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Meir-Keeler theorem in b-rectangular metric spaces

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Abstract

In this paper, we prove a Meir-Keeler theorem in b-rectangular metric spaces. Thus, we answer the open question raised by Ding et al. [H. S. Ding, V. Ozturk, S. Radenović, J. Nonlinear Sci. Appl., **8** (2015), 378–386].

Keywords: Fixed point, b-metric space, rectangular metric space, b-rectangular metric space.
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1. Introduction

To prove a fixed point theorem, researchers must consider contractive condition and underlying space. A large number of weaker contractive conditions have been put forward since Banach contraction principle was published in 1922. For example, in a comprehensive overview of contractive definitions, Rhoades [9] compared 250 contractive definitions in 1977. In the recent forty years, the theory of fixed point has been grown rapidly (see [2, 7, 8, 10, 12, 14, 15] and the references therein for others). In the meantime, the underlying spaces have been extended from usual metric spaces to generalized metric spaces such as b-metric spaces [1, 4], rectangular metric spaces [3], b-rectangular metric spaces [5, 6] and so on. Ding et al. in [5, 6] discussed some fixed point results in b-rectangular metric spaces and put forward the following open question [6]:

Prove or disprove the following (Meir-Keeler theorem): let (X, d) be a b-rectangular metric space with coefficient $s > 1$, and let $f, g : X \to X$ be two self-maps such that $f(X) \subseteq g(X)$, and one of these two subsets of X being complete. Assume that the following condition holds:

for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \leqslant d(gx, gy) < \varepsilon + \delta$ implies $sd(fx, fy) < \varepsilon$, and $fx = fy$ whenever $gx = gy$.

Then f and g have a unique point of coincidence, say $\omega \in X$. Moreover, for each $x_0 \in X$, the corresponding Jungck sequence $\{y_n\}$ can be chosen such that $\lim_{n\to\infty} y_n = \omega$. In addition, if f and g are weakly compatible, then they have a unique common fixed point.

In this paper, we answer the open question affirmatively.

Let recall some definitions and lemmas that will be used in the paper.

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Definition 1.1 ([1, 4])**.** Let X be a nonempty set, $s \geqslant 1$ be a given real number and let $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$ (b-triangular inequality).

Then the pair (X, d) is called a b-metric space (metric type space).

For all definitions of notions as b-convergence, b-completeness, and b-Cauchy in the frame of b-metric spaces see [1, 4].

Definition 1.2 ([3])**.** Let X be a nonempty set, and let $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from x and y:

(r1) $d(x, y) = 0$ if and only if $x = y$;
(r2) $d(x, y) = d(y, x)$;
(r3) $d(x, y) \leqslant d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

Then (X, d) is called a rectangular metric space or generalized metric space.

For all definitions of notions in the frame of rectangular metric spaces see [3].

Definition 1.3 ([5, 6])**.** Let X be a nonempty set, $s \geqslant 1$ be a given real number and let $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from x and y:

(rb1) $d(x, y) = 0$ if and only if $x = y$;
(rb2) $d(x, y) = d(y, x)$;
(rb3) $d(x, y) \leqslant s[d(x, u) + d(u, v) + d(v, y)]$ (b-rectangular inequality).

Then (X, d) is called a b-rectangular metric space or b-generalized metric space.

From the above definitions, we know that every metric space is a rectangular metric space and a b-metric space. Also, every rectangular metric space or every b-metric space is a b-rectangular metric space. However the converse is not necessarily true [11, 13]. To illustrate it, we give the following example which is a modification of example of [13].

Example 1.4. Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$, and $X = A \bigcup B$. Define $d : X \times X \longrightarrow [0, +\infty)$ as follows:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \text{ and } \{x, y\} \subset A \text{ or } \{x, y\} \subset B, \\ y^2, & \text{if } x \in A, y \in B, \\ x^2, & \text{if } x \in B, y \in A. \end{cases}$$

Then (X, d) is a complete b-rectangular metric space with coefficient $s = 3$, but which is neither a b-metric space nor a rectangular metric space. Meanwhile, it is easy to see that [13]:

(i) the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ converges to both 0 and 2, and it is not a Cauchy sequence;
(ii) there is no $r > 0$ such that $B_r(0) \bigcap B_r(2) = \emptyset$. Hence, the corresponding topology is not Hausdorff;
(iii) $B_{1/3}(\frac{1}{3}) = \{0, 2, \frac{1}{3}\}$, however, there does not exist $r > 0$ such that $B_r(0) \subseteq B_{1/3}(\frac{1}{3})$;
(iv) $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\lim_{n\to\infty} d(\frac{1}{n}, \frac{1}{2}) \neq d(0, \frac{1}{2})$. Hence, d is not a continuous function.

