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Existence of homoclinic orbits for a higher order difference system

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Abstract

By using critical point theory, some new criteria are obtained for the existence of a nontrivial homoclinic orbit to a higher order difference system containing both many advances and retardations. The proof is based on the mountain pass lemma in combination with periodic approximations. Related results in the literature are generalized and improved.

Keywords: Homoclinic orbits, higher order difference systems, critical point theory, advances and retardations.

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1. Introduction

In the theory of differential equations, the trajectories which are asymptotic to a constant state as the time variable $|t| \to \infty$ are called homoclinic orbits (or homoclinic solutions). Such orbits have been found in various models of continuous dynamical systems and frequently have tremendous effects on the dynamics of such nonlinear systems. So homoclinic orbits have been extensively studied since the time of Poincaré, see [7, 8, 17] and the references therein. Recently, Ma and Guo [14, 15] have found that the trajectories which are asymptotic to a constant state as the time variable $|k| \to \infty$ also exists in discrete dynamical systems [2–6, 11–15, 20–24, 26]. These trajectories are also called homoclinic orbits (or homoclinic solutions).

We denote by $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{R}$ the sets of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}, \mathbf{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a < b$. In the following and in the sequel, for any $n \in \mathbf{N}$, we will denote the Euclidean norm in $\mathbf{R}^n$ by $|\cdot|$, and defined as

$$|X| = \left(\sum_{i=1}^{n} X_i^2\right)^{\frac{1}{2}}, \quad \forall X = (X_1, X_2, \cdots, X_n) \in \mathbf{R}^n.$$

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In this paper, we consider the following higher order nonlinear difference system

$$\sum_{i=0}^{n} r_i(X_{k-i}+X_{k+i})+\chi_k X_k = f(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}), n \in \mathbf{N}, k \in \mathbf{Z}, \tag{1.1}$$

where r_i is real-valued for each $i \in \mathbf{Z}$, χ_k is positive real-valued for each $k \in \mathbf{Z}$, Γ is a given nonnegative integer, m is a given positive integer, $f=(f_1, f_2, \cdots, f_m)^* \in C(\mathbf{R}^{2\Gamma+2} \times \mathbf{R}^m, \mathbf{R})$, χ_k and $f(k, Y_\Gamma, \cdots, Y_0, \cdots, Y_{-\Gamma})$ are T-periodic in k for a given positive integer T.

Difference equations represent the discrete counterpart of ordinary differential equations and are usually studied in connection with numerical analysis. For the general background of difference equations, one can refer to monographs [1, 19]. We may regard (1.1) as being a discrete analog of the following $2n$th-order differential equation

$$\left[r(t)X^{(n)}\right]^{(n)} + \chi(t)X(t) = f(t, X(t+\Gamma), \cdots, X(t), \cdots, X(t-\Gamma)), t \in \mathbf{R}. \tag{1.2}$$

Smets and Willem [25] had proved the existence of solitary waves with prescribed speed on infinite lattices of particles with nearest neighbor interaction for the following forward and backward differential difference equation

$$c^2u''(t) = V'\left(u(t+1)-u(t)\right) - V'\left(u(t)-u(t-1)\right),\ t \in \mathbf{R}.$$

Equations similar in structure to (1.2) arise in the study of the existence of homoclinic orbits for functional differential equations, see [7, 8].

When $m = 1$, $n = 2$ and $\Gamma = 0$, (1.1) reduces to the following special case

$$\Delta\left(p_k \Delta x_{k-1}\right) - A_k x_k + b_k V(x_k) = 0, k \in \mathbf{Z}. \tag{1.3}$$

In 2009, Deng et al. [3] applied the critical point theory to prove the existence of one homoclinic orbit for (1.3).

In 2015, Liu et al. [12] considered the existence of a nontrivial homoclinic orbit for the following equation

$$Lu_k - \omega u_k = f(k, u_{k+1}, u_k, u_{k-1}), k \in \mathbf{Z},$$

containing both advance and retardation [27] by using the mountain pass lemma in combination with periodic approximations.

Recently, Shi et al. [22] studied the existence of a nontrivial homoclinic orbit for second order p-Laplacian difference equations containing both advance and retardation

$$\Delta\left(\varphi_p\left(\Delta u_{k-1}\right)\right) - q_k \varphi_p\left(u_k\right) + f(k, u_{k+M}, u_k, u_{k-M}) = 0, k \in \mathbf{Z},$$

by using the critical point theory.

By establishing a proper variational framework and using the critical point theory, Chen and Tang [2] obtained some new existence criteria to guarantee the $2n$th-order nonlinear difference equation containing both many advances and retardations

$$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right) + q_k u_k = f(k, u_{k+n}, \cdots, u_k, \cdots, u_{k-n}), n \in \mathbf{Z}(3), k \in \mathbf{Z},$$

has at least one or infinitely many homoclinic orbits.

