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On the existence of coupled best proximity point and best proximity point for Suzuki type α^+-θ-proximal multivalued mappings

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Abstract

Following the study for the best proximity points of the Suzuki type α^+-θ-proximal single-valued mappings given by Hussain et al., we deal with the Suzuki type α^+-θ-proximal multivalued mappings satisfying generalized conditions of existence, some novel existence results of best proximity point and coupled best proximity points are established. Our results improve and extend some recent results in the previous work. ©2017 All rights reserved.

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1. Introduction

Employing the limit of Picard iterates, the classical Banach's contractive principal verifies the existence and uniqueness of the fixed point equation $Tx = x$, where T is a self-mapping on a complete metric space X and $x \in X$. However, if T a non-self-mapping from A to B, where A, B are two nonempty subsets of X, then $d(x, Tx) \geqslant \text{dist}(A, B)$, the equation $Tx = x$ has no solution, naturally, in this case, one usually hopes to find an element $x \in A$ closing Tx such that the error $d(x, Tx)$ is minimal, that is, to consider the existence of optimal approximate solutions for such fixed point equations. A problem arises: Is there such point satisfying $d(x, Tx) = \text{dist}(A, B)$?

In other words, does an element $x \in A$ with the minimum $\text{dist}(A, B)$ error $d(x, Tx)$ exist?

For the sake of clearness in the introduction, we give some notations which will be used in the sequel: Let (X, d) be a metric space and $A, B \subset X$,

$$\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}, D(x, B) = \inf\{d(x, y) : y \in B\},$$
$$A_0 = \{a \in A : d(a, b) = \text{dist}(A, B) \text{ for some } b \in B\},$$
$$B_0 = \{b \in B : d(a, b) = \text{dist}(A, B) \text{ for some } a \in A\}.$$

For any two nonempty sets A and B in a metric space (X, d), the point $a \in A_0$ is called a best proximity

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point of the mapping $T : A \to B$, if $d(a, Ta) = \text{dist}(A, B)$. In 1969, addressing the problem above, Fan [5] established a classical best approximation theorem. Afterward, the existence and convergence of best proximity points have become interesting topics of optimization theory which recently attracted the attention of many authors, they devote themselves to the investigate of best proximity points of mappings in a variety of settings and have given many extended results for Fan's Theorem [2–4, 8, 10, 14–19, 21, 22, 24, 26]. These extensions mainly lie in two aspects, the one comes from the development of spaces, the other comes from the change of mappings, that is, under various of contractive conditions for certain spaces, the proximity points are considered. Especially, best proximity theorems also serve as a natural generalization of fixed point theorems, if the mapping under consideration is a self-mapping. If we take T as $T : A \times A \to B$, that is, T is a two-variable mapping, such pair point (x, y) satisfying $d(x, T(x, y)) = d(y, T(y, x)) = \text{dist}(A, B)$ is called coupled best proximity point. As a result, some best proximity results were further generalized into coupled best proximity results when considering the two-variable mappings. If ones extend the class of single-valued mappings to the class of multivalued mappings $T : A \to 2^B \setminus \{\emptyset\}$ and moreover, the class of one-variable multivalued mappings to the class of two-variable multivalued mappings $T : A \times A \to 2^B \setminus \{\emptyset\}$, then the situation is getting more complicated and the results of proximity points are more generalized. In this paper, we focus our attentions on the study of one-variable and two-variable multivalued mappings.

Some problems of fixed points of either single-valued or multivalued mappings involving α-admissible have become a hotspot research since Samet et al. [20] introduced the notion of α-admissible in 2012, for example, following Samet's definition, Latif et al. [9] defined the concept of (α, ψ)-Meir-Keeler self-mappings. Redjel et al. [13] introduced a concept of (α, ψ)-Meir-Keeler-Khan mappings, we also defined the class of (α, ψ)-Meir-Keeler-Khan multivalued mappings recently [25]. Hussain et al. [6] introduced the concept of proximal α^+-admissible and introduced new Suzuki type contractions and proved new best proximity results for these contractions in the setting of a metric space, they dealt with best proximity theorems for Suzuki contractions involving the θ function which was introduced by Jleli and Samet [7]. Sintunavarat and Kumam [23] introduced the concept of coupled best proximity point and proved the coupled best proximity theorem for involving cyclic contractions. Recently, Nantadilok [11] established coupled best proximity point theorems for multivalued mappings via α-admissible notion and ψ function.

In this paper, we devote ourselves to study the existence of coupled best proximity point and best proximity point of the new types of proximal multivalued mappings. In Section 2, we present some preliminaries and recall the results for α^+-θ-proximal single-valued mappings. In Section 3, inspired and motivated by Hussain et al. [6], Sintunavarat et al. [23], Jleli et al. [7], and Nantadilok [11], we define the notion of α^+-θ-proximal multivalued mapping for one-variable and two-variable and explore the best proximity point theorem and the coupled best proximity point theorem for the class of α^+-θ-proximal multivalued mappings in this paper. The existence theorems of coupled best proximity points for two-variable α^+-θ-proximal multivalued mappings with the hypothesis of continuity or regularity and the best proximity point results for one-variable α^+-θ-proximal multivalued mappings with the hypothesis of continuity or regularity are established, respectively. These more generalized results extend and generalize the main results of Hussain et al., Nantadilok, and Jleli et al. in the literatures [6, 7, 11, 23, 24]. We also provide an example to show the generality and effectiveness of our results.

2. Preliminaries

As the preliminaries, we review some definitions (see [6, 9, 12] and references therein).

Definition 2.1 ([6]). Let T be a self-mapping on a nonempty set X and $\alpha : X \times X \to [-\infty, +\infty)$ be a mapping. The mapping T is said to be proximal α^+-admissible, if the following condition holds:

$$\left.\begin{array}{r} \alpha(x, y) \geqslant 0, \\ d(u_1, Tx_1) = \text{dist}(A, B), \\ d(u_2, Tx_2) = \text{dist}(A, B), \end{array}\right\} \Rightarrow \alpha(u_1, u_2) \geqslant 0,$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 2.2 ([12])**.** For nonempty subsets A, B of a metric space (X, d) with $A_0 \neq \emptyset$, we say the pair (A, B) satisfies

(a) the P-property, if
$$\left.\begin{array}{l} d(x_1, y_1) = \text{dist}(A, B), \\ d(x_2, y_2) = \text{dist}(A, B), \end{array}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$
for all $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$;

(b) the weak P-property, if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,
$$\left.\begin{array}{l} d(x_1, y_1) = \text{dist}(A, B), \\ d(x_2, y_2) = \text{dist}(A, B), \end{array}\right\} \Rightarrow d(x_1, x_2) \leqslant d(y_1, y_2).$$

Jleli and Samet [7] introduced a new type of contractive maps related to θ functions and established some fixed point results for such mappings. In this section, we deal with coupled best proximity theorems and best proximity theorems for Suzuki type contractions involving the θ functions.

We denote by Θ the family of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is nondecreasing;

(Θ_2) for each sequence $\{\beta_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\lim\limits_{n \to \infty} \beta_n = 0^+$ if and only if $\lim\limits_{n \to \infty} \theta(\beta_n) = 1$;

(Θ_3) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim\limits_{\beta \to 0^+} \dfrac{\theta(\beta) - 1}{\beta^r} = \ell$.

Definition 2.3 ([6])**.** Let (X, d) be a metric space. $T : A \to B$ is called a Suzuki type α^+-θ-proximal mapping, if there exist two functions $\theta \in \Theta$ and $\alpha : X \times X \to [-\infty, +\infty]$ such that for all $x, y \in A$,

$$\frac{1}{2} d^*(x, Tx) \leqslant d(x, Tx) \Rightarrow \alpha(x, y) + \theta(d(Tx, Ty)) \leqslant [\theta(M(x, y))]^k, \tag{2.1}$$

where

$$d^*(x, Tx) = d(x, Tx) - \text{dist}(A, B),$$

and

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2} - \text{dist}(A, B), \frac{d(x, Ty) + d(y, Tx)}{2} - \text{dist}(A, B) \right\}.$$

Hussian et al. [6] established existence theorem for best proximity points of Suzuki type α^+-θ-proximal mappings with continuity assumption or regularity on the mappings.

Theorem 2.4 ([6])**.** *Suppose* A *and* B *are nonempty closed subsets of a metric space* (X, d) *with* $A_0 \neq \emptyset$. *Let* $T : A \to B$ *satisfy* (2.1) *together with the following assertions:*

(i) $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the weak* P-*property;*

(ii) T *is proximal* α^+-*admissible;*

(iii) *there exist* $x_0, x_1 \in A_0$ *such that*
$$d(x, Tx) = \text{dist}(A, B) \text{ and } \alpha(x_0, x_1) \geqslant 0;$$

(iv) T *is continuous; or*

(v) A *is* α-*regular, that is, if* $\{x_n\}$ *is a sequence in* A *such that* $\alpha(x_n, x_{n+1}) \geqslant 0$ *and* $x_n \to x \in A$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geqslant 0$, *for all* $n \in \mathbb{N}$.

Then there exists $x^* \in A_0$ *such that* $d(x^*, Tx^*) = \text{dist}(A, B)$.

