^{\alpha_i}+\frac{2\varphi_i(R,c_iR^{\gamma_i})}{\alpha_i}[\omega(h_i,\varepsilon)]^{\alpha_i},
\end{aligned}
$$

where

$$
\begin{aligned}
U_\varepsilon = \sup\Big\{ & |K_i(\tau_1,s,y_1,y_2)-K_i(\tau_2,s,y_1,y_2)| :\ \tau_1,\tau_2,s\in[a,b],\ |\tau_1-\tau_2|\leqslant\varepsilon,\\
& |y_1|\leqslant\|x\|_E,\ |y_2|\leqslant c_i\|x\|_E^{\gamma_i}\Big\},
\end{aligned}
$$

and

$$
\omega(h_i,\varepsilon)=\sup\{|h_i(\tau_2)-h_i(\tau_1)| :\ \tau_1,\tau_2\in[a,b],\ |\tau_1-\tau_2|\leqslant\varepsilon\}.
$$

Therefore,

$$
|(B_ix)(t_2)-(B_ix)(t_1)|\leqslant \frac{U_\varepsilon}{\alpha_i}(h_i(b)-h_i(a))^{\alpha_i}+\frac{2\varphi_i(R,c_iR^{\gamma_i})}{\alpha_i}[\omega(h_i,\varepsilon)]^{\alpha_i}. \tag{4.9}
$$

Now, combining (4.8) and (4.9), we obtain the estimate

$$
\begin{aligned}
&|(S_ix)(t_2)(B_ix)(t_2)-(S_ix)(t_1)(B_ix)(t_1)|\\
&\quad\leqslant \frac{U_\varepsilon(\theta_i+\kappa_iR^{\phi_i})}{\alpha_i}(h_i(b)-h_i(a))^{\alpha_i}+\frac{2\varphi_i(R,c_iR^{\gamma_i})(\theta_i+\kappa_iR^{\phi_i})}{\alpha_i}[\omega(h_i,\varepsilon)]^{\alpha_i}\\
&\qquad+\frac{\varphi_i(R,c_iR^{\gamma_i})(h_i(b)-h_i(a))^{\alpha_i}\omega(S_ix,\varepsilon)}{\alpha_i}.
\end{aligned}
$$

Therefore, by (4.7), we obtain

$$
\begin{aligned}
&|(A_ix)(t_2)-(A_ix)(t_1)|\\
&\quad\leqslant \omega(g_i,\varepsilon)+\frac{U_\varepsilon(\theta_i+\kappa_iR^{\phi_i})}{\Gamma(\alpha_i+1)}(h_i(b)-h_i(a))^{\alpha_i}+\frac{2\varphi_i(R,c_iR^{\gamma_i})(\theta_i+\kappa_iR^{\phi_i})}{\Gamma(\alpha_i+1)}[\omega(h_i,\varepsilon)]^{\alpha_i}\\
&\qquad+\frac{\varphi_i(R,c_iR^{\gamma_i})(h_i(b)-h_i(a))^{\alpha_i}\omega(S_ix,\varepsilon)}{\Gamma(\alpha_i+1)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\omega(A_iM,\varepsilon)\leqslant{}& \omega(g_i,\varepsilon)+\frac{U_\varepsilon(\theta_i+\kappa_iR^{\phi_i})}{\Gamma(\alpha_i+1)}(h_i(b)-h_i(a))^{\alpha_i}+\frac{2\varphi_i(R,c_iR^{\gamma_i})(\theta_i+\kappa_iR^{\phi_i})}{\Gamma(\alpha_i+1)}[\omega(h_i,\varepsilon)]^{\alpha_i}\\
&+\frac{\varphi_i(R,c_iR^{\gamma_i})(h_i(b)-h_i(a))^{\alpha_i}\omega(S_iM,\varepsilon)}{\Gamma(\alpha_i+1)}.
\end{aligned}
$$

Passing to the limit as $\varepsilon\to 0$, we obtain

$$
\mu(A_iM)\leqslant \frac{\varphi_i(R,c_iR^{\gamma_i})(h_i(b)-h_i(a))^{\alpha_i}\mu(S_iM)}{\Gamma(\alpha_i+1)}.
$$

In view of assumption (C4), it follows that

$$\mu(A_iM) \leqslant \frac{\nu_i \varphi_i(R, c_i R^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \mu(M).$$

Therefore, we proved the following result.