However, to the best of our knowledge, since (1.1) contains both many advances and retardations, there are very few manuscripts dealing with this subject. The main purpose of this paper is to develop a new approach to above problem without the classical Ambrosetti-Rabinowitz condition. Motivated by the above papers [3, 22], the intention of this paper is to consider problem (1.1) in a more general sense. More exactly, our results represent the extensions to a higher order nonlinear difference system containing both many advances and retardations. We establish some new existence criteria to guarantee that (1.1) has a nontrivial homoclinic orbit. Some existing results are generalized and improved. In fact, one can see the following Remarks 1.3 and 1.4 for details.

Throughout the paper, for a function F, we let $F_i'(Y_1, \cdots, Y_i \cdots, Y_n)$ denote the partial derivative of F on the i variable. Let

$$\underline{\chi} = \min_{k\in \mathbf{Z}(1,T)} \{\chi_k\}, \bar{\chi} = \max_{k\in \mathbf{Z}(1,T)} \{\chi_k\}.$$

Our main results are the following theorems.

Theorem 1.1. *Assume that* $T \geqslant 2n+1$ *and the following hypotheses are satisfied:*

(r) $r_0 + \sum\limits_{s=1}^{n} |r_s| < 0$;

(F_1) *there exists a function* $F(t, Y_\Gamma, \cdots, Y_0) \in C^1(\boldsymbol{R}^{\Gamma+2} \times \boldsymbol{R}^m, \boldsymbol{R})$ *such that*

$$F(t+T, Y_\Gamma, \cdots, Y_0) = F(t, Y_\Gamma, \cdots, Y_0), \quad \sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t+i, Y_{\Gamma+i}, \cdots, Y_i) = f(t, Y_\Gamma, \cdots, Y_0, \cdots, Y_{-\Gamma});$$

(F_2) *there exist positive constants* ρ *and* $a < \frac{\underline{\chi}}{2(\Gamma+1)}$ *such that*

$$|F(t, Y_\Gamma, \cdots, Y_0)| \leqslant a \left(|Y_\Gamma|^2 + \cdots + |Y_0|^2\right)$$

for all $t \in \boldsymbol{R}$ *and* $\sqrt{|Y_\Gamma|^2 + \cdots + |Y_0|^2} \leqslant \rho$;

(F_3) *there exist constants* $\rho, c > \frac{\bar{\chi}+\lambda_{\max}}{2(\Gamma+1)}$ *and* b *such that*

$$|F(t, Y_\Gamma, \cdots, Y_0)| \geqslant c \left(|Y_\Gamma|^2 + \cdots + |Y_0|^2\right) + b$$

for all $t \in \boldsymbol{R}$ *and* $\sqrt{|Y_\Gamma|^2 + \cdots + |Y_0|^2} \geqslant \rho$;

(F_4) *for all* $(t, Y_\Gamma, \cdots, Y_0) \in \boldsymbol{R}^{\Gamma+2} \setminus \{(0, \cdots, 0)\}$,

$$\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t+i, Y_\Gamma, \cdots, Y_0) Y_{-i} - 2F(t, Y_\Gamma, \cdots, Y_0) > 0;$$

(F_5) $\sum\limits_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t+i, Y_\Gamma, \cdots, Y_0) Y_{-i} - 2F(t, Y_\Gamma, \cdots, Y_0) \to +\infty$ *as* $\sqrt{|Y_\Gamma|^2 + \cdots + |Y_0|^2} \to +\infty$, *where* $\lambda_{\max}$ *can be referred to* (2.4).

Then (1.1) *has a nontrivial homoclinic orbit.*

Remark 1.2. By (F_3), it is easy to see that there exists a constant $\zeta > 0$ such that

(F_3')
$$|F(t, Y_\Gamma, \cdots, Y_0)| \geqslant c \left(|Y_\Gamma|^2 + \cdots + |Y_0|^2\right) + b - \zeta, \quad \forall (t, Y_\Gamma, \cdots, Y_0) \in \mathbf{R}^{\Gamma+2}.$$

As a matter of fact, let

$$\zeta = \sup \left\{ \left| F(t, Y_\Gamma, \cdots, Y_0) - c\left(|Y_\Gamma|^2 + \cdots + |Y_0|^2\right) - b \right| : t \in \mathbf{R}, \sqrt{|Y_\Gamma|^2 + \cdots + |Y_0|^2} \leqslant \rho \right\},$$

we can easily get the desired result.

Remark 1.3. Theorem 1.1 extends Theorem 1.1 in [12] which is the special case of our Theorem 1.1 by letting $m = 1$ and $n = 2$.

Remark 1.4. In many studies (see, e.g., [3, 4, 9, 11, 12, 14, 15, 23]) of second order difference equations, the following classical Ambrosetti-Rabinowitz condition is assumed

(AR) there exists a constant $\beta > 2$ such that $0 < \beta F(k,u) \leqslant u f(k,u)$ for all $k \in \mathbf{Z}$ and $u \in \mathbf{R} \setminus \{0\}$.