3. Coupled best proximity points and best proximity points for Suzuki type α^+-θ-proximal mappings

In the sequel, $\mathbb{N}$ denotes the set of all nonnegative integers, $\mathrm{Bpp}(T)$ denotes the set of best proximity points of T, $\mathrm{CBpp}(T)$ denotes the set of coupled best proximity points of T, and $\mathcal{CL}(X)$ denotes the family of nonempty closed subsets of X.

For any $A, B \in \mathcal{CL}(X)$, let the mapping $H(\cdot, \cdot)$ be the generalized Hausdorff distance with respect to d defined by

$$H(A, B) = \begin{cases} \max \left\{ \sup_{x \in A} \mathrm{dist}(x, B), \sup_{y \in B} \mathrm{dist}(y, A) \right\} & \text{if it exists,} \\ \infty & \text{otherwise.} \end{cases}$$

Definition 3.1 ([1, 11]). An element $x^* \in A$ is said to be the best proximity point of a multivalued non-self-mapping $T : A \to 2^B \backslash \{\emptyset\}$, if $D(x^*, Tx^*) = \mathrm{dist}(A, B)$.

Definition 3.2. Let (X, d) be a metric space and $A, B \in \mathcal{CL}(X)$. A multivalued mapping $T : A \to 2^B \backslash \{\emptyset\}$ is called proximal α^+-admissible, if there exists a mapping $\alpha : A \times A \to [-\infty, +\infty)$ such that for any $x_1, x_2, u_1, u_2 \in A$ and $y_1 \in Tx_1, y_2 \in Tx_2$,

$$\left. \begin{array}{r} \alpha(x_1, x_2) \geqslant 0, \\ d(u_1, y_1) = \mathrm{dist}(A, B), \\ d(u_2, y_2) = \mathrm{dist}(A, B), \end{array} \right\} \Rightarrow \alpha(u_1, u_2) \geqslant 0.$$

Definition 3.3. Let (X, d) be a metric space and $A, B \in \mathcal{CL}(X)$. A mapping $T : A \times A \to 2^B \backslash \{\emptyset\}$ is called proximal α^+-admissible, if there exists a mapping $\alpha : A \times A \to [-\infty, +\infty)$ such that for any $x_1, x_2, w_1, w_2, w'_1, w'_2, y_1, y_2 \in A$ and $u_1 \in T(x_1, y_1), u_2 \in T(x_2, y_2), v_1 \in T(y_1, x_1), v_2 \in T(y_2, x_2)$,

$$\left. \begin{array}{r} \alpha(x_1, x_2) \geqslant 0, \\ d(w_1, u_1) = \mathrm{dist}(A, B), \\ d(w_2, u_2) = \mathrm{dist}(A, B), \end{array} \right\} \Rightarrow \alpha(w_1, w_2) \geqslant 0,$$

and

$$\left. \begin{array}{r} \alpha(y_1, y_2) \geqslant 0, \\ d(w'_1, v_1) = \mathrm{dist}(A, B), \\ d(w'_2, v_2) = \mathrm{dist}(A, B), \end{array} \right\} \Rightarrow \alpha(w'_1, w'_2) \geqslant 0.$$

Definition 3.4 ([11]). Let (X, d) be a complete metric space and $A, B \in \mathcal{CL}(X)$. An element $(x^*, y^*) \in (A \times A)$ is said to be the coupled best proximity point of a multivalued mapping $T : A \times A \to \mathcal{CL}(B)$, if $D(x^*, T(x^*, y^*)) = \mathrm{dist}(A, B)$ and $D(y^*, T(y^*, x^*)) = \mathrm{dist}(A, B)$.

Lemma 3.5 ([11]). *Let B be nonempty closed subsets of a metric space (X, d). Then, for each $x \in X$ with $D(x, B) > 0$ and $q > 1$, there exists an element $b \in B$ such that*

$$d(x, b) < qD(x, B).$$

Definition 3.6. Let $T : A \to \mathcal{CL}(B)$ be a mapping on a metric space (X, d). T is called a one-variable Suzuki type α^+-θ-proximal multivalued mapping, if there exist two functions $\theta \in \Theta, \alpha : X \times X \to [-\infty, +\infty)$, and $0 < k < 1, q > 1$ such that for all $x, y \in A$,

$$\frac{1}{2} D^*(x, Tx) \leqslant d(x, y) \Rightarrow \alpha(x, y) + \theta(qH(Tx, Ty)) \leqslant \big[\theta(M(x, y)\big]^k,$$

when $H(Tx, Ty) > 0$, where

$$D^*(x, Tx) = D(x, Tx) - \mathrm{dist}(A, B),$$

and

$$M(x, y) = \max \left\{ d(x, y), \frac{D(x, Tx) + D(y, Ty)}{2} - \mathrm{dist}(A, B), \frac{D(x, Ty) + D(y, Tx)}{2} - \mathrm{dist}(A, B) \right\}.$$

Definition 3.7. Let $T : A \times A \to \mathcal{CL}(B)$ be a mapping on a metric space (X, d). T is called a two-variable Suzuki type α^+-θ-proximal multivalued mapping, if there exist two functions $\theta \in \Theta$, $\alpha : X \times X \to [-\infty, +\infty)$, and $0 < k < 1$, $q > 1$ such that for all $x, y, x', y' \in A$,

$$\left. \begin{array}{l} \frac{1}{2}D^*(x, T(x, x')) \leqslant d(x, y), \\ \frac{1}{2}D^*(x', T(x', x)) \leqslant d(x', y'), \\ d(x, y) = d(x', y') = 0, \text{ or} \\ d(x, y) > 0, d(x', y') > 0, \end{array} \right\} \Rightarrow \alpha(x, y) + \theta(qH(T(x, x'), T(y, y'))) \leqslant \left[\theta(M(x, y, x', y'))\right]^k, \qquad (3.1)$$

when $H(T(x, x'), T(y, y')) > 0$ and $H(T(x', x), T(y', y)) > 0$, where

$$D^*(x, T(x, x')) = D(x, T(x, x')) - \text{dist}(A, B),$$

and

$$M(x, y, x', y') = \max\left\{ d(x, y), \frac{D(x, T(x, x')) + D(y, T(y, y'))}{2} - \text{dist}(A, B), \right.$$
$$\left. \frac{D(y, T(x, x')) + D(x, T(y, y'))}{2} - \text{dist}(A, B)\right\}.$$

The following coupled best proximity result can be stated:

Theorem 3.8. *Let (X, d) be a complete metric space and $A, B \in \mathcal{CL}(X)$ with $A_0 \neq \emptyset$. $\theta \in \Theta$ and $T : A \times A \to \mathcal{CL}(B)$ be a two-variable Suzuki type α^+-θ- proximal multivalued mapping. Suppose that the following conditions hold:*

(i) *$T(x, y) \subseteq B_0$ for $(x, y) \in A_0 \times A_0$ and (A, B) satisfies the weak P-property;*

(ii) *T is α^+-proximal admissible;*

(iii) *there exist elements $(x_0, y_0), (x_1, y_1) \in (A_0 \times A_0)$ and $u_1 \in T(x_0, y_0), v_1 \in T(y_0, x_0)$ such that*

$$d(x_1, u_1) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geqslant 0,$$
$$d(y_1, v_1) = \text{dist}(A, B), \quad \alpha(y_0, y_1) \geqslant 0;$$

(iv) *if $D(x, T(x, y)) = 0$ or $D(y, T(y, x)) = 0$, for any $x, y \in A_0$, then $D(x, T(x, y)) = D(y, T(y, x)) = 0$;*

(v) *T is continuous; or*

(vi) *if $\{x_n\}_{n=0}^\infty$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geqslant 0$ and $x_n \to x^* \in A$ as $n \to \infty$, then $\alpha(x_n, x^*) \geqslant 0$ for all $n \in \mathbb{N}$.*

Then $CBpp(T)$ *is nonempty.*

Proof. By condition (iii), there exist elements $(x_0, y_0), (x_1, y_1) \in (A_0 \times A_0)$ and $u_1 \in T(x_0, y_0), v_1 \in T(y_0, x_0)$ such that

$$d(x_1, u_1) = \text{dist}(A, B), \quad \alpha(x_0, x_1) \geqslant 0,$$
$$d(y_1, v_1) = \text{dist}(A, B), \quad \alpha(y_0, y_1) \geqslant 0. \qquad (3.2)$$

We consider the following four cases:
(a) $u_1 \in T(x_1, y_1), v_1 \notin T(y_1, x_1)$,   (b) $u_1 \notin T(x_1, y_1), v_1 \in T(y_1, x_1)$,
(c) $u_1 \in T(x_1, y_1), v_1 \in T(y_1, x_1)$,   (d) $u_1 \notin T(x_1, y_1), v_1 \notin T(y_1, x_1)$.