Lemma 4.4. *For every $\emptyset \neq M \subset \overline{B(0_E, R)}$, we have*

$$\mu(A_iM) \leqslant \frac{\nu_i \varphi_i(R, c_i R^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \mu(M), \quad i = 1, \cdots, q.$$

Next, let us define the operator $A : \overline{B(0_E, R)} \to E$ by

$$(Ax)(t) = \prod_{i=1}^{q} (A_i x)(t), \quad t \in [a, b]. \tag{4.10}$$

Lemma 4.5. *The operator $A : \overline{B(0_E, R)} \to \overline{B(0_E, R)}$ is well-defined.*

Proof. The result follows immediately from (4.6). □

Now, we are able to state and prove the main result in this paper.

Theorem 4.6. *Suppose that all conditions* (C1)-(C9) *are satisfied. Moreover, suppose that*

$$R^{1-\frac{1}{q}} \sum_{i=1}^{q} \frac{\nu_i \varphi_i(R, c_i R^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)} < 1. \tag{4.11}$$

Then Equation (1.2) *has at least one solution $x^* \in \overline{B(0_E, R)}$.*

Proof. Observe that $x \in \overline{B(0_E, R)}$ is a solution to Equation (1.2) if and only if x is a solution to (3.3), where $D = \overline{B(0_E, R)}$ and A is given by (4.10). From Lemmas 4.1, 4.2, 4.3, and 4.5, assumptions (i)-(iii) of Theorem 3.2 are satisfied. From Lemma 4.4, assumption (iv) of Theorem 3.2 is satisfied with

$$k_i = \frac{\nu_i \varphi_i(R, c_i R^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \quad i = 1, \cdots, q.$$

On the other hand, using (4.6) and (4.11), we have

$$\begin{aligned} \sum_{i=1}^{q} k_i \prod_{j=1, j \neq i}^{q} \|A_j \overline{B(0_E, R)}\| &\leqslant \sum_{i=1}^{q} k_i \prod_{j=1, j \neq i}^{q} R^{1/q} \\ &= R^{1-\frac{1}{q}} \sum_{i=1}^{q} \frac{\nu_i \varphi_i(R, c_i R^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \\ &< 1. \end{aligned}$$

Therefore, assumption (v) of Theorem 3.2 is satisfied. Finally, an application of Theorem 3.2 yields the desired result. □

5. Particular cases

In this section, we present existence results for some special cases of equation (1.2).

5.1. A functional equation with supremum in the kernel

For $i = 1, \cdots, q$, define the operator $T_i : E \to E$ by

$$(T_i x)(t) = \max\{|x(\tau)| : a \leqslant \tau \leqslant t\}, \quad t \in [a, b], \quad x \in E.$$

Clearly, we have

$$\|T_i x\|_E \leqslant \|x\|_E, \quad x \in E.$$

Now, let us consider a pair of elements $(x, y) \in E \times E$, and let $t \in [a, b]$. We have

$$|(T_i x)(t) - (T_i y)(t)| = |\max\{|x(\tau)| : a \leqslant \tau \leqslant t\} - \max\{|y(\tau)| : a \leqslant \tau \leqslant t\}| .$$

On the other hand, by Heine-Borel theorem, there exist $\tau_1, \tau_2 \in [a, t]$ such that

$$\max\{|x(\tau)| : a \leqslant \tau \leqslant t\} = |x(\tau_1)|,$$

and

$$\max\{|y(\tau)| : a \leqslant \tau \leqslant t\} = |y(\tau_2)|.$$

Therefore, we obtain

$$|(T_i x)(t) - (T_i y)(t)| = \Big| |x(\tau_1)| - |y(\tau_2)| \Big|.$$

Without loss of the generality, we may suppose that $|x(\tau_1)| \geqslant |y(\tau_2)|$. Thus, we obtain

$$|(T_i x)(t) - (T_i y)(t)| = |x(\tau_1)| - |y(\tau_2)| \leqslant |x(\tau_1)| - |y(\tau_1)| \leqslant |x(\tau_1) - y(\tau_1)| \leqslant \|x - y\|_E.$$

Then

$$\|T_i x - T_i y\|_E \leqslant \|x - y\|_E, \quad (x, y) \in E \times E.$$

Hence, T_i satisfies assumptions (C2) and (C3) with

$$c_i = \gamma_i = L_i = \ell_i = 1.$$

Consider now the following assumptions:

(i) $g_i \in E$, $i = 1, \cdots, q$.