Note that (F_3)-(F_5) are much weaker than (AR). Thus our result improves the existing ones.

Theorem 1.5. *Assume that* $T \geqslant 2n+1$, (r) *and* (F_1)-(F_5) *and the following hypothesis are satisfied:*

(F_6) $\chi_{-k} = \chi_k, F(-k, Y_\Gamma, \cdots, Y_0) = F(k, Y_\Gamma, \cdots, Y_0)$.

Then (1.1) *has a nontrivial even homoclinic orbit.*

For basic knowledge of variational methods, the reader is referred to [16, 18].

2. Variational structure

Our main tool is the critical point theory. We shall establish the corresponding variational framework for (1.1). We start by some basic notations for the reader's convenience.

Let S be the set of sequences $X = (\cdots, X_{-k}, \cdots, X_{-1}, X_0, X_1, \cdots, X_k, \cdots) = \{X_k\}_{k=-\infty}^{+\infty}$, where $X_k = (X_{k,1}, X_{k,2}, \cdots, X_{k,m}) \in \mathbf{R}^m$.

For any $X, Y \in S$, $a, b \in \mathbf{R}$, $aX + bY$ is defined by

$$aX + bY := \{aX_k + bY_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers p and T, E_p is defined as a subspace of S by

$$E_p = \{X \in S | X_{k+2pT} = X_k, \forall k \in \mathbf{Z}\}.$$

E_p can be equipped with the inner product $\langle X, Y \rangle$ and norm $\|X\|$ as follows,

$$\langle X, Y \rangle := \sum_{j=-pT}^{pT-1} X_j \cdot Y_j, \quad \forall X, Y \in E_p,$$

and

$$\|X\| := \left(\sum_{j=-pT}^{pT-1} |X_j|^2 \right)^{\frac{1}{2}}, \quad \forall X \in E_p,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^m$, and $X_j \cdot Y_j$ denotes the usual scalar product in $\mathbf{R}^m$.

Define a linear map $M : E_p \to \mathbf{R}^{2pmT}$ by

$$MX := \left(X_{-pT,1}, \cdots, X_{pT-1,1}, X_{-pT,2}, \cdots, X_{pT-1,2}, \cdots, X_{-pT,m}, \cdots, X_{pT-1,m}\right)^*, \tag{2.1}$$

where $X = \{X_k\}$, $X_k = (X_{k,1}, X_{k,2}, \cdots, X_{k,m})^*$, $k \in \mathbf{Z}(-pT, pT-1)$.

It is easy to see that the map M defined in (2.1) is a linear homeomorphism with $\|X\| = |MX|$, and $(E_p, \langle \cdot, \cdot \rangle)$ is a Hilbert space, which can be identified with $\mathbf{R}^{2pmT}$.

For all $X \in E_p$, define the functional J on E_p as follows:

$$J(X) := \frac{1}{2} \sum_{k=-pT}^{pT-1} \sum_{i=0}^{n} r_i \left(X_{k-i} + X_{k+i}\right) X_k + \frac{1}{2} \sum_{k=-pT}^{pT-1} \chi_k |X_k|^2 - \sum_{k=-pT}^{pT-1} F(k, X_{k+\Gamma}, \cdots, X_k).$$

Since E_p is linearly homeomorphic to $\mathbf{R}^{2pmT}$, J can be viewed as a continuously differentiable functional defined on a finite dimensional Hilbert space. That is, $J \in C^1(E_p, \mathbf{R})$. Furthermore, $J'(X) = 0$ if and