Case (a). Since $u_1 \in A_0, T(x_1, y_1) \subseteq B_0$, then

$$0 = D(u_1, T(x_1, y_1)) \geqslant \text{dist}(A_0, B_0) \geqslant \text{dist}(A, B) \geqslant 0,$$

hence,

$$D(u_1, T(x_1, y_1)) = \text{dist}(A, B) = 0.$$

From (3.2), we get

$$d(x_1, u_1) = 0, \ d(y_1, v_1) = 0,$$

that is, $x_1 = u_1, y_1 = v_1$, hence, applying (iv), we have

$$D(x_1, T(x_1, y_1)) = D(u_1, T(x_1, y_1)) = \text{dist}(A, B) = 0,$$
$$D(y_1, T(y_1, x_1)) = D(v_1, T(y_1, x_1)) = \text{dist}(A, B) = 0.$$

On the other hand, since $v_1 \notin T(y_1, x_1)$ and $T(y_1, x_1) \in \mathcal{CL}(B)$, therefore $D(y_1, T(y_1, x_1)) = D(v_1, T(y_1, x_1)) > 0$, which contradicts to $D(y_1, T(y_1, x_1)) = 0$, thus, Case (a) is not true. Similarly, Case (b) is not true, too.

Case (c). When $u_1 \in T(x_1, y_1), v_1 \in T(y_1, x_1)$, as proved above, we can get that

$$D(x_1, T(x_1, y_1)) = D(u_1, T(x_1, y_1)) = \text{dist}(A, B) = 0,$$
$$D(y_1, T(x_1, y_1)) = D(v_1, T(y_1, x_1)) = \text{dist}(A, B) = 0,$$

which imply that (x_1, y_1) is the coupled best proximity point. So we only consider Case (d), that is, $u_1 \notin T(x_1, y_1), v_1 \notin T(y_1, x_1)$. Because $T(x_1, y_1)$ and $T(y_1, x_1)$ are closed in B, therefore

$$D(u_1, T(x_1, y_1)) > 0, \ D(v_1, T(y_1, x_1)) > 0.$$

Since $u_1 \in T(x_0, y_0), v_1 \in T(y_0, x_0)$, it is obvious that

$$0 < D(u_1, T(x_1, y_1)) \leqslant H(T(x_0, y_0), T(x_1, y_1)), \tag{3.3}$$

and

$$0 < D(v_1, T(y_1, x_1)) \leqslant H(T(y_0, x_0), T(y_1, x_1)). \tag{3.4}$$

Applying Lemma 3.5, we obtain that for $q_0 > 1, q_0' > 1$, there exist $u_2 \in T(x_1, y_1), v_2 \in T(y_1, x_1)$ such that

$$0 < d(u_1, u_2) < q_0 D(u_1, T(x_1, y_1)), \ 0 < d(v_1, v_2) < q_0' D(v_1, T(y_1, x_1)). \tag{3.5}$$

On the other hand, as $u_2 \in T(x_1, y_1) \subseteq B_0, v_2 \in T(y_1, x_1) \subseteq B_0$, there exist $x_2 \neq x_1, y_2 \neq y_1 \in A_0$, for otherwise (x_1, y_1) is the coupled best proximity point, such that

$$d(x_2, u_2) = d(y_2, v_2) = \text{dist}(A, B). \tag{3.6}$$

Since T is an α^+-proximal admissible, $u_2 \in T(x_1, y_1), v_2 \in T(y_1, x_1)$ and $\alpha(x_0, x_1) \geqslant 0, \alpha(y_0, y_1) \geqslant 0$, using (3.2), we obtain $\alpha(x_1, x_2) \geqslant 0$ and $\alpha(y_1, y_2) \geqslant 0$, that is,

$$\begin{aligned} d(x_2, u_2) &= \text{dist}(A, B), \quad \alpha(x_1, x_2) \geqslant 0, \\ d(y_2, v_2) &= \text{dist}(A, B), \quad \alpha(y_1, y_2) \geqslant 0. \end{aligned} \tag{3.7}$$

Because (A, B) satisfies the weak P-property and in combination with (3.2), (3.6), we have

$$d(x_1, x_2) \leqslant d(u_1, u_2), \ d(y_1, y_2) \leqslant d(v_1, v_2). \tag{3.8}$$

From (3.3), (3.4), (3.5) and (3.8), we derive

$$d(x_1, x_2) \leqslant d(u_1, u_2) < q_0 D(u_1, T(x_1, y_1)) \leqslant q_0 H(T(x_0, y_0), T(x_1, y_1)),$$
$$d(y_1, y_2) \leqslant d(v_1, v_2) < q_0' D(v_1, T(y_1, x_1)) \leqslant q_0' H(T(y_0, x_0), T(y_1, x_1)).$$

Likewise, assuming that $u_2 \notin T(x_2, y_2), v_2 \notin T(y_2, x_2)$; for otherwise, condition (iv) is not true or (x_2, y_2)

is the coupled best proximity point. Because $T(x_2, y_2)$ and $T(y_2, x_2)$ are closed in B, therefore

$$D(u_2, T(x_2, y_2)) > 0, \quad D(v_2, T(y_2, v_2)) > 0.$$

Thus, by $u_2 \in (x_1, y_1), v_2 \in (y_1, x_1)$, we have

$$\begin{aligned} 0 < D(u_2, T(x_2, y_2)) &\leqslant H(T(x_1, y_1), T(x_2, y_2)), \\ 0 < D(v_2, T(y_2, x_2)) &\leqslant H(T(y_1, x_1), T(y_2, x_2)). \end{aligned} \tag{3.9}$$

Applying Lemma 3.5, we obtain that for $q_1 > 1, q_1' > 1$, there exist $u_3 \in T(x_2, y_2), v_3 \in T(y_2, x_2)$ such that

$$0 < d(u_2, u_3) < q_1 D(u_2, T(x_2, y_2)), \quad 0 < d(v_2, v_3) < q_1' D(v_2, T(y_2, x_2)). \tag{3.10}$$

On the other hand, as $u_3 \in T(x_2, y_2) \subseteq B_0, v_3 \in T(y_2, x_2) \subseteq B_0$, there exist $x_3 \neq x_2, y_3 \neq y_2 \in A_0$, for otherwise (x_2, y_2) is the coupled best proximity point, such that

$$d(x_3, u_3) = d(y_3, v_3) = \text{dist}(A, B).$$

Again, since T is an α^+-proximal admissible, $u_3 \in T(x_2, y_2), v_3 \in T(y_2, x_2)$ and $\alpha(x_1, x_2) \geqslant 0, \alpha(y_1, y_2) \geqslant 0$, we obtain $\alpha(x_2, x_3) \geqslant 0$ and $\alpha(y_2, y_3) \geqslant 0$, that is,

$$\begin{aligned} d(x_3, u_3) &= \text{dist}(A, B), \quad \alpha(x_2, x_3) \geqslant 0, \\ d(y_3, v_3) &= \text{dist}(A, B), \quad \alpha(y_2, y_3) \geqslant 0. \end{aligned} \tag{3.11}$$

Because (A, B) satisfies the weak P-property and in combination with (3.7), (3.11), we have

$$d(x_2, x_3) \leqslant d(u_2, u_3), \quad d(y_2, y_3) \leqslant d(v_2, v_3). \tag{3.12}$$

From (3.9), (3.10) and (3.12), we have

$$\begin{aligned} d(x_2, x_3) &\leqslant d(u_2, u_3) < q_1 D(u_2, T(x_2, y_2)) \leqslant q_1 H(T(x_1, y_1), T(x_2, y_2)), \\ d(y_2, y_3) &\leqslant d(v_2, v_3) < q_1' D(v_2, T(y_2, x_2)) \leqslant q_1' H(T(y_1, x_1), T(y_2, x_2)), \end{aligned}$$

for all $n \in \mathbb{N} \backslash \{0\}$. Inductively, we can obtain sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \subseteq A_0$ and $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty} \subseteq B_0$ satisfying $x_n \neq x_{n+1}, y_n \neq y_{n+1}, D(u_n, T(x_n, y_n)) > 0, D(v_n, T(y_n, x_n)) > 0, u_{n+1} \in T(x_n, y_n)$ and $v_{n+1} \in T(y_n, x_n), n \in \mathbb{N}$ such that

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &= \text{dist}(A, B), \quad \alpha(x_n, x_{n+1}) \geqslant 0, \quad \forall n \in \mathbb{N}, \\ d(y_{n+1}, v_{n+1}) &= \text{dist}(A, B), \quad \alpha(y_n, y_{n+1}) \geqslant 0, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant d(u_n, u_{n+1}), \quad \forall n \in \mathbb{N}, \\ d(y_n, y_{n+1}) &\leqslant d(v_n, v_{n+1}), \quad \forall n \in \mathbb{N}, \end{aligned}$$

and

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant d(u_n, u_{n+1}) < q_{n-1} D(u_n, T(x_n, y_n)) \leqslant q_{n-1} H(T(x_{n-1}, y_{n-1}), T(x_n, y_n)), \\ d(y_n, y_{n+1}) &\leqslant d(v_n, v_{n+1}) < q_{n-1}' D(v_n, T(y_n, x_n)) \leqslant q_{n-1}' H(T(y_{n-1}, x_{n-1}), T(y_n, x_n)), \end{aligned} \tag{3.14}$$

for all $n \in \mathbb{N} \backslash \{0\}$. Since $x_n \in A_0, T(x_{n-1}, y_{n-1}) \subseteq B_0$ and $A_0 \subseteq A, B_0 \subseteq B$, from the definition of D and (3.13), we have

$$\begin{aligned} \text{dist}(A, B) &= d(x_n, u_n) \geqslant D(x_n, T(x_{n-1}, y_{n-1})) \geqslant \text{dist}(A_0, B_0) \geqslant \text{dist}(A, B), \\ \text{dist}(A, B) &= d(y_n, v_n) \geqslant D(y_n, T(y_{n-1}, x_{n-1})) \geqslant \text{dist}(A_0, B_0) \geqslant \text{dist}(A, B), \end{aligned}$$

for all $n \in \mathbb{N} \backslash \{0\}$, hence,

$$\begin{aligned} d(x_n, u_n) &= D(x_n, T(x_{n-1}, y_{n-1})) = \text{dist}(A, B), \quad \alpha(x_n, x_{n+1}) \geqslant 0, \\ d(y_n, v_n) &= D(y_n, T(y_{n-1}, x_{n-1})) = \text{dist}(A, B), \quad \alpha(y_n, y_{n+1}) \geqslant 0. \end{aligned} \tag{3.15}$$