(ii) $S_i : E \to E$, $i = 1, \cdots, q$, is continuous, and

$$\mu(S_i M) \leqslant \nu_i \mu(M), \quad M \in \mathcal{P}_b(E),$$

where $\nu_i > 0$ is a constant and μ is given by (4.1).

(iii) For all $i = 1, \cdots, q$, there exist constants $\theta_i > 0$, $\kappa_i > 0$ and $\phi_i > 0$ such that

$$\|S_i x\|_E \leqslant \theta_i + \kappa_i \|x\|_E^{\phi_i}, \quad x \in E.$$

(iv) $K_i : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, q$, is continuous.

(v) There exists $\varphi_i : [0, \infty) \times [0, \infty) \to [0, \infty)$, $i = 1, \cdots, q$, a nondecreasing function with respect to each variable, such that

$$|K_i(t, s, u, v)| \leqslant \varphi_i(|u|, |v|), \quad (t, s, u, v) \in [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R}.$$

(vi) $h_i \in E \cap C^1((a, b]; \mathbb{R})$ is nondecreasing, $i = 1, \cdots, q$.

(vii) There exists $R > 0$ such that

$$\|g_i\|_E + \frac{(\theta_i + \kappa_i R^{\varphi_i})\varphi_i(R,R)(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \leqslant R^{1/q}, \quad i = 1, \cdots, q.$$

(viii) We suppose that

$$R^{1-\frac{1}{q}} \sum_{i=1}^{q} \frac{\nu_i \varphi_i(R,R)(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)} < 1.$$

From Theorem 4.6, we deduce the following existence result.

Corollary 5.1. *Suppose that assumptions* (i)-(viii) *are satisfied. Then the functional equation*

$$x(t) = \prod_{i=1}^{q} \left(g_i(t) + \frac{(S_i x)(t)}{\Gamma(\alpha_i)} \int_a^t \frac{h_i'(s) K_i\left(t, s, x(s), \max_{a \leqslant \tau \leqslant s} |x(\tau)|\right)}{(h_i(t) - h_i(s))^{1-\alpha_i}} \, ds \right), \quad t \in [a,b], \tag{5.1}$$

has at least one solution $x^* \in \overline{B(0_E, R)}$.

5.2. A functional equation involving Riemann-Liouville frational integrals

In what follows, as a particular case of (1.2), we consider a functional equation involving Riemann-Liouville frational integrals. More precisely, we deal with the functional equation

$$x(t) = \prod_{i=1}^{q} \left(g_i(t) + \frac{(S_i x)(t)}{\Gamma(\alpha_i)} \int_a^t \frac{K_i\left(t, s, x(s), (T_i x)(s)\right)}{(t-s)^{1-\alpha_i}} \, ds \right), \quad t \in [a,b], \tag{5.2}$$

where $0 < \alpha_i < 1$.

Clearly, (5.2) is a special case of (1.2) with

$$h_i(t) = t, \quad i = 1, \cdots q.$$

We consider the following assumptions:

(i) $g_i \in E$, $i = 1, \cdots, q$.

(ii) $T_i : E \to E$, $i = 1, \cdots, q$, and

$$\|T_i x\|_E \leqslant c_i \|x\|_E^{\gamma_i}, \quad x \in E,$$

where $c_i > 0$ and $\gamma_i > 0$ are constants.

(iii) For all $i = 1, \cdots, q$,

$$\|T_i x - T_i y\|_E \leqslant L_i \|x - y\|_E^{\ell_i}, \quad (x,y) \in E \times E,$$

where $L_i > 0$ and $\ell_i > 0$ are constants.

(iv) $S_i : E \to E$, $i = 1, \cdots, q$, is continuous, and

$$\mu(S_i M) \leqslant \nu_i \mu(M), \quad M \in \mathcal{P}_b(E),$$

where $\nu_i > 0$ is a constant and μ is given by (4.1).