only if
$$\frac{\partial J(X)}{\partial X_{k,l}} = 0, l \in \mathbf{Z}(1, m), k \in \mathbf{Z}(-pT, pT-1).$$
If we define $X_{-pT} := X_{pT}$, then for all $l \in \mathbf{Z}(1, m)$, $k \in \mathbf{Z}(-pT, pT-1)$,
$$\frac{\partial J(X)}{\partial X_{k,l}} = \sum_{i=0}^{n} r_i (X_{k-i,l} + X_{k+i,l}) + \chi_k X_{k,l} - f_l(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}).$$
Therefore, $X \in E_p$ is a critical point of J, i.e., $J'(X) = 0$ if and only if
$$\sum_{i=0}^{n} r_i (X_{k-i,l} + X_{k+i,l}) + \chi_k X_{k,l} = f_l(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}), l \in \mathbf{Z}(1, m),\ k \in \mathbf{Z}(-pT, pT-1).$$
That is,
$$\sum_{i=0}^{n} r_i (X_{k-i} + X_{k+i}) + \chi_k X_k = f(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}), k \in \mathbf{Z}(-pT, pT-1).$$
On the other hand, $\{X_k\}_{k\in\mathbf{Z}} \in E_p$ is 2pT-periodic in k and $f(k, Y_\Gamma, \cdots, Y_0, \cdots, Y_{-\Gamma})$ is 2pT-periodic in k. So $X \in E_p$ is a critical point of J if and only if
$$\sum_{i=0}^{n} r_i (X_{k-i} + X_{k+i}) + \chi_k X_k = f(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}), \ \ \forall k \in \mathbf{Z}.$$
Thus, we reduce the problem of finding 2pT-periodic solutions of (1.1) to that of seeking critical points of the functional J in E_p. For all $X \in E_p$, J can be rewritten as
$$J(X) = -\frac{1}{2}\langle DMX, MX\rangle + \frac{1}{2}\sum_{k=-pT}^{pT-1} \chi_k |X_k|^2 - \sum_{k=-pT}^{pT-1} F(k, X_{k+\Gamma}, \cdots, X_k), \tag{2.2}$$
where $X = \{X_k\} \in E_p,\ X_k = (X_{k,1}, X_{k,2}, \cdots, X_{k,m})^*, k \in \mathbf{Z}(-pT, pT-1)$, and
$$D = \begin{pmatrix} P & & & 0 \\ & P & & \\ & & \ddots & \\ 0 & & & P \end{pmatrix}_{2pmT\times 2pmT},$$
$$-P = \begin{pmatrix}
2r_0 & r_1 & r_2 & \cdots & r_{n-1} & r_n & 0 & 0 & \cdots & 0 & r_n & r_{n-1} & \cdots & r_2 & r_1 \\
r_1 & 2r_0 & r_1 & \cdots & r_{n-2} & r_{n-1} & r_n & 0 & \cdots & 0 & 0 & r_n & \cdots & r_3 & r_2 \\
r_2 & r_1 & 2r_0 & \cdots & r_{n-3} & r_{n-2} & r_{n-1} & r_n & \cdots & 0 & 0 & 0 & \cdots & r_4 & r_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
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\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
r_2 & r_3 & r_4 & \cdots & 0 & 0 & 0 & 0 & \cdots & r_n & r_{n-1} & r_{n-2} & \cdots & 2r_0 & r_1 \\
r_1 & r_2 & r_3 & \cdots & r_n & 0 & 0 & 0 & \cdots & r_{n-1} & r_{n-2} & r_{n-3} & \cdots & r_1 & 2r_0
\end{pmatrix}$$

is a $2pT \times 2pT$ matrix. Assume that the eigenvalues of P are $\lambda_1, \lambda_2, \cdots, \lambda_T$, and P is a circulant matrix [12] denoted by

$$P := \mathrm{Circ}\{-2r_0, -r_1, -r_2, \cdots, -r_n, 0, \cdots, 0, -r_n, -r_{n-1}, \cdots, -r_2, -r_1\}.$$

By [10], the eigenvalues of P are

$$\lambda_j = -2r_0 - \sum_{s=1}^{n} r_s \left\{\exp i\frac{j\pi}{pT}\right\}^s - \sum_{s=1}^{n} r_s \left\{\exp i\frac{j\pi}{pT}\right\}^{2pT-s} = -2\sum_{s=0}^{n} r_s \cos\left(\frac{js\pi}{pT}\right), \tag{2.3}$$

where $j = 1, 2, \cdots, 2pT$. By (2.3), we know that

$$-2r_0 - 2\sum_{s=1}^{n} |r_s| \leqslant \lambda_j \leqslant -2r_0 + 2\sum_{s=1}^{n} |r_s|, \ j = 1, 2, \cdots, 2pT.$$

It follows from (r) that $\lambda_j > 0$ for all $j \in \mathbf{Z}(1, 2pT)$. Denote

$$\lambda_{\max} = \max\{\lambda_j | \lambda_j \neq 0, j = 1, 2, \cdots, 2pT\}. \tag{2.4}$$

Let E be a real Banach space, and $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{X^{(n)}\}_{n\in\mathbf{N}} \subset E$ for which $\{J(X^{(n)})\}_{n\in\mathbf{N}}$ is bounded and $J'(X^{(n)}) \to 0$ $(n \to \infty)$ possesses a convergent subsequence in E.

3. Main lemmas

In order to apply critical point theory to study the existence of a nontrivial homoclinic orbit of (1.1), we shall state some lemmas, which will be used in the proofs of our main results.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 3.1 (Mountain pass lemma [16, 18]). *Let E be a real Banach space and $J \in C^1(E, \mathbf{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*

(J_1) *there exist constants ρ, $\alpha > 0$ such that $J|_{\partial B_\rho} \geqslant \alpha$, and*
(J_2) *there exists $e \in E \setminus B_\rho$ such that $J(e) \leqslant 0$,*

then J possesses a critical value $c \geqslant \alpha$ given by

$$c = \inf_{g\in\Upsilon} \max_{s\in[0,1]} J(g(s)), \tag{3.1}$$

where

$$\Upsilon = \{g \in C([0,1], E) | g(0) = 0, \ g(1) = e\}. \tag{3.2}$$

Lemma 3.2. *Assume that $T \geqslant 2n+1$, (r) and (F_1)-(F_5) are satisfied. Then J satisfies the P.S. condition.*