In addition, we deduce that

$$
\begin{aligned}
\frac{1}{2}D^*(x_{n-1}, T(x_{n-1}, y_{n-1})) &= \frac{1}{2}\big[D(x_{n-1}, T(x_{n-1}, y_{n-1})) - \mathrm{dist}(A, B)\big] \\
&\leqslant \frac{1}{2}\big[d(x_{n-1}, x_n) + D(x_n, T(x_{n-1}, y_{n-1})) - \mathrm{dist}(A, B)\big] \\
&= \frac{1}{2}d(x_{n-1}, x_n) \\
&\leqslant d(x_{n-1}, x_n),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}D^*(y_{n-1}, T(y_{n-1}, x_{n-1})) &= \frac{1}{2}\big[D(y_{n-1}, T(y_{n-1}, x_{n-1})) - \mathrm{dist}(A, B)\big] \\
&\leqslant \frac{1}{2}\big[d(y_{n-1}, y_n) + D(y_n, T(y_{n-1}, x_{n-1})) - \mathrm{dist}(A, B)\big] \\
&= \frac{1}{2}d(y_{n-1}, y_n) \\
&\leqslant d(y_{n-1}, y_n),
\end{aligned}
$$

for all $n \in \mathbb{N}\setminus\{0\}$. Thus, by (3.1), we get

$$
\begin{aligned}
\theta(qH(T(x_{n-1}, y_{n-1}), T(x_n, y_n))) &\leqslant \alpha(x_{n-1}, x_n) + \theta(qH(T(x_{n-1}, y_{n-1}), T(x_n, y_n))) \\
&\leqslant [\theta(M(x_{n-1}, x_n, y_{n-1}, y_n))]^k, \\
\theta(qH(T(y_{n-1}, x_{n-1}), T(y_n, x_n))) &\leqslant \alpha(y_{n-1}, y_n) + H(qT(y_{n-1}, x_{n-1}), T(y_n, x_n)) \\
&\leqslant [\theta(M(y_{n-1}, y_n, x_{n-1}, x_n))]^k.
\end{aligned} \tag{3.16}
$$

We check

$$
\begin{aligned}
&M(x_{n-1}, x_n, y_{n-1}, y_n) \\
&\quad = \max\Big\{ d(x_{n-1}, x_n), \frac{D(x_{n-1}, T(x_{n-1}, y_{n-1})) + D(x_n, T(x_n, y_n))}{2} \\
&\quad\quad - \mathrm{dist}(A, B), \frac{D(x_n, T(x_{n-1}, y_{n-1})) + D(x_{n-1}, T(x_n, y_n))}{2} - \mathrm{dist}(A, B) \Big\} \\
&\quad \leqslant \max\Big\{ d(x_{n-1}, x_n), \frac{1}{2}\big[d(x_{n-1}, x_n) + D(x_n, T(x_{n-1}, y_{n-1}))\big] \\
&\quad\quad + \frac{1}{2}\big[d(x_n, x_{n+1}) + D(x_{n+1}, T(x_n, y_n))\big] - \mathrm{dist}(A, B) \\
&\quad\quad \frac{1}{2}\big[\mathrm{dist}(A, B) + d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + D(x_{n+1}, T(x_n, y_n))\big] - \mathrm{dist}(A, B) \Big\} \\
&\quad \leqslant \max\Big\{ d(x_{n-1}, x_n), \frac{1}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big], \frac{1}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big] \Big\} \\
&\quad \leqslant \max\Big\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \Big\},
\end{aligned} \tag{3.17}
$$

for all $n \in \mathbb{N}\setminus\{0\}$. Similarly, we get

$$
M(y_{n-1}, y_n, x_{n-1}, x_n) \leqslant \max\Big\{ d(y_{n-1}, y_n), d(y_n, y_{n+1}) \Big\}. \tag{3.18}
$$

Set $q_{n-1} = q'_{n-1} = q$ for every $n \in \mathbb{N}\setminus\{0\}$. From (3.14), (3.15), (3.16), (3.17) and (3.18), we get

$$
\begin{aligned}
\theta(d(x_n, x_{n+1})) &\leqslant \theta(d(u_n, u_{n+1})) \\
&\leqslant \theta\big(qH(T(x_{n-1}, y_{n-1}), T(x_n, y_n))\big) \\
&\leqslant \Big[\theta\big(M(x_{n-1}, x_n, y_{n-1}, y_n)\big)\Big]^k \\
&= \Big[\theta\big(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\big)\Big]^k,
\end{aligned}
$$

and

$$\theta(d(y_n, y_{n+1})) \leqslant \theta(d(v_n, v_{n+1}))$$
$$\leqslant \theta\big(qH(T(y_{n-1}, x_{n-1}), T(y_n, x_n))\big)$$
$$\leqslant \Big[\theta\big(M(y_{n-1}, y_n, x_{n-1}, x_n)\big)\Big]^k$$
$$= \Big[\theta\big(\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}\big)\Big]^k.$$

If we assume that

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1}),$$
$$\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} = d(y_n, y_{n+1}),$$

then we get the contradictive inequalities

$$\theta(d(x_n, x_{n+1})) \leqslant \Big[\theta\big(M(x_{n-1}, x_n, y_{n-1}, y_n)\big)\Big]^k$$
$$= \Big[\theta\big(d(x_n, x_{n+1})\big)\Big]^k$$
$$< \theta\big(d(x_n, x_{n+1})\big),$$

and

$$\theta(d(y_n, y_{n+1})) \leqslant \Big[\theta\big(M(y_{n-1}, y_n, x_{n-1}, x_n)\big)\Big]^k$$
$$= \Big[\theta\big(d(y_n, y_{n+1})\big)\Big]^k$$
$$< \theta\big(d(y_n, y_{n+1})\big),$$

thus,

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n),$$
$$\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} = d(y_{n-1}, y_n), \tag{3.19}$$

hence,

$$\theta(d(x_n, x_{n+1})) \leqslant \theta(d(u_n, u_{n+1})) \leqslant \big[\theta\big(d(x_{n-1}, x_n)\big)\big]^k,$$

and

$$\theta(d(y_n, y_{n+1})) \leqslant \theta(d(v_n, v_{n+1})) \leqslant \big[\theta\big(d(y_{n-1}, y_n)\big)\big]^k,$$

for all $n \in \mathbb{N}\backslash\{0\}$. Therefore, we obtain

$$1 \leqslant \theta(d(x_n, x_{n+1})) \leqslant \big[\theta\big(d(x_{n-1}, x_n)\big)\big]^k$$
$$\leqslant \big[\theta\big(d(x_{n-2}, x_{n-1})\big)\big]^{k^2} \leqslant \ldots \leqslant \big[\theta\big(d(x_0, x_1)\big)\big]^{k^n}, \tag{3.20}$$

$$1 \leqslant \theta(d(y_n, y_{n+1})) \leqslant \big[\theta\big(d(y_{n-1}, y_n)\big)\big]^k$$
$$\leqslant \big[\theta\big(d(y_{n-2}, y_{n-1})\big)\big]^{k^2} \leqslant \ldots \leqslant \big[\theta\big(d(y_0, y_1)\big)\big]^{k^n}, \tag{3.21}$$

$$1 \leqslant \theta(d(u_n, u_{n+1})) \leqslant \big[\theta\big(d(x_{n-1}, x_n)\big)\big]^k$$
$$\leqslant \big[\theta\big(d(x_{n-2}, x_{n-1})\big)\big]^{k^2} \leqslant \ldots \leqslant \big[\theta\big(d(x_0, x_1)\big)\big]^{k^n}, \tag{3.22}$$

and

$$1 \leqslant \theta(qd(v_n, v_{n+1})) \leqslant \big[\theta\big(d(y_{n-1}, y_n)\big)\big]^k$$
$$\leqslant \big[\theta\big(d(y_{n-2}, y_{n-1})\big)\big]^{k^2} \leqslant \ldots \leqslant \big[\theta\big(d(y_0, y_1)\big)\big]^{k^n}. \tag{3.23}$$