(v) For all $i = 1, \cdots, q$, there exist constants $\theta_i > 0$, $\kappa_i > 0$ and $\phi_i > 0$ such that

$$\|S_i x\|_E \leqslant \theta_i + \kappa_i \|x\|_E^{\phi_i}, \quad x \in E.$$

(vi) $K_i : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, q$, is continuous.

(vii) There exists $\varphi_i : [0, \infty) \times [0, \infty) \to [0, \infty)$, $i = 1, \cdots, q$, a nondecreasing function with respect to each variable, such that

$$|K_i(t, s, u, v)| \leqslant \varphi_i(|u|, |v|), \quad (t, s, u, v) \in [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R}.$$

(viii) There exists $R > 0$ such that

$$\|g_i\|_E + \frac{(\theta_i + \kappa_i R^{\phi_i})\varphi_i(R, c_i R_E^{\gamma_i})(b-a)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \leqslant R^{1/q}, \quad i = 1, \cdots, q.$$

(ix) We suppose that

$$R^{1-\frac{1}{q}} \sum_{i=1}^{q} \frac{\nu_i \varphi_i(R, c_i R^{\gamma_i})(b-a)^{\alpha_i}}{\Gamma(\alpha_i + 1)} < 1.$$

From Theorem 4.6, we deduce the following existence result.

Corollary 5.2. *Suppose that assumptions* (i)-(ix) *are satisfied. Then* (5.2) *has at least one solution* $x^* \in \overline{B(0_E, R)}$.

5.3. A functional equation involving Hadamard frational integrals

In what follows, as a particular case of (1.2), we consider a functional equation involving Hadamard frational integrals. More precisely, we deal with the functional equation

$$x(t) = \prod_{i=1}^{q} \left(g_i(t) + \frac{(S_i x)(t)}{\Gamma(\alpha_i)} \int_a^t \frac{1}{s} \left(\ln \frac{t}{s} \right)^{\alpha_i - 1} K_i\left(t, s, x(s), (T_i x)(s)\right) \, ds \right), \quad t \in [a, b], \tag{5.3}$$

where $0 < \alpha_i < 1$ and $0 < a < b$.

Clearly, (5.2) is a special case of (1.2) with

$$h_i(t) = \ln t, \quad i = 1, \cdots q.$$

Equation (5.3) is investigated under the following assumptions:

(i) $g_i \in E$, $i = 1, \cdots, q$.

(ii) $T_i : E \to E$, $i = 1, \cdots, q$, and

$$\|T_i x\|_E \leqslant c_i \|x\|_E^{\gamma_i}, \quad x \in E,$$

where $c_i > 0$ and $\gamma_i > 0$ are constants.

(iii) For all $i = 1, \cdots, q$,

$$\|T_i x - T_i y\|_E \leqslant L_i \|x - y\|_E^{\ell_i}, \quad (x, y) \in E \times E,$$

where $L_i > 0$ and $\ell_i > 0$ are constants.

(iv) $S_i : E \to E$, $i = 1, \cdots, q$, is continuous, and

$$\mu(S_i M) \leqslant \nu_i \mu(M), \quad M \in \mathcal{P}_b(E),$$

where $\nu_i > 0$ is a constant and μ is given by (4.1).

(v) For all $i = 1, \cdots, q$, there exist constants $\theta_i > 0$, $\kappa_i > 0$ and $\phi_i > 0$ such that

$$\|S_i x\|_E \leqslant \theta_i + \kappa_i \|x\|_E^{\phi_i}, \quad x \in E.$$

(vi) $K_i : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, q$, is continuous.

(vii) There exists $\varphi_i : [0, \infty) \times [0, \infty) \to [0, \infty)$, $i = 1, \cdots, q$, a nondecreasing function with respect to each variable, such that

$$|K_i(t, s, u, v)| \leqslant \varphi_i(|u|, |v|), \quad (t, s, u, v) \in [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R}.$$

(viii) There exists $R > 0$ such that

$$\|g_i\|_E + \frac{(\theta_i + \kappa_i R^{\phi_i})\varphi_i(R, c_i R^{\gamma_i}) \left(\ln \frac{b}{a}\right)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \leqslant R^{1/q}, \quad i = 1, \cdots, q.$$

(ix) We suppose that

$$R^{1-\frac{1}{q}} \sum_{i=1}^{q} \frac{\nu_i \varphi_i(R, c_i R^{\gamma_i}) \left(\ln \frac{b}{a}\right)^{\alpha_i}}{\Gamma(\alpha_i + 1)} < 1.$$

From Theorem 4.6, we deduce the following existence result.