Proof. Let $\{X^{(n)}\}_{n\in\mathbf{N}} \subset E_p$ be such that $\{J(X^{(n)})\}_{n\in\mathbf{N}}$ is bounded and $J'(X^{(n)}) \to 0$ as $n \to \infty$. Then there exists a positive constant K such that $-K \leqslant J(X^{(n)})$. By (F_3'), we have

$$\begin{aligned}-K \leqslant J\left(X^{(n)}\right) &\leqslant \frac{\lambda_{\max}}{2}\left\|X^{(n)}\right\|^2 + \frac{\bar{\chi}}{2}\left\|X^{(n)}\right\|^2 - \sum_{k=-pT}^{pT-1}\left[c\left(\left|X_{k+\Gamma}^{(n)}\right|^2 + \cdots + \left|X_k^{(n)}\right|^2\right) + b - \zeta\right] \\ &\leqslant \left[\frac{\lambda_{\max}}{2} + \frac{\bar{\chi}}{2} - (\Gamma+1)c\right]\left\|X^{(n)}\right\|^2 + 2pT(\zeta - b).\end{aligned}$$

Therefore,

$$\left[(\Gamma+1)c - \frac{\lambda_{\max}}{2} - \frac{\bar{\chi}}{2}\right]\left\|X^{(n)}\right\|^2 \leqslant 2pT(\zeta - b) + K. \tag{3.3}$$

Since $c > \frac{\bar{\chi}+\lambda_{\max}}{2(\Gamma+1)}$, (3.3) implies that $\{X^{(n)}\}_{n\in\mathbf{N}}$ is bounded in E_p. As a consequence, it has a convergent subsequence. □

Lemma 3.3. *Assume that* $T \geqslant 2n+1$, (r) *and* (F_1)-(F_5) *are satisfied. Then for any given positive integer* p, (1.1) *possesses a* $2pT$*-periodic solution* $X^{(p)} \in E_p$.

Proof. In our case, it is clear that $J(0) = 0$. By Lemma 3.1, J satisfies the P.S. condition. By (F_2), we have

$$J(X) \geqslant \frac{\underline{\chi}}{2}\|X\|^2 - a\sum_{k=-pT}^{pT-1}\left(|X_{k+\Gamma}|^2 + \cdots + |X_k|^2\right) \geqslant \frac{\underline{\chi}}{2}\|X\|^2 - a(\Gamma+1)\|X\|^2 = \left[\frac{\underline{\chi}}{2} - a(\Gamma+1)\right]\|X\|^2.$$

Taking $\alpha = \left[\frac{\underline{\chi}}{2} - a(\Gamma+1)\right]\rho^2 > 0$, we obtain

$$J(X)|_{\partial B_\rho} \geqslant \alpha > 0,$$

which implies that J satisfies the condition (J_1) of the mountain pass lemma.

Next, we shall verify the condition (J_2).

There exists a sufficiently large number $\varepsilon > \max\{\rho, \rho\}$ such that

$$\left[(\Gamma+1)c - \frac{\lambda_{max}}{2} - \frac{\bar{\chi}}{2}\right]\varepsilon^2 \geqslant |b|. \tag{3.4}$$

Let $\vartheta \in E_m$ and

$$\vartheta_k = \begin{cases} \varepsilon, & \text{if } k = 0, \\ 0, & \text{if } k \in \{j \in \mathbf{Z} : -pT \leqslant j \leqslant pT-1 \text{ and } j \neq 0\}, \end{cases}$$
$$\vdots$$
$$\vartheta_{k+\Gamma} = \begin{cases} \varepsilon, & \text{if } k = 0, \\ 0, & \text{if } k \in \{j \in \mathbf{Z} : -pT \leqslant j \leqslant pT-1 \text{ and } j \neq 0\}. \end{cases}$$

Then

$$F(k, \vartheta_{k+\Gamma}, \cdots, \vartheta_k) = \begin{cases} F(0, \varepsilon, \cdots, \varepsilon), & \text{if } k = 0, \\ 0, & \text{if } k \in \{j \in \mathbf{Z} : -pT \leqslant j \leqslant pT-1 \text{ and } j \neq 0\}. \end{cases}$$

With (3.4) and (F_3), we have

$$\begin{aligned} J(\vartheta) &= -\frac{1}{2}\langle DM\vartheta, M\vartheta\rangle + \frac{1}{2}\sum_{k=-pT}^{pT-1}\chi_k|\vartheta_k|^2 - \sum_{k=-pT}^{pT-1}F(k, \vartheta_{k+\Gamma}, \cdots, \vartheta_k) \\ &\leqslant \frac{\lambda_{max}}{2}\|\vartheta\|^2 + \frac{\bar{\chi}}{2}\|\vartheta\|^2 - (\Gamma+1)c\|\vartheta\|^2 - b \\ &= -\left[(\Gamma+1)c - \frac{\lambda_{max}}{2} - \frac{\bar{\chi}}{2}\right]\varepsilon^2 - b \leqslant 0. \end{aligned}$$