Lemma 1.5 ([5])**.** *Let (X, d) be a b-rectangular metric space with $s \geqslant 1$, and let $f, g : X \to X$ be two self-maps such that $f(X) \subseteq g(X)$. If Jungck sequence $y_n = fx_n = gx_{n+1}$ and $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$ satisfies*

$$d(y_n, y_{n+1}) < \lambda d(y_{n-1}, y_n)$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, 1)$, then $y_n \neq y_m$ whenever $n \neq m$.

Lemma 1.6 ([5, 6])**.** *Let (X, d) be a b-rectangular metric space with $s \geqslant 1$, and let $\{y_n\}$ be a Cauchy sequence in X such that $y_n \neq y_m$ whenever $n \neq m$. Then $\{y_n\}$ can converge to at most one point.*

2. Main results

Theorem 2.1. *Let* (X, d) *be a* b*-rectangular metric space with coefficient* $s > 1$*, and let* $f, g : X \to X$ *be two self-maps such that* $f(X) \subseteq g(X)$*, and one of these two subsets of* X *being complete. Assume that the following condition holds: for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leqslant d(gx, gy) < \varepsilon + \delta \textit{ implies } sd(fx, fy) < \varepsilon, \textit{ and } fx = fy \textit{ whenever } gx = gy. \tag{2.1}$$

Then f *and* g *have a unique point of coincidence, say* $\omega \in X$*. Moreover, for each* $x_0 \in X$*, the corresponding Jungck sequence* $\{y_n\}$ *can be chosen such that* $\lim_{n\to\infty} y_n = \omega$*. In addition, if* f *and* g *are weakly compatible, then they have a unique common fixed point.*

Proof. First of all, by (2.1), we point out that: for all $x, y \in X$, and $gx \neq gy$,

$$sd(fx, fy) < d(gx, gy). \tag{2.2}$$

Suppose $x_0 \in X$ be an arbitrary point, since $f(X) \subseteq g(X)$, we can choose sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = fx_n = gx_{n+1}, n = 0, 1, 2, \ldots$.

If $y_{n+1} = y_n$ for some $n = p \in N$, then $gy_{p+1} = y_p = y_{p+1} = fx_{p+1}$, so f and g have a point of coincidence. Therefore, we can suppose $y_{n+1} \neq y_n$ for each $n \in N$.

Making use of the inequality (2.2) with $x = x_{n+1}$ and $y = x_n$, we can get

$$sd(y_n, y_{n+1}) < d(y_{n-1}, y_n). \tag{2.3}$$

Since $s > 1$, $\{d(y_n, y_{n+1})\}$ is a decreasing sequence, it is easy to prove that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.4}$$

By (2.3) and Lemma 1.5, for $n \neq m$, we have $y_n \neq y_m$.

Now making use of the inequality (2.2) repeatedly with initial value $x = x_{n+k}$ and $y = x_{m+k}$, we obtain

$$s^k d(y_{n+k}, y_{m+k}) < d(y_n, y_m). \tag{2.5}$$

In what follows, we prove that $\{y_n\}$ is a Cauchy sequence in X. For any $\varepsilon > 0$, we can choose an N (large enough) such that whence $n \geqslant N$,

$$d(y_{n+1}, y_n) \leqslant \frac{\varepsilon - \frac{\varepsilon}{s}}{1+s}.$$

Put $K(y_N, \varepsilon) = \{y \in \{y_n\} : d(y, y_N) \leqslant \varepsilon\}$. Define the map $H : \{y_n\} \to \{y_n\}$ by $H(y_n) = y_{n+1}$.

If $y_m \in K(y_N, \varepsilon)$ with $m > N$, then $y_m \neq y_N$,

$$\begin{aligned} d(H^2 y_m, y_N) &\leqslant s(d(H^2 y_m, H^2 y_N) + d(H^2 y_N, Hy_N) + d(Hy_N, y_N)) \\ &= s(d(y_{m+2}, y_{N+2}) + d(y_{N+2}, y_{N+1}) + d(y_{N+1}, y_N)) \\ &\leqslant s(\frac{1}{s^2} d(y_m, y_N) + (\frac{1}{s} + 1) d(y_{N+1}, y_N)) \\ &\leqslant \frac{1}{s} d(y_m, y_N) + s(\frac{1}{s} + 1) d(y_{N+1}, y_N) \\ &\leqslant \frac{\varepsilon}{s} + (1+s)\frac{\varepsilon - \frac{\varepsilon}{s}}{1+s} = \varepsilon. \end{aligned}$$

That is to say, H^2 maps $K(y_N, \varepsilon)$ into itself. Since $y_{N+1} \in K(y_N, \varepsilon)$, then $y_{N+3}, y_{N+5} \in K(y_N, \varepsilon)$.