Corollary 5.3. *Suppose that assumptions* (i)-(ix) *are satisfied. Then* (5.3) *has at least one solution* $x^* \in \overline{B(0_E, R)}$.

5.4. A functional equation involving Erdélyi-Kober frational integrals

Now, we deal with a functional equation involving Erdélyi-Kober fractional integrals. More precisely, we are interested in studying the existence of solutions to the functional equation

$$x(t) = \prod_{i=1}^{q} \left(g_i(t) + \frac{(S_i x)(t)}{\Gamma(\alpha_i)} \int_a^t \frac{\beta_i s^{\beta_i - 1} K_i(t, s, x(s), (T_i x)(s))}{(t^{\beta_i} - s^{\beta_i})^{1-\alpha_i}} \, ds \right), \quad t \in [a, b], \tag{5.4}$$

where $\beta_i > 0$, $0 < \alpha_i < 1$ and $0 \leqslant a \leqslant b$.

Clearly, (5.4) is a special case of (1.2) with

$$h_i(t) = t^{\beta_i}, \quad i = 1, \cdots q.$$

Equation (5.4) is investigated under the following assumptions:

(i) $g_i \in E$, $i = 1, \cdots, q$.

(ii) $T_i : E \to E$, $i = 1, \cdots, q$, and

$$\|T_i x\|_E \leqslant c_i \|x\|_E^{\gamma_i}, \quad x \in E,$$

where $c_i > 0$ and $\gamma_i > 0$ are constants.

(iii) For all $i = 1, \cdots, q$,

$$\|T_i x - T_i y\|_E \leqslant L_i \|x - y\|_E^{\ell_i}, \quad (x, y) \in E \times E,$$

where $L_i > 0$ and $\ell_i > 0$ are constants.

(iv) $S_i : E \to E$, $i = 1, \cdots, q$, is continuous, and

$$\mu(S_i M) \leqslant \nu_i \mu(M), \quad M \in \mathcal{P}_b(E),$$

where $\nu_i > 0$ is a constant and μ is given by (4.1).

(v) For all $i = 1, \cdots, q$, there exist constants $\theta_i > 0$, $\kappa_i > 0$ and $\phi_i > 0$ such that

$$\|S_i x\|_E \leqslant \theta_i + \kappa_i \|x\|_E^{\phi_i}, \quad x \in E.$$

(vi) $K_i : [a,b] \times [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, q$, is continuous.

(vii) There exists $\varphi_i : [0,\infty) \times [0,\infty) \to [0,\infty)$, $i = 1, \cdots, q$, a nondecreasing function with respect to each variable, such that

$$|K_i(t,s,u,v)| \leqslant \varphi_i(|u|,|v|), \quad (t,s,u,v) \in [a,b] \times [a,b] \times \mathbb{R} \times \mathbb{R}.$$

(viii) There exists $R > 0$ such that

$$\|g_i\|_E + \frac{(\theta_i + \kappa_i R^{\varphi_i})\varphi_i(R, c_i R^{\gamma_i}) \left(b^{\beta_i} - a^{\beta_i}\right)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \leqslant R^{1/q}, \quad i = 1, \cdots, q.$$

(ix) We suppose that

$$R^{1-\frac{1}{q}} \sum_{i=1}^{q} \frac{\nu_i \varphi_i(R, c_i R^{\gamma_i}) \left(b^{\beta_i} - a^{\beta_i}\right)^{\alpha_i}}{\Gamma(\alpha_i + 1)} < 1.$$

From Theorem 4.6, we deduce the following existence result.

Corollary 5.4. *Suppose that assumptions* (i)-(ix) *are satisfied. Then* (5.4) *has at least one solution* $x^* \in \overline{B(0_E, R)}$.