Letting $n \to \infty$ in (3.20), (3.21), (3.22), and (3.23), we get

$$\theta(d(x_n, x_{n+1})) \to 1, \quad \theta(d(y_n, y_{n+1})) \to 1,$$
$$\theta(d(u_n, u_{n+1})) \to 1, \quad \theta(d(v_n, v_{n+1})) \to 1,$$

as $n \to \infty$. According to the properties of function θ, there exist $0 < r < 1$ and $0 < \ell \leqslant \infty$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} = \lim_{n\to\infty} \frac{\theta(d(y_n, y_{n+1})) - 1}{[d(y_n, y_{n+1})]^r} = \ell,$$
$$\lim_{n\to\infty} \frac{\theta(d(u_n, u_{n+1})) - 1}{[d(u_n, u_{n+1})]^r} = \lim_{n\to\infty} \frac{\theta(d(v_n, v_{n+1})) - 1}{[d(v_n, v_{n+1})]^r} = \ell,$$

when $\ell < \infty$, taking $M = \frac{\ell}{2} > 0$ and using the definition of the limit, there exists $n_0 \in \mathbb{N}\backslash\{0\}$ such that

$$\left| \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} - \ell \right| \leqslant M, \quad \left| \frac{\theta(d(y_n, y_{n+1})) - 1}{[d(y_n, y_{n+1})]^r} - \ell \right| \leqslant M,$$

$$\left| \frac{\theta(d(u_n, u_{n+1})) - 1}{[d(u_n, u_{n+1})]^r} - \ell \right| \leqslant M, \quad \left| \frac{\theta(d(v_n, v_{n+1})) - 1}{[d(v_n, v_{n+1})]^r} - \ell \right| \leqslant M,$$

for all $n \geqslant n_0$. Thus

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geqslant M, \quad \frac{\theta(d(y_n, y_{n+1})) - 1}{[d(y_n, y_{n+1})]^r} \geqslant M,$$

and

$$\frac{\theta(d(u_n, u_{n+1})) - 1}{[d(u_n, u_{n+1})]^r} \geqslant M, \quad \frac{\theta(d(v_n, v_{n+1})) - 1}{[d(v_n, v_{n+1})]^r} \geqslant M,$$

for all $n \geqslant n_0$. Subsequently,

$$n[d(x_n, x_{n+1})]^r \leqslant nM'[\theta(d(x_n, x_{n+1})) - 1],$$

$$n[d(y_n, y_{n+1})]^r \leqslant nM'[\theta(d(y_n, y_{n+1})) - 1],$$

$$n[d(u_n, u_{n+1})]^r \leqslant nM'[\theta(d(u_n, u_{n+1})) - 1],$$

and

$$n[d(v_n, v_{n+1})]^r \leqslant nM'[\theta(d(v_n, v_{n+1})) - 1],$$

for all $n \geqslant n_0$, where $M' = \frac{1}{M}$. Suppose now that $\ell = \infty$. Let $M > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n \in \mathbb{N}$ such that

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geqslant M, \quad \frac{\theta(d(y_n, y_{n+1})) - 1}{[d(y_n, y_{n+1})]^r} \geqslant M,$$

and

$$\frac{\theta(d(u_n, u_{n+1})) - 1}{[d(u_n, u_{n+1})]^r} \geqslant M, \quad \frac{\theta(d(v_n, v_{n+1})) - 1}{[d(v_n, v_{n+1})]^r} \geqslant M,$$

for all $n \geqslant n_0$. This implies

$$n[d(x_n, x_{n+1})]^r \leqslant nM'[\theta(d(x_n, x_{n+1})) - 1],$$

$$n[d(y_n, y_{n+1})]^r \leqslant nM'[\theta(d(y_n, y_{n+1})) - 1],$$

$$n[d(u_n, u_{n+1})]^r \leqslant nM'[\theta(d(u_n, u_{n+1})) - 1],$$

and

$$n[d(v_n, v_{n+1})]^r \leqslant nM'[\theta(d(v_n, v_{n+1})) - 1],$$

for all $n \geqslant n_0$, where $M' = \frac{1}{M}$. Thus either of $\ell = \infty$ and $\ell < \infty$, the inequalities

$$n[d(x_n, x_{n+1})]^r \leqslant nM'[\theta(d(x_n, x_{n+1})) - 1],$$

$$n[d(y_n, y_{n+1})]^r \leqslant nM'[\theta(d(y_n, y_{n+1})) - 1],$$

$$n[d(u_n, u_{n+1})]^r \leqslant nM'[\theta(d(u_n, u_{n+1})) - 1],$$

and

$$n[d(v_n, v_{n+1})]^r \leqslant nM'[\theta(d(v_n, v_{n+1})) - 1],$$

are true for all $n \geqslant n_0$. Applying (3.20), (3.21), (3.22), (3.23) to above inequalities, we get

$$n[d(x_n, x_{n+1})]^r \leqslant nM'\left(\left[\theta(d(x_n, x_{n+1}))\right]^{k^n} - 1\right),$$

$$n[d(y_n, y_{n+1})]^r \leqslant nM'\left(\left[\theta(d(y_n, y_{n+1}))\right]^{k^n} - 1\right),$$

$$n[d(u_n, u_{n+1})]^r \leqslant nM'\left(\left[\theta(d(u_n, u_{n+1}))\right]^{k^n} - 1\right),$$

and

$$n[d(v_n, v_{n+1})]^r \leqslant nM'\left(\left[\theta(d(v_n, v_{n+1}))\right]^{k^n} - 1\right),$$

for all $n \geqslant n_0$. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} n[d(x_n, x_{n+1})]^r = 0, \quad \lim_{n \to \infty} n[d(y_n, y_{n+1})]^r = 0,$$
$$\lim_{n \to \infty} n[d(u_n, u_{n+1})]^r = 0, \quad \lim_{n \to \infty} n[d(v_n, v_{n+1})]^r = 0.$$

Subsequently, there exists $n_1 \in \mathbb{N}$ such that when $n \geqslant n_1$,

$$n[d(x_n, x_{n+1})]^r \leqslant 1, \quad n[d(y_n, y_{n+1})]^r \leqslant 1,$$
$$n[d(u_n, u_{n+1})]^r \leqslant 1, \quad n[d(v_n, v_{n+1})]^r \leqslant 1.$$

Hence, when $n \geqslant n_1$,

$$d(x_n, x_{n+1}) \leqslant \frac{1}{n^{1/r}}, \quad d(y_n, y_{n+1}) \leqslant \frac{1}{n^{1/r}},$$
$$d(u_n, u_{n+1}) \leqslant \frac{1}{n^{1/r}}, \quad d(v_n, v_{n+1}) \leqslant \frac{1}{n^{1/r}}.$$

Letting $n > m > n_1$ and applying the triangle inequality repeatedly, we get

$$d(x_m, x_n) \leqslant \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \leqslant \sum_{i=m}^{n-1} \frac{1}{i^{1/r}}, \tag{3.24}$$

$$d(x_m, x_n) \leqslant \sum_{i=m}^{n-1} d(y_i, y_{i+1}) \leqslant \sum_{i=m}^{n-1} \frac{1}{i^{1/r}}, \tag{3.25}$$

$$d(x_m, x_n) \leqslant \sum_{i=m}^{n-1} d(u_i, u_{i+1}) \leqslant \sum_{i=m}^{n-1} \frac{1}{i^{1/r}}, \tag{3.26}$$

and

$$d(x_m, x_n) \leqslant \sum_{i=m}^{n-1} d(v_i, v_{i+1}) \leqslant \sum_{i=m}^{n-1} \frac{1}{i^{1/r}}. \tag{3.27}$$

Because $0 < r < 1$, therefore $\sum_{i=m}^{\infty} \frac{1}{i^{1/r}}$ is convergent. Thus letting $m, n \to \infty$ in (3.24), (3.25), (3.26), (3.27), we deduce that $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are Cauchy sequence in (X, d). Since A and B are closed subsets of complete metric space (X, d), thus, there exists $(x^*, y^*) \in A \times A$ such that

$$x_k \xrightarrow{d} x^* \text{ as } k \to \infty, \quad \text{and} \quad y_k \xrightarrow{d} y^* \text{ as } k \to \infty,$$

likewise, there exists $(u^*, v^*) \in A \times A$ such that

$$u_k \xrightarrow{d} u^* \text{ as } k \to \infty, \quad \text{and} \quad v_k \xrightarrow{d} v^* \text{ as } k \to \infty.$$

If (v) holds, then from (3.15), noting that $u_n \in T(x_{n-1}, x_n)$ and $v_n \in T(y_{n-1}, y_n)$, it is easy to derive that

$$d(x^*, u^*) = D(x^*, T(x^*, y^*)) = \text{dist}(A, B),$$
$$d(y^*, v^*) = D(y^*, T(y^*, x^*)) = \text{dist}(A, B).$$

If (vi) holds, then $\alpha(x_k, x^*) \geqslant 0$. We can conclude that

$$\frac{1}{2} D^*(x_n, T(x_n, y_n)) \leqslant d(x_n, x^*),$$
$$\frac{1}{2} D^*(y_n, T(y_n, x_n)) \leqslant d(y_n, y^*),$$

(3.28)

or

$$\frac{1}{2} D^*(x_{n+1}, T(x_{n+1}, y_{n+1})) \leqslant d(x_{n+1}, x^*),$$
$$\frac{1}{2} D^*(y_{n+1}, T(y_{n+1}, x_{n+1})) \leqslant d(y_{n+1}, y^*),$$