All the assumptions of the mountain pass lemma have been verified. Consequently, J possesses a critical value c_p given by (3.1) and (3.2) with $E = E_p$ and $\Upsilon = \Upsilon_m$, where

$$\Upsilon_p = \{g_m \in C([0,1], E_p) | g_p(0) = 0,\ g_p(1) = \vartheta, \vartheta \in E_p \backslash B_\varepsilon\}.$$

Let $X^{(p)}$ denote the corresponding critical point of J on E_p. Note that $\left\|X^{(p)}\right\| \neq 0$ since $c_p > 0$. □

Lemma 3.4. *Assume that* $T \geqslant 2n+1$, (r) *and* (F_1)-(F_5) *are satisfied. Then there exist positive constants* ρ *and* η *independent of* m *such that*

$$\rho \leqslant \left\|X^{(p)}\right\|_\infty \leqslant \eta. \tag{3.5}$$

Proof. The continuity of $\alpha F(t, Y_\Gamma, \cdots, Y_0) - \sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t, Y_\Gamma, \cdots, Y_0)Y_{-i}$ with respect to the variable from Y_Γ to Y_0 implies that there exists a constant $\tau > 0$ such that $|F(t, Y_\Gamma, \cdots, Y_0)| \leqslant \tau$ for $\sqrt{|Y_\Gamma|^2 + \cdots + |Y_0|^2} \leqslant \rho$. It is clear that

$$J\left(X^{(p)}\right) \leqslant \max_{0\leqslant s\leqslant 1} \left\{ -\frac{1}{2}\langle DM(s\vartheta), M(s\vartheta)\rangle + \frac{1}{2}\sum_{k=-pT}^{pT-1} \chi_k |(s\vartheta)_k|^2 - \sum_{k=-pT}^{pT-1} F(k, (s\vartheta)_{k+\Gamma}, \cdots, (s\vartheta)_k) \right\}$$
$$\leqslant \frac{(\lambda_{max} + \bar{\chi})}{2}\|\vartheta\|^2 + \tau = \frac{(\lambda_{max} + \bar{\chi})}{2}\varepsilon^2 + \tau.$$

Let $\xi = \frac{(\lambda_{max}+\bar{\chi})}{2}\varepsilon^2 + \tau$, we have that $J\left(X^{(p)}\right) \leqslant \xi$, which is independent of p. From (2.2), we have

$$J\left(X^{(p)}\right) = \frac{1}{2}\sum_{k=-pT}^{pT-1}\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma+i}, \cdots, X^{(p)}_{k+i}\right) X^{(p)}_k - \sum_{k=-pT}^{pT-1} F\left(k, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_k\right)$$
$$= \frac{1}{2}\sum_{k=-pT}^{pT-1}\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right) X^{(p)}_{k-i} - \sum_{k=-pT}^{pT-1} F\left(k, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_k\right) \leqslant \xi.$$

By (F_4) and (F_5), there exists a constant $\eta > 0$ such that for all $t \in \mathbf{R}$ and $\sqrt{|Y_\Gamma|^2 + \cdots + |Y_0|^2} \geqslant \eta$,

$$\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t+i, Y_\Gamma, \cdots, Y_0)Y_{-i} - 2F(t, Y_\Gamma, \cdots, Y_0) > \xi,$$

which implies that $\left|X^{(p)}_k\right| \leqslant \eta$ for all $t \in \mathbf{R}$, that is $\left\|X^{(p)}\right\|_\infty \leqslant \eta$. From the definition of J, we have

$$0 = \left\langle J'\left(X^{(p)}\right), X^{(p)}\right\rangle \geqslant \underline{\chi}\sum_{k=-pT}^{pT-1}\left|X^{(p)}_k\right|^2 - \sum_{k=-pT}^{pT-1}\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma+i}, \cdots, X^{(p)}_{k+i}\right) X^{(p)}_k$$
$$\geqslant \underline{\chi}\sum_{k=-pT}^{pT-1}\left|X^{(p)}_k\right|^2 - \sum_{k=-pT}^{pT-1}\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right) X^{(p)}_{k-i}.$$

Therefore, combined with (F_2), we get

$$\underline{\chi}\left\|X^{(p)}\right\|^2 \leqslant \sum_{k=-pT}^{pT-1}\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right) X^{(p)}_{k-i}$$
$$\leqslant \sum_{i=-\Gamma}^{0}\left[\sum_{k=-pT}^{pT-1}\left|F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right)\right|^2\right]^{\frac{1}{2}}\left\|X^{(p)}\right\|.$$