Remark 5.5. For $q = 1$, Corollary 5.4 is a generalization of the existence result obtained recently in [5].

5.5. A functional equation with mixed Hadamard and Erdélyi-Kober frational integrals

In this section, a functional equation with mixed Hadamard and Erdélyi-Kober frational integrals is investigated. More precisely, we are interested in the existence of solutions to the functional equation

$$\begin{aligned} x(t) = &\left(g_1(t) + \frac{(S_1x)(t)}{\Gamma(\alpha_1)} \int_a^t \frac{1}{s}\left(\ln \frac{t}{s}\right)^{\alpha_1 - 1} K_1\left(t, s, x(s), (T_1x)(s)\right) ds\right) \\ &\times \left(g_2(t) + \frac{(S_2x)(t)}{\Gamma(\alpha_2)} \int_a^t \frac{\beta s^{\beta-1} K_2(t,s,x(s),(T_2x)(s))}{(t^\beta - s^\beta)^{1-\alpha_2}} ds\right), \quad t \in [a,b], \end{aligned} \tag{5.5}$$

where $0 < \alpha_i < 1$, $i = 1, 2$, $\beta > 0$, and $0 < a < b$.

Observe that (5.5) is a special case of (1.2) with

$$q = 2, \quad h_1(t) = \ln t, \quad h_2(t) = t^\beta.$$

Equation (5.5) is investigated under the following assumptions:

(i) $g_i \in E$, $i = 1, 2$.

(ii) $T_i : E \to E$, $i = 1, 2$, and

$$\|T_i x\|_E \leqslant c_i \|x\|_E^{\gamma_i}, \quad x \in E,$$

where $c_i > 0$ and $\gamma_i > 0$ are constants.

(iii) For all $i = 1, 2$,

$$\|T_i x - T_i y\|_E \leqslant L_i \|x - y\|_E^{\ell_i}, \quad (x,y) \in E \times E,$$

where $L_i > 0$ and $\ell_i > 0$ are constants.

(iv) $S_i : E \to E$, $i = 1, 2$, is continuous, and

$$\mu(S_i M) \leqslant \nu_i \mu(M), \quad M \in \mathcal{P}_b(E),$$

where $\nu_i > 0$ is a constant and μ is given by (4.1).

(v) For all $i = 1, 2$, there exist constants $\theta_i > 0$, $\kappa_i > 0$ and $\phi_i > 0$ such that

$$\|S_i x\|_E \leqslant \theta_i + \kappa_i \|x\|_E^{\phi_i}, \quad x \in E.$$

(vi) $K_i : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, is continuous.

(vii) There exists $\varphi_i : [0, \infty) \times [0, \infty) \to [0, \infty)$, $i = 1, 2$, a nondecreasing function with respect to each variable, such that

$$|K_i(t, s, u, v)| \leqslant \varphi_i(|u|, |v|), \quad (t, s, u, v) \in [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R}.$$

(viii) There exists $R > 0$ such that

$$\|g_i\|_E + \frac{(\theta_i + \kappa_i R^{\phi_i})\varphi_i(R, c_i R^{\gamma_i})(h_i(b) - h_i(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \leqslant \sqrt{R}, \quad i = 1, 2,$$

where $h_1(t) = \ln t$ and $h_2(t) = t^\beta$.

(ix) We suppose that

$$\sqrt{R}\left(\frac{\nu_1 \varphi_1(R, c_1 R^{\gamma_1})\left(\ln \frac{b}{a}\right)^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{\nu_2 \varphi_2(R, c_2 R^{\gamma_2})(b^\beta - a^\beta)^{\alpha_2}}{\Gamma(\alpha_2 + 1)}\right) < 1.$$

From Theorem 4.6, we deduce the following existence result.

Corollary 5.6. *Suppose that assumptions* (i)-(ix) *are satisfied. Then* (5.5) *has at least one solution* $x^* \in \overline{B(0_E, R)}$.

6. Numerical examples

In order to illustrate our theoretical results, some numerical examples are presented in this section.