(3.29)

hold for all $n \in \mathbb{N}$. In fact, assume that

$$\frac{1}{2} D^*(x_n, T(x_n, y_n)) > d(x_n, x^*), \frac{1}{2} D^*(y_n, T(y_n, x_n)) > d(y_n, y^*),$$

and

$$\frac{1}{2} D^*(x_{n+1}, T(x_{n+1}, y_{n+1})) > d(x_{n+1}, x^*), \frac{1}{2} D^*(y_{n+1}, T(y_{n+1}, x_{n+1})) > d(y_{n+1}, y^*),$$

are true for some $n \in \mathbb{N}$. Since $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$, then $d(x_n, x_{n+1}) > 0$ and $d(y_n, y_{n+1}) > 0$, for all $n \in \mathbb{N}$. Moreover, using (3.15), (3.19), we derive the following contradictive inequalities

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leqslant d(x_n, x^*) + d(x_{n+1}, x^*) \\
&< \frac{1}{2} \big[D^*(x_n, T(x_n, y_n)) + D^*(x_{n+1}, T(x_{n+1}, y_{n+1})) \big] \\
&= \frac{1}{2} \big[D(x_n, T(x_n, y_n)) + D(x_{n+1}, T(x_{n+1}, y_{n+1})) - 2\text{dist}(A, B) \big] \\
&\leqslant \frac{1}{2} \big[d(x_n, x_{n+1}) + D(x_{n+1}, T(x_n, y_n)) + d(x_{n+1}, x_{n+2}) \\
&\quad + D(x_{n+2}, T(x_{n+1}, y_{n+1})) - 2\text{dist}(A, B) \big] \\
&= \frac{1}{2} \big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \big] \\
&< d(x_n, x_{n+1}),
\end{aligned}
$$

$$
\begin{aligned}
d(y_n, y_{n+1}) &\leqslant d(y_n, y^*) + d(y_{n+1}, x^*) \\
&< \frac{1}{2} \big[D^*(y_n, T(y_n, x_n)) + D^*(y_{n+1}, T(y_{n+1}, x_{n+1})) \big] \\
&= \frac{1}{2} \big[D(y_n, T(y_n, x_n)) + D(y_{n+1}, T(y_{n+1}, x_{n+1})) - 2\text{dist}(A, B) \big] \\
&= \frac{1}{2} \big[d(y_n, y_{n+1}) + D(y_{n+1}, T(y_n, x_n)) + d(y_{n+1}, y_{n+2}) \\
&\quad + D(y_{n+2}, T(y_{n+1}, x_{n+1})) - 2\text{dist}(A, B) \big] \\
&\leqslant \frac{1}{2} \big[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) \big] \\
&< d(y_n, y_{n+1}),
\end{aligned}
$$

hence, either (3.28) or (3.29) holds, notice that $\{x_{n+1}\}_{n=0}^{\infty}$ is a subsequence of $\{x_n\}_{n=0}^{\infty}$, consequently, we can verify that there exists at least a subsequence $\{x_{n_k}\}$ of $\{x_n\}_{n=0}^{\infty}$ such that for all $k \in \mathbb{N}$,

$$\frac{1}{2} D^*(x_{n_k}, T(x_{n_k}, y_{n_k})) \leqslant d(x_{n_k}, x^*), \frac{1}{2} D^*(y_{n_k}, T(y_{n_k}, x_{n_k})) \leqslant d(y_{n_k}, y^*),$$

and $\alpha(x_{n_k}, x^*) \geqslant 0$, $\alpha(y_{n_k}, y^*) \geqslant 0$ hold. From (3.1), we obtain

$$\theta(qH(T(x_{n_k}, y_{n_k}), T(x^*, y^*))) \leqslant \left[\theta(M(x_{n_k}, x^*, y_{n_k}, y^*))\right]^k$$
$$< \theta(M(x_{n_k}, x^*, y_{n_k}, y^*)),$$

$$\theta(qH(T(y_{n_k}, x_{n_k}), T(y^*, x^*))) \leqslant \left[\theta(M(y_{n_k}, y^*, x_{n_k}, x^*))\right]^k$$
$$< \theta(M(y_{n_k}, y^*, x_{n_k}, x^*)),$$

which implies

$$0 < qH(T(x_{n_k}, y_{n_k}), T(x^*, y^*)) < M(x_{n_k}, x^*, y_{n_k}, y^*),$$
$$0 < qH(T(y_{n_k}, x_{n_k}), T(y^*, x^*)) < M(y_{n_k}, y^*, x_{n_k}, x^*),$$

hence,

$$0 < H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)) < M(x_{n_k}, x^*, y_{n_k}, y^*),$$
$$0 < H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)) < M(y_{n_k}, y^*, x_{n_k}, x^*).$$

Subsequently,

$$\lim_{k \to \infty} H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)) \leqslant \lim_{k \to \infty} M(x_{n_k}, x^*, y_{n_k}, y^*),$$
$$\lim_{k \to \infty} H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)) \leqslant \lim_{k \to \infty} M(y_{n_k}, y^*, x_{n_k}, x^*).$$

Moreover,

$$M(x_{n_k}, x^*, y_{n_k}, y^*) = \max \left\{ d(x_{n_k}, x^*), \frac{D(x_{n_k}, T(x_{n_k}, y_{n_k})) + D(x^*, T(x^*, y^*))}{2} \right.$$
$$- \text{dist}(A, B), \frac{D(x^*, T(x_{n_k}, y_{n_k})) + D(x_{n_k}, T(x^*, y^*))}{2} - \text{dist}(A, B) \right\}$$
$$\leqslant \max \left\{ d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + D(x_{n_k+1}, T(x_{n_k}, y_{n_k})) + D(x^*, T(x^*, y^*))}{2} \right.$$
$$- \text{dist}(A, B), \frac{d(x^*, x_{n_k+1}) + D(x_{n_k+1}, T(x_{n_k}, y_{n_k}))}{2}$$
$$+ \frac{D(x_{n_k}, x^*) + D(x^*, T(x^*, y^*))}{2} - \text{dist}(A, B) \right\}$$
$$\leqslant \max \left\{ d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + \text{dist}(A, B) + D(x^*, T(x^*, y^*))}{2} - \text{dist}(A, B), \right.$$
$$\frac{d(x^*, x_{n_k+1}) + \text{dist}(A, B) + D(x_{n_k}, x^*) + D(x^*, T(x^*, y^*))}{2} - \text{dist}(A, B) \right\},$$

which implies that

$$\lim_{k \to \infty} M(x_{n_k}, x^*, y_{n_k}, y^*) \leqslant \frac{D(x^*, T(x^*, y^*)) - \text{dist}(A, B)}{2}.$$

Similarly, we get

$$\lim_{k \to \infty} M(y_{n_k}, y^*, x_{n_k}, x^*) \leqslant \frac{D(y^*, T(y^*, x^*)) - \text{dist}(A, B)}{2}.$$

Additionally, we have

$$D(x^*, T(x^*, y^*)) \leqslant d(x^*, x_{n_k}) + d(x_{n_k}, u_{n_k}) + D(u_{n_k}, T(x^*, y^*))$$
$$\leqslant d(x^*, x_{n_k}) + d(x_{n_k}, u_{n_k}) + H(T(x_{n_k}, y_{n_k}), T(x^*, y^*))$$
$$\leqslant d(x^*, x_{n_k}) + \text{dist}(A, B) + H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)),$$

and

$$D(y^*, T(y^*, x^*)) \leqslant d(y^*, y_{n_k}) + d(y_{n_k}, v_{n_k}) + D(v_{n_k}, T(y^*, x^*))$$
$$\leqslant d(y^*, y_{n_k}) + d(y_{n_k}, v_{n_k}) + H(T(y_{n_k}, x_{n_k}), T(y^*, x^*))$$
$$\leqslant d(y^*, y_{n_k}) + \text{dist}(A, B) + H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)),$$

which implies

$$D(x^*, T(x^*, y^*)) - \text{dist}(A, B) \leqslant d(x^*, x_{n_k}) + H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)),$$

and

$$D(y^*, T(y^*, x^*)) - \text{dist}(A, B) \leqslant d(y^*, y_{n_k}) + H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)).$$

Letting $k \to \infty$ in , we derive

$$D(x^*, T(x^*, y^*)) - \text{dist}(A, B) \leqslant \lim_{k \to \infty} H(T(x_{n_k}, y_{n_k}), T(x^*, y^*)),$$
$$D(y^*, T(y^*, x^*)) - \text{dist}(A, B) \leqslant \lim_{k \to \infty} H(T(y_{n_k}, x_{n_k}), T(y^*, x^*)).$$

Therefore, the inequalities

$$D(x^*, T(x^*, y^*)) - \text{dist}(A, B) \leqslant \frac{D(x^*, T(x^*, y^*)) - \text{dist}(A, B)}{2},$$

and

$$D(y^*, T(y^*, x^*)) - \text{dist}(A, B) \leqslant \frac{D(y^*, T(y^*, x^*)) - \text{dist}(A, B)}{2},$$

hold, which implies that

$$D(x^*, T(x^*, y^*)) = \text{dist}(A, B), \ D(y^*, T(y^*, x^*)) = \text{dist}(A, B),$$

that is, $\text{CBpp}(T)$ is nonempty.