That is,

$$\underline{\chi}\left\|X^{(p)}\right\| \leqslant \sum_{i=-\Gamma}^{0}\left[\sum_{k=-pT}^{pT-1}\left|F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right)\right|^2\right]^{\frac{1}{2}}.$$

Thus,

$$\underline{\chi}^2\left\|X^{(p)}\right\|^2 \leqslant \left\{\sum_{i=-\Gamma}^{0}\left[\sum_{k=-pT}^{pT-1}\left|F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right)\right|^2\right]^{\frac{1}{2}}\right\}^2. \tag{3.6}$$

Combined with (F_2), we get

$$\underline{\chi}^2 \left\|X^{(p)}\right\|^2 \leqslant \left\{ \sum_{i=-\Gamma}^{0} \left\{ \sum_{k=-pT}^{pT-1} \left[2a \left|X^{(p)}_{k+\Gamma+i}\right|\right]^2 \right\}^{\frac{1}{2}} \right\}^2 \leqslant 4(\Gamma+1)^2 a^2 \left\|X^{(p)}\right\|^2.$$

Thus, we have $X^{(p)} = 0$. But this contradicts $\left\|X^{(p)}\right\| \neq 0$, which shows that

$$\left\|X^{(p)}\right\|_\infty \geqslant \rho,$$

and the proof of Lemma 3.3 is finished. □

4. Proof of the main results

Now, we shall finish out main results by using the critical point method.

Proof of Theorem 1.1. In the following, we shall give the existence of a nontrivial homoclinic orbit.

Consider the sequence $\left\{X^{(p)}_k\right\}_{k\in\mathbf{Z}}$ of 2pT-periodic solutions found in Lemma 3.3. First, by (3.5), for any $p \in \mathbf{N}$, there exists a constant $k_p \in \mathbf{Z}$ independent of p such that

$$\left|X^{(p)}_{k_p}\right| \geqslant \rho. \tag{4.1}$$

Since χ_k and $f(k, Y_\Gamma, \cdots, Y_0, \cdots, Y_{-\Gamma})$ are all T-periodic in k, $\left\{X^{(p)}_{k+jT}\right\}$ $(\forall j \in \mathbf{N})$ is also 2pT-periodic solution of (1.1). Hence, making such shifts, we can assume that $k_p \in \mathbf{Z}(0, T-1)$ in (4.1). Moreover, passing to a subsequence of ps, we can even assume that $k_p = k_0$ is independent of p.

Next, we extract a subsequence, still denote by $X^{(p)}$, such that

$$X^{(p)}_k \to X_k,\ p \to \infty,\ \forall k \in \mathbf{Z}.$$

Inequality (4.1) implies that $|X_{k_0}| \geqslant \xi$ and, hence, $X = \{X_k\}$ is a nonzero sequence. Moreover,

$$\begin{aligned}
&\sum_{i=0}^{n} r_i(X_{k-i} + X_{k+i}) + \chi_k X_k - f(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}) \\
&\quad = \lim_{n\to\infty} \left[\sum_{i=0}^{n} r_i \left(X^{(p)}_{k-i} + X^{(p)}_{k+i} \right) + \chi_k X^{(p)}_k - f\left(k, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_k, \cdots, X^{(p)}_{k-\Gamma}\right) \right] = 0.
\end{aligned}$$

So $X = \{X_k\}$ is a solution of (1.1). Finally, for $X_p \in E_p$, let

$$R_p = \left\{ k \in \mathbf{Z} : \left|X^{(p)}_k\right| < \frac{\sqrt{\Gamma+1}}{\Gamma+1}\rho, -pT \leqslant k \leqslant pT-1 \right\},$$
$$S_p = \left\{ k \in \mathbf{Z} : \left|X^{(p)}_k\right| \geqslant \frac{\sqrt{\Gamma+1}}{\Gamma+1}\rho, -pT \leqslant k \leqslant pT-1 \right\}.$$

Since $F(t, Y_\Gamma, \cdots, Y_0) \in C^1(\mathbf{R}^{\Gamma+2} \times \mathbf{R}^m, \mathbf{R})$, there exist constants $\bar{\xi} > 0$, $\underline{\xi} > 0$ such that

$$\max \left\{ \sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(k+i, Y_\Gamma, \cdots, Y_0) : \rho \leqslant \sqrt{|Y_\Gamma|^2 + \cdots + |Y_0|^2} \leqslant \eta, k \in \mathbf{Z} \right\}^2 \leqslant \bar{\xi},$$
$$\min \left\{ \sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(k+i, Y_\Gamma, \cdots, Y_0)Y_{-i} - F(k, Y_\Gamma, \cdots, Y_0) : \rho \leqslant \sqrt{|Y_\Gamma|^2 + \cdots + |Y_0|^2} \leqslant \eta, k \in \mathbf{Z} \right\} \geqslant \underline{\xi}.$$