Example 6.1. We deal with the solvability of the nonlinear integral equation

$$x(t) = \frac{t}{10} + \frac{x(\sin t) + t^2}{2} \int_0^t \frac{\ln\left(1 + \frac{|x(s)| + \max_{0 \leqslant \tau \leqslant s} |x(\tau)|}{4}\right)}{(1 + t + s)\sqrt{t - s}} \, ds, \quad 0 \leqslant t \leqslant 1. \tag{6.1}$$

Observe that (6.1) is a special case of (5.1) with $(a, b) = (0, 1)$, $q = 1$, $h_1(t) = t$, $\alpha_1 = \frac{1}{2}$, $g_1(t) = \frac{t}{10}$,

$$(S_1 x)(t) = \Gamma\left(\frac{1}{2}\right)\left(\frac{x(\sin t) + t^2}{2}\right), \quad t \in [0, 1], \, x \in C([0, 1]; \mathbb{R}),$$

and

$$K_1(t, s, u, v) = \ln\left(1 + \frac{|u| + |v|}{4}\right), \quad (t, s, u, v) \in [0, 1] \times [0, 1] \times \mathbb{R} \times \mathbb{R}.$$

In order to obtain an existence result for the integral equation (6.1), we shall apply Corollary 5.1 after checking that the required assumptions (i)-(viii) are satisfied. Clearly, $g_1 \in E$, where $E = C([0, 1]; \mathbb{R})$. Moreover, we have

$$\|g_1\|_E = \frac{1}{10}.$$

Therefore, assumption (i) is satisfied. Obviously, $S_1 : E \to E$ is a well-defined and continuous operator. Let $M \in \mathcal{P}_b(E)$. Fix $\varepsilon > 0$, $x \in M$, and take $t_1, t_2 \in [a, b]$ such that $|t_1 - t_2| \leqslant \varepsilon$. We have

$$|(S_1 x)(t_1) - (S_1 x)(t_2)| = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\left|x(\sin t_1) - x(\sin t_2) + t_1^2 - t_2^2\right|$$

$$
\begin{aligned}
&\leqslant \Gamma\left(\frac{3}{2}\right)\left(|x(\sin t_1) - x(\sin t_2)| + 2\varepsilon\right) \\
&\leqslant \Gamma\left(\frac{3}{2}\right)\left(\omega(x \circ \sin, \varepsilon) + 2\varepsilon\right) \\
&\leqslant \Gamma\left(\frac{3}{2}\right)\left(\omega(x, \varepsilon) + 2\varepsilon\right) \\
&\leqslant \Gamma\left(\frac{3}{2}\right)\left(\omega(M, \varepsilon) + 2\varepsilon\right).
\end{aligned}
$$

Therefore, we have

$$\omega(S_1 M, \varepsilon) \leqslant \Gamma\left(\frac{3}{2}\right)\left(\omega(M, \varepsilon) + 2\varepsilon\right),$$

which yields

$$\mu(S_1 M) \leqslant \Gamma\left(\frac{3}{2}\right)\mu(M), \quad M \in \mathcal{P}_b(E).$$

Then assumption (ii) is satisfied with

$$\nu_1 = \Gamma\left(\frac{3}{2}\right).$$

For every $x \in E$, we have

$$|(S_1 x)(t)| \leqslant \Gamma\left(\frac{3}{2}\right)\left(|x(\sin t)| + 1\right) \leqslant \Gamma\left(\frac{3}{2}\right)\left(\|x\|_E + 1\right), \quad t \in [0, 1].$$

Then

$$\|S_1 x\|_E \leqslant \Gamma\left(\frac{3}{2}\right) + \Gamma\left(\frac{3}{2}\right)\|x\|_E, \quad x \in E.$$

Therefore, assumption (iii) is satisfied with

$$\theta_1 = \kappa_1 = \Gamma\left(\frac{3}{2}\right), \quad \phi_1 = 1.$$

Clearly, K_1 is a continuous function and

$$|K_1(t, s, u, v)| \leqslant \varphi_1(|u|, |v|), \quad (t, s, u, v) \in [0, 1] \times [0, 1] \times \mathbb{R} \times \mathbb{R},$$

where

$$\varphi_1(r_1, r_2) = \ln\left(1 + \frac{r_1 + r_2}{4}\right), \quad r_1, r_2 \geqslant 0.$$