$\square$

The following example is provided to support the results of Theorem 3.8, we take the same θ function as that in example of [7].

Example 3.9. Let $X = R^2$ be equipped with the metric

$$d(P, Q) = \sqrt{|p - q|^2 + |p' - q'|^2},$$

for any $P = (p, p'), Q = (q, q') \in X$. Let $A = A_1 \cup A_2 \cup A_3$, $B = \{P = (p, p') \in X : p = 2, 0 \leqslant p' \leqslant 1\}$, where

$$A_1 = \{P = (p, p') \in X : p = 0, 0 \leqslant p' \leqslant 1\},$$
$$A_2 = \{P = (p, p') \in X : p = 1, 0 \leqslant p' \leqslant 1\},$$
$$A_3 = \{P = (p, p') \in X : p = \frac{1}{2}, p' = 0\}.$$

Obviously, $\text{dist}(A, B) = 1$, $A_0 = \{(0, 1), (1, 1)\}$, $B_0 = \{(0, 2), (1, 2)\}$. Let

$$B_1 = \{P = (p, p') \in X : 0 \leqslant p = 2, 0 \leqslant p' \leqslant \frac{1}{2}\},$$
$$B_2 = \{P = (p, p') \in X : 0 \leqslant p = 2, \frac{1}{2} \leqslant p' \leqslant 1\}.$$

Define the mapping $T : A \times A \to \mathcal{CL}(B)$, for $P, Q \in A$,

$$T(P, Q) = \begin{cases} \{(0, 2)\}, & \text{if } P = (0, 1), \\ \{(1, 2)\}, & \text{if } P = (1, 1), \\ \{(\frac{1}{2}, 2)\}, & \text{if } P = (\frac{1}{2}, 0), \\ B_1, & \text{if } P \in A_1 \backslash \{(0, 1)\}, \\ B_2, & \text{if } P \in A_2 \backslash \{(1, 1)\}. \end{cases}$$

Obviously, $T(A_0 \times A_0) \subseteq B_0$. Let $P_1 = (p_1, p'_1), P_2 = (p_2, p'_2) \in A_0$, $Q_1 = (q_1, q'_1), Q_2 = (q_2, q'_2) \in B_0$. We obtain that

$$d(P_1, P_2) = 1 = d(Q_1, Q_1).$$

Consequently, $d(P_1, P_2) \leqslant d(Q_1, Q_2)$, for $P_1, P_2 \in A_0, Q_1, Q_2 \in B_0$. So (A, B) satisfy the weak P-property.

Now, consider functions $\theta : (0, +\infty) \to (1, +\infty)$ by [7]

$$\theta(t) := e^{\sqrt{te^t}}.$$

It is not difficult to show $\theta \in \Theta$. Define function $\alpha : A \times A \to [-\infty, +\infty)$

$$\alpha(x, y) := \begin{cases} 0, & x, y \in \{(0,1), (1,1), (\frac{1}{2}, 0)\}, \\ -\infty, & \text{otherwise.} \end{cases}$$

Suppose that for any $x_1, x_2, w_1, w_2, w_1', w_2', y_1, y_2 \in A$ and $u_1 \in T(x_1, y_1), u_2 \in T(x_2, y_2), v_1 \in T(y_1, x_1), v_2 \in T(y_2, x_2)$,

$$\begin{cases} \alpha(x_1, x_2) \geqslant 0, \\ d(w_1, u_1) = \text{dist}(A, B), \\ d(w_2, u_2) = \text{dist}(A, B), \end{cases}$$

and

$$\begin{cases} \alpha(y_1, y_2) \geqslant 0, \\ d(w_1', v_1) = \text{dist}(A, B), \\ d(w_2', v_2) = \text{dist}(A, B), \end{cases}$$

we deduce by the definition of α that $x_i, y_i \in \{(0,1), (1,1), (\frac{1}{2}, 0)\}$ for $i = 1, 2$. Moreover, $d(w_i, u_i) = \text{dist}(A, B)$, if and only if $w_i = (0,1), u_i = (0,2)$ or $w_i = (1,1), u_i = (1,2)$ for $i = 1, 2$. It follows that $\alpha(w_1, w_2) \geqslant 0$. Similarly, $\alpha(w_1', w_2') \geqslant 0$. Thus, T is a proximal α^+-admissible mapping.

Assume that $H(T(P, Q), T(U, V)) > 0, H(T(U, V), T(P, Q)) > 0$, and

$$\left. \begin{array}{l} \frac{1}{2} D^*(P, T(P, Q)) \leqslant d(P, U), \\ \frac{1}{2} D^*(Q, T(Q, P)) \leqslant d(Q, V), \\ d(P, U) = d(Q, V) = 0, \text{ or} \\ d(P, U) > 0, d(Q, V) > 0, \end{array} \right\} \quad \text{and} \quad \alpha(P, U) \geqslant 0, \alpha(Q, V) \geqslant 0,$$

for $P, Q, U, V \in A$. When $P = Q = U = V = (1,1)$ or $P = Q = U = V = (0,1)$,

$$D^*(P, T(P, Q)) = D^*(Q, T(Q, P)) = d(P, U) = d(Q, V) = 0,$$

in this case, it is clear that both $((1,1), (1,1))$ and $((0,1), (0,1))$ are the coupled best proximity point. The remainder cases satisfying (3.13) are as follows:

(i). $\begin{cases} P \in \{(\frac{1}{2}, 0)\}, U \in \{(0,1)\}, \\ Q \in \{(\frac{1}{2}, 0)\}, V \in \{(1,1)\}, \end{cases}$ or $\begin{cases} P \in \{(\frac{1}{2}, 0)\}, U \in \{(1,1)\}, \\ Q \in \{(\frac{1}{2}, 0)\}, V \in \{(0,1)\}, \end{cases}$

(ii). $\begin{cases} P \in \{(\frac{1}{2}, 0)\}, U \in \{(0,1), (1,1)\}, \\ Q \in \{(0,1)\}, V \in \{(1,1), (\frac{1}{2}, 0)\}, \end{cases}$

(iii). $\begin{cases} P \in \{(\frac{1}{2}, 0)\}, U \in \{(0,1)\}, \\ Q \in \{(1,1)\}, V \in \{(\frac{1}{2}, 0)\}, \end{cases}$ or $\begin{cases} P \in \{(\frac{1}{2}, 0)\}, U \in \{(1,1)\}, \\ Q \in \{(1,1)\}, V \in \{(0,1)\}, \end{cases}$

(iv). $\begin{cases} P \in \{(0,1)\}, U \in \{(1,1), (\frac{1}{2}, 0)\}, \\ Q \in \{(\frac{1}{2}, 0)\}, V \in \{(0,1), (1,1)\}, \end{cases}$

(v). $\begin{cases} P \in \{(0,1)\}, U \in \{(1,1), (\frac{1}{2}, 0)\}, \\ Q \in \{(1,1)\}, V \in \{(0,1), (\frac{1}{2}, 0)\}, \end{cases}$

(vi). $\begin{cases} P \in \{(1,1)\}, U \in \{(\frac{1}{2}, 0)\}, \\ Q \in \{(1,1)\}, V \in \{(0,1)\}, \end{cases}$ or $\begin{cases} P \in \{(1,1)\}, U \in \{(0,1)\}, \\ Q \in \{(1,1)\}, V \in \{(\frac{1}{2}, 0)\}, \end{cases}$

(vii). $\begin{cases} P \in \{(1,1)\}, U \in \{(\frac{1}{2}, 0), (0,1)\}, \\ Q \in \{(0,1)\}, V \in \{(\frac{1}{2}, 0), (1,1)\}, \end{cases}$

(viii). $\begin{cases} P \in \{(1,1)\}, U \in \{(\frac{1}{2}, 0)\}, \\ Q \in \{(\frac{1}{2}, 0)\}, V \in \{(0,1)\}, \end{cases}$ or $\begin{cases} P \in \{(1,1)\}, U \in \{(0,1)\}, \\ Q \in \{(\frac{1}{2}, 0)\}, V \in \{(1,1)\}. \end{cases}$

We discuss these eight cases:

Case (i). $d(P, U) = d(Q, V) = \frac{\sqrt{5}}{2}$.

Case (ii). $d(P, U) = \frac{\sqrt{5}}{2}$ and $d(Q, V) = \frac{\sqrt{5}}{2}$ or $d(Q, V) = 1$.

Case (iii). $d(P, U) = \frac{\sqrt{5}}{2}$ and $d(Q, V) = 1$ or $d(Q, V) = \frac{\sqrt{5}}{2}$.

Case (iv). $d(P, U) = \frac{\sqrt{5}}{2}$ or $d(P, U) = 1$ and $d(Q, V) = \frac{\sqrt{5}}{2}$.

Case (v). $d(P, U) = \frac{\sqrt{5}}{2}$ or $d(P, U) = 1$ and $d(Q, V) = \frac{\sqrt{5}}{2}$ or $d(Q, V) = 1$.

Case (vi). $d(P, U) = 1$ or $d(P, U) = \frac{\sqrt{5}}{2}$ and $d(Q, V) = \frac{\sqrt{5}}{2}$ or $d(Q, V) = 1$.

Case (vii). $d(P, U) = 1$ or $d(P, U) = \frac{\sqrt{5}}{2}$ and $d(Q, V) = 1$ or $d(Q, V) = \frac{\sqrt{5}}{2}$.