For $k \in R_p$,

$$\left[\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right)\right]^2$$
$$\leqslant \frac{\overline{\xi}}{\underline{\xi}}\left\{\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right) X^{(p)}_{k-i} - F\left(k, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right)\right\}.$$

By (3.6), we have

$$\begin{aligned}
\underline{\chi}^2\left\|X^{(p)}\right\|^2 &\leqslant \left\{\sum_{i=-\Gamma}^{0}\left[\sum_{k\in R_m}\left|F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right)\right|^2\right]^{\frac{1}{2}}\right\}^2 \\
&\quad + \left\{\sum_{i=-\Gamma}^{0}\left[\sum_{k\in Q_m}\left|F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right)\right|^2\right]^{\frac{1}{2}}\right\}^2 \\
&\leqslant \left\{\sum_{i=-\Gamma}^{0}\left\{\sum_{k\in R_m}\left[2a\left|X^{(p)}_{k+\Gamma+i}\right|\right]^2\right\}^{\frac{1}{2}}\right\}^2 \\
&\quad + \frac{\overline{\xi}}{\underline{\xi}}\left\{\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}\left(k+i, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right) X^{(p)}_{k-i} - F\left(k, X^{(p)}_{k+\Gamma}, \cdots, X^{(p)}_{k}\right)\right\} \\
&\leqslant 4(\Gamma+1)^2 a^2\left\|X^{(p)}\right\|^2 + \frac{\overline{\xi}\xi}{\underline{\xi}}.
\end{aligned}$$

Thus,

$$\left\|X^{(p)}\right\|^2 \leqslant \frac{\overline{\xi}\xi}{\underline{\xi}\left[\underline{\chi}^2 - 4(\Gamma+1)^2 a^2\right]}.$$

For any fixed $D \in \mathbf{Z}$ and p large enough, we have that

$$\sum_{k=-D}^{D}\left|X^{(p)}_k\right|^2 \leqslant \left\|X^{(p)}\right\|^2 \leqslant \frac{\overline{\xi}\xi}{\underline{\xi}\left[\underline{\chi}^2 - 4(\Gamma+1)^2 a^2\right]}.$$

Since $\overline{\xi}$, $\underline{\xi}$, ξ, $\underline{\chi}$ and a are constants independent of p, passing to the limit, we have that

$$\sum_{k=-D}^{D}|X_k|^2 \leqslant \frac{\overline{\xi}\xi}{\underline{\xi}\left[\underline{\chi}^2 - 4(\Gamma+1)^2 a^2\right]}.$$

Due to the arbitrariness of D, X satisfies $X_k \to 0$ as $|k| \to \infty$. The existence of a nontrivial homoclinic orbit is obtained. □

Proof of Theorem 1.5. Consider the following boundary problem:

$$\begin{cases}
\sum\limits_{i=0}^{n} r_i(X_{k-i}+X_{k+i}) + \chi_k X_k = f(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}), & k \in \mathbf{Z}(-pT, pT), \\
\gamma_{-pT} = \gamma_{pT} = 0,\ \chi_{-pT} = \chi_{pT} = 0, \quad \gamma_{-k} = \gamma_k,\ \chi_{-k} = \chi_k, & k \in \mathbf{Z}(-pT, pT).
\end{cases}$$

Let S be the set of sequences $X = (\cdots, X_{-k}, \cdots, X_{-1}, X_0, X_1, \cdots, X_k, \cdots) = \{X_k\}_{k=-\infty}^{+\infty}$, where $X_k = (X_{k,1}, X_{k,2}, \cdots, X_{k,m}) \in \mathbf{R}^m$. For any $X, Y \in S$, $a, b \in \mathbf{R}$, $aX + bY$ is defined by

$$aX + bY := \{aX_k + bY_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space. For any given positive integers p and T, $\tilde{E}_p$ is defined as a subspace of S by

$$\tilde{E}_p = \{X \in S | X_{-k} = X_k, \ \forall k \in \mathbf{Z}\}.$$

$\tilde{E}_p$ can be equipped with the inner product $\langle X, Y\rangle$ and norm $\|X\|$ as follows,

$$\langle X, Y\rangle := \sum_{j=-pT}^{pT} X_j \cdot Y_j, \ \forall X, Y \in \tilde{E}_{pT},$$

and

$$\|X\| := \left(\sum_{j=-pT}^{pT} |X_j|^2\right)^{\frac{1}{2}}, \ \forall X \in \tilde{E}_{pT},$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^m$, and $X_j \cdot Y_j$ denotes the usual scalar product in $\mathbf{R}^m$. It is obvious that $\tilde{E}_p$ is linearly homeomorphic to $\mathbf{R}^{2pmT+1}$.

The techniques of the proof of Theorem 1.5 are just the same as those carried out in the proof of Theorem 1.1. For simplicity, we do not repeat them here. □

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