Clearly, φ_1 is nondecreasing with respect to each variable. Then assumption (v) is satisfied. In order to check assumptions (vii) and (viii), we have to find $R > 0$ such that

$$\begin{cases} \frac{1}{10} + (1 + R)\ln\left(1 + \frac{R}{2}\right) \leqslant R, \\ \ln\left(1 + \frac{R}{2}\right) < 1. \end{cases} \tag{6.2}$$

Observe that for $R = \frac{1}{2}$, we have

$$\begin{cases} \frac{1}{10} + \frac{3}{2}\ln\left(\frac{5}{4}\right) \approx 0.434 \leqslant \frac{1}{2}, \\ \ln\left(\frac{5}{4}\right) \approx 0.223 < 1. \end{cases}$$

Therefore, $R = \frac{1}{2}$ is a solution to (6.2). Finally, by Corollary 5.1, (6.1) has at least one solution $x^* \in \overline{B(0_E, 1/2)}$.

Example 6.2. In this example, we study the existence of solutions to the functional equation

$$\begin{aligned} x(t) &= \left(\frac{t}{10e} + \frac{x(t)}{2} \int_1^t \frac{1}{s} \left(\ln \frac{t}{s} \right)^{-1/2} \max_{1 \leqslant \tau \leqslant s} |x(\tau)| \, ds \right) \\ &\quad \times \left(\frac{t}{15e} + \frac{x(t)}{2e^2\sqrt{e-1}} \int_1^t \frac{ts|x(s)|}{\sqrt{t-s}} \, ds \right), \quad t \in [1, e]. \end{aligned} \tag{6.3}$$

Observe that (6.3) is a special case of (5.5) with $(a, b) = (1, e)$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\beta = 1$, $g_1(t) = \frac{t}{10e}$, $g_2(t) = \frac{t}{15e}$,

$$\begin{aligned} (T_1x)(t) &= \max_{1 \leqslant \tau \leqslant t} |x(\tau)|, \quad t \in [1, e],\ x \in C([1, e]; \mathbb{R}), \\ (T_2x)(t) &= |x(t)|, \quad t \in [1, e],\ x \in C([1, e]; \mathbb{R}), \\ (S_1x)(t) &= \frac{\Gamma\left(\frac{1}{2}\right) x(t)}{2}, \quad t \in [1, e],\ x \in C([1, e]; \mathbb{R}), \\ (S_2x)(t) &= \frac{\Gamma\left(\frac{1}{2}\right) x(t)}{2e^2\sqrt{e-1}}, \quad t \in [1, e],\ x \in C([1, e]; \mathbb{R}), \end{aligned}$$

and

$$\begin{aligned} K_1(t, s, u, v) &= v, \quad (t, s, u, v) \in [1, e] \times [1, e] \times \mathbb{R} \times \mathbb{R}, \\ K_2(t, s, u, v) &= tsv, \quad (t, s, u, v) \in [1, e] \times [1, e] \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

We argue as in the previous example to check that assumptions (i)-(vii) of Corollary 5.6 are satisfied with $c_1 = c_2 = \gamma_1 = \gamma_2 = 1$, $L_1 = L_2 = \ell_1 = \ell_2 = 1$, $\kappa_1 = \nu_1 = \Gamma\left(\frac{3}{2}\right)$, $\kappa_2 = \nu_2 = \frac{\Gamma\left(\frac{3}{2}\right)}{e^2\sqrt{e-1}}$, $\theta_1 = \theta_2 = 0$, $\phi_1 = \phi_2 = 1$ and

$$\begin{aligned} \varphi_1(r_1, r_2) &= r_2, \quad r_1, r_2 \geqslant 0, \\ \varphi_2(r_1, r_2) &= e^2 r_2, \quad r_1, r_2 \geqslant 0. \end{aligned}$$

In order to check assumptions (viii) and (ix), we have to find $R > 0$ such that

$$\left\{ \begin{array}{lll} \frac{1}{10} + R^2 & \leqslant & \sqrt{R}, \\ 2R\sqrt{R} & < & 1. \end{array} \right. \tag{6.4}$$

Observe that $R = \frac{1}{4}$ is a solution to (6.4). Therefore, by Corollary 5.6, the functional equation (6.3) has at least one solution $x^* \in \overline{B(0_E, 1/4)}$.

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