Using the b-rectangular inequality, and by (2.5),

$$d(y_N, y_{N+2}) \leqslant s(d(y_N, y_{N+3}) + d(y_{N+3}, y_{N+5}) + d(y_{N+5}, y_{N+2}))$$

$$\begin{aligned}&\leqslant s(d(y_N, y_{N+3}) + \frac{1}{s^3}d(y_N, y_{N+2}) + \frac{1}{s^2}d(y_{N+3}, y_N))\\&\leqslant s(\varepsilon + \frac{1}{s^3}d(y_N, y_{N+2}) + \frac{1}{s^2}\varepsilon).\end{aligned}$$

Therefore, we have

$$d(y_N, y_{N+2}) \leqslant \frac{s^3+s}{s^2-1}\varepsilon.$$

Put $\varepsilon' = \frac{s^3+s}{s^2-1}\varepsilon$ and $K(y_N, \varepsilon') = \{y \in \{y_n\} : d(y, y_N) \leqslant \varepsilon'\}$. Then we can verify that H^2 maps $K(y_N, \varepsilon')$ into itself in a similar way. Since $\varepsilon' > \varepsilon$, then $y_{N+1}, y_{N+2} \in K(y_N, \varepsilon')$. Thus $\{y_{N+1}, y_{N+3}, y_{N+5}, \cdots\} \subset K(y_N, \varepsilon')$ and $\{y_{N+2}, y_{N+4}, y_{N+6}, \cdots\} \subset K(y_N, \varepsilon')$. That is to say, $\{y_n : n \geqslant N\} \subset K(y_N, \varepsilon')$.

For $n > m > N$, since $y_n, y_m \in K(y_N, \varepsilon')$, we have

$$d(y_n, y_m) \leqslant s(d(y_n, y_{n+1}) + d(y_{n+1}, y_N) + d(y_N, y_m)) \leqslant s(\frac{\varepsilon - \frac{\varepsilon}{s}}{1+s} + \varepsilon' + \varepsilon') \leqslant 3s\varepsilon' = \frac{3s^4+3s^2}{s^2-1}\varepsilon.$$

Thus $\{y_n\}$ is a Cauchy sequence in X.

Since $g(X)$ or $f(X)$ is complete, and $f(X) \subseteq g(X)$, then $\{y_n\}$ converges to some point ω in gX. Thus, there exists a point $z \in X$ such that $gz = \omega$. In order to prove $fz = gz$, we suppose that $fz \neq gz$.

By b-rectangular inequality, (2.2), and (2.4),

$$\begin{aligned}d(fz, gz) &\leqslant s(d(fz, fx_{n+1}) + d(fx_{n+1}, fx_n) + d(fx_n, gz))\\&\leqslant s(\frac{1}{s}d(gz, gx_{n+1}) + d(fx_n, fx_{n+1}) + d(fx_{n+1}, gz))\\&= d(gz, y_n) + sd(y_n, y_{n+1}) + sd(y_{n+1}, gz).\end{aligned}$$

Passing to limit as $n \to \infty$, we have

$$d(fz, gz) \leqslant 0,$$

which is a contradiction. Thus, $fz = gz = \omega$.

Next, we shall show that the point of coincidence of f and g is unique.

Suppose $\mu \neq \omega$ is another point of coincidence of f and g, so there exists $t \in X$ such that $ft = gt = \mu$. Then

$$d(\omega, \mu) = d(fz, ft) < \frac{1}{s}d(gz, gt) = \frac{1}{s}d(\omega, \mu),$$

which is a contradiction. Thus, point of coincidence of f and g is unique. If f and g are weakly compatible, it is easy to prove that ω is the unique common fixed point. □

Finally, we give an example to support our result, which is a modification of Example 1.4.

Example 2.2. Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$, $C = [5, +\infty)$, and $X = A \bigcup B \bigcup C$. Define $d : X \times X \longrightarrow [0, +\infty)$ as follows:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y,\\ 1, & \text{if } x \neq y \text{ and } \{x, y\} \subset A \text{ or } \{x, y\} \subset B,\\ y^2, & \text{if } x \in A, y \in B,\\ x^2, & \text{if } x \in B, y \in A,\\ |x - y|, & \text{otherwise.}\end{cases}$$

Then (X, d) is a complete b-rectangular metric space with coefficient $s = 3$, but which is neither a b-metric space nor a rectangular metric space as pointed out in Example 1.4.

Now, define

$$f(x) = \begin{cases} 5, & \text{if } x \in A \bigcup B,\\ 5 + \frac{x-5}{6}, & \text{if } x \in C,\end{cases}$$

and

$$g(x) = \begin{cases} 5, & \text{if } x \in A \bigcup B, \\ 5 + \frac{x-5}{2}, & \text{if } x \in C. \end{cases}$$

Then for $\varepsilon > 0$, pick $\delta = \varepsilon$. We can easily show that f, g satisfy all the conditions of Theorem 2.1. Let $x_0 = 10$, then $x_n = 5 + \frac{5}{3^n}$, and $y_n = 5 + \frac{5}{2\times 3^n} \longrightarrow 5$. Obviously $\omega = 5$ is the unique point of coincidence of f and g.

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