Case (viii). $d(P, U) = 1$ or $d(P, U) = \frac{\sqrt{5}}{2}$ and $d(Q, V) = \frac{\sqrt{5}}{2}$.

For all cases (i)-(viii), we obtain that $H(T(P, Q), T(U, V)) = H(T(U, V), T(P, Q)) = \frac{1}{2}$ and $\min\left\{d(P, U), d(Q, V)\right\} = 1$. Calculating

$$\theta\left(\tfrac{10}{9}H(T(P, Q), T(U, V))\right) = \theta(\tfrac{5}{9}) = e^{\sqrt{\frac{5}{9}e^{\frac{5}{9}}}},$$
$$[\theta\left(\min\left\{d(P, U), d(Q, V)\right\}\right)]^{8/9} = \left(\theta(1)\right)^{8/9} = \left(e^{\sqrt{e}}\right)^{8/9},$$

since

$$e^{\sqrt{\frac{5}{9}e^{\frac{5}{9}}}} \leqslant \left(e^{\sqrt{e}}\right)^{8/9} \Leftarrow \sqrt{\frac{5}{9}e^{\frac{5}{9}}} \leqslant \frac{8\sqrt{e}}{9}$$
$$\Leftarrow \frac{5}{9}e^{\frac{5}{9}} \leqslant \frac{64e}{81}$$
$$\Leftarrow \frac{45}{64}e^{\frac{5}{9}} \leqslant e$$
$$\Leftarrow \frac{45}{64} < 1 < e^{\frac{4}{9}},$$

hence,

$$\theta\left(\tfrac{10}{9}H(T(P, Q), T(U, V))\right) \leqslant [\theta\left(\min\left\{d(P, U), d(Q, V)\right\}\right)]^{8/9} \leqslant [\theta(M(P, Q, U, V))]^{8/9},$$

and

$$\theta\left(\tfrac{10}{9}H(T(U, V), T(P, Q))\right) \leqslant [\theta\left(\min\left\{d(P, U), d(Q, V)\right\}\right)]^{8/9} \leqslant [\theta(M(U, V, P, Q))]^{8/9}.$$

Comprehensively, for all cases (i)-(viii), we obtain that when $H(T(P, Q), T(U, V)) > 0$, $H(T(U, V), T(P, Q)) > 0$, $\alpha(P, U) \geqslant 0$, $\alpha(Q, V) \geqslant 0$, and $q = \frac{10}{9}$, $k = 8/9$,

$$\left.\begin{aligned}&\tfrac{1}{2}D^*(P, T(P, Q)) \leqslant d(P, U),\\ &\tfrac{1}{2}D^*(Q, T(Q, P)) \leqslant d(Q, V),\\ &d(P, U) = d(Q, V) = 0, \text{ or}\\ &d(P, U) > 0, d(Q, V) > 0,\end{aligned}\right\} \Rightarrow \alpha(P, U) + \theta(qH(T(P, Q), T(U, V))) \leqslant [\theta(M(P, Q, U, V))]^k.$$

In addition, $D(P, T(P, Q)) \neq 0$ for all cases, hence, the condition (iv) holds in Theorem 3.8. Thus all the assumptions of Theorem 3.8 are satisfied and there exists $((0,1),(1,1)) \in A_0 \times A_0$ such that

$$D\Big((0,1), T((0,1),(1,1))\Big) = 1 = \text{dist}(A, B),$$
$$D\Big((1,1), T((1,1),(0,1))\Big) = 1 = \text{dist}(A, B),$$

that is, $\text{CBpp}(T) = \big\{((0,1),(1,1)), ((1,1),(1,1)), ((0,1),(0,1))\big\}$.

If we replace the condition that T is a two-variable Suzuki type α^+-θ-proximal multivalued mapping with

$$\alpha(x, y) + \theta(qH(T(x, x'), T(y, y'))) \leqslant [\theta(M(x, y, x', y'))]^k,$$

then the next result can be deduced easily from Theorem 3.8.

Corollary 3.10. *Let* X, A, A_0, *and* B *be as in Theorem* 3.8. *Assume that* $T : A \times A \to \mathcal{CL}(B)$ *satisfies the assertions* (i)-(vi) *in Theorem* 3.8 *and*

$$\alpha(x, y) + \theta(qH(T(x, x'), T(y, y'))) \leqslant [\theta(M(x, y, x', y'))]^k,$$

holds for all $x, y, x', y\prime \in A$, *where* $\alpha : A \times A \to [-\infty, +\infty)$, $q > 1$, *and* $\theta \in \Theta$. *Then* $\text{CBpp}(T)$ *is nonempty.*

Especially, letting $\alpha(x, y) = 0$ for all $x, y \in A$ in Theorem 3.8, obviously, (ii), (iii), and (vi) hold in Theorem 3.8, thus we state the following new result.

Corollary 3.11. *Suppose that* X, A, A_0, *and* B *are as defined in Theorem* 3.8 *and* $\theta \in \Theta$. $T : A \times A \to \mathcal{CL}(B)$ *satisfies the following assertions:*

(i) $T(x, y) \subseteq B_0$ *for* $(x, y) \in A_0 \times A_0$ *and* (A, B) *satisfies the weak* P-*property;*

(ii) *there exist* $q > 1$ *and* $0 < k < 1$ *such that*

$$\left. \begin{array}{l} \dfrac{1}{2}D^*(x, T(x, x')) \leqslant d(x, y), \\ \dfrac{1}{2}D^*(x', T(x', x)) \leqslant d(x', y'), \\ d(x, y) = d(x', y') = 0, \text{ or} \\ d(x, y) > 0, d(x', y') > 0, \end{array} \right\} \Rightarrow \theta(qH(T(x, x'), T(y, y'))) \leqslant [\theta(M(x, y, x', y'))]^k,$$

holds for all $x, y, x', y\prime \in A$;

(iii) *if* $D(x, T(x, y)) = 0$ *or* $D(y, T(y, x)) = 0$ *for any* $x, y \in A_0$, *then* $D(x, T(x, y)) = D(y, T(y, x)) = 0$.

Then $\text{CBpp}(T)$ *is nonempty.*

If we reduce the two variables of the mapping $T : A \times A \to \mathcal{CL}(B)$ to one-variable in Theorem 3.8, Corollary 3.10, and Corollary 3.11, respectively, applying Definitions 3.2, 3.6 to T and simplifying proof processes of Theorem 3.8, Corollary 3.10, and Corollary 3.11 to one-variable version of T, it is easy to see that these results are still true, thus, we state the following results with the omitting proofs.

Theorem 3.12. *Let* (X, d) *be a complete metric space and* $A, B \in \mathcal{CL}(X)$ *with* $A_0 \neq \emptyset$. $\theta \in \Theta$ *and* $T : A \to \mathcal{CL}(B)$ *be a one-variable Suzuki type* α^+-θ-*proximal multivalued mapping. Suppose that the following conditions hold:*

(i) $Tx \subseteq B_0$ *for* $x \in A_0$ *and* (A, B) *satisfies the weak* P-*property;*

(ii) T *is* α^+-*proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A_0$, and $u_1 \in Tx_0$ such that*

$$d(x_1, u_1) = dist(A, B), \quad \alpha(x_0, x_1) \geqslant 0;$$

(iv) T *is continuous; or*

(v) *if $\{x_n\}_{n=0}^{\infty}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geqslant 0$ and $x_n \to x^* \in A$ as $n \to \infty$, then $\alpha(x_n, x^*) \geqslant 0$ for all $n \in \mathbb{N}$.*

Then $Bpp(T)$ is nonempty.

Corollary 3.13. *Let X, A, A_0, and B be as in Theorem 3.12. Assume that $T : A \to \mathcal{CL}(B)$ satisfies the assertions* (i)-(v) *in Theorem 3.12 and*

$$\alpha(x, y) + \theta(qH(Tx, Ty)) \leqslant [\theta(M(x, y))]^k,$$

holds for all $x, y \in A$, where $\alpha : A \times A \to [-\infty, +\infty)$ and $q > 1$. Then $Bpp(T)$ is nonempty.

Corollary 3.14. *Suppose that X, A, A_0, and B are as defined in Theorem 3.12 and $\theta \in \Theta$. $T : A \to \mathcal{CL}(B)$ satisfies the following assertions:*

(i) $Tx \subseteq B_0$ *for $x \in A_0$ and (A, B) satisfies the weak P-property;*

(ii)

$$\frac{1}{2}D^*(x, Tx) \leqslant d(x, y) \Rightarrow \theta(qH(Tx, Ty)) \leqslant [\theta(M(x, y)]^k,$$

holds for all $x, y \in A$.

Then $Bpp(T)$ is nonempty.

Remark 3.15.

(i) The results proved in Section 3 generalize the corresponding results of Suzuki [24], Hussain et al. [6] and many others. They also enrich and develop existing conclusions on coupled best proximity point [11, 23]. Meanwhile some creative and developed techniques are used to the proofs of our results well.

(ii) If we choose some other function Θ and some more choices for the forms of α, $M(x, y)$ or $M(x, y, x', y')$ in the results of the above sections, then more fruitful results will be obtained.

Acknowledgment

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