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Integral inequalities of the Hermite–Hadamard type for (α, m)-GA-convex functions

Ye Shuang[a], Feng Qi[b,c,*]

[a] College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, China.
[b] Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China.
[c] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China.

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Abstract

In this paper, the authors introduce a notion "(α, m)-GA-convex function" and establish some Hermite–Hadamard type inequalities for this kind of convex functions.

Keywords: Hermite–Hadamard type integral inequality, (α, m)-convex function, (α, m)-GA-convex function.
2010 MSC: 26A51, 26D15, 41A55.

1. Introduction

The following definitions are well-known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2. A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be GA-convex, if

$$f\left(x^t y^{1-t}\right) \leqslant tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.3 ([11])**.** For $f : [0, b] \to \mathbb{R}$, $b > 0$, and $m \in (0, 1]$, if

$$f(tx + m(1-t)y) \leqslant tf(x) + m(1-t)f(y),$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m-convex function on $[0, b]$.

*Corresponding author
Email addresses: shuangye152300@sina.com (Ye Shuang), qifeng618@msn.com (Feng Qi)

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Definition 1.4 ([6]). For $f : [0, b] \to \mathbb{R}$, $b > 0$, and $(\alpha, m) \in (0, 1]^2$, if

$$f(\lambda x + m(1-\lambda)y) \leqslant \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y),$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

Now we recall some Hermite–Hadamard type inequalities for several kinds of convex functions.

Theorem 1.5 ([3, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.6 ([7, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ and $q \geqslant 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q},$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

Theorem 1.7 ([4]). *Let $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be m-convex and $m \in (0, 1]$. If $f \in L_1([a, b])$ for $a, b \in \mathbb{R}_0$ and $a < b$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{mf(a/m)+f(b)}{2}\right\}.$$

Theorem 1.8 ([2, Theorem 2.2]). *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. If $a, b \in \mathbb{R}_0$, $a < b$, and $f \in L_1([a, b])$, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \frac{f(x)+mf(x/m)}{2}\,dx \leqslant \frac{m+1}{4}\left[\frac{f(a)+f(b}{2} + m\frac{f(a/m)+f(b/m)}{2}\right].$$

Theorem 1.9 ([5, Theorem 3.1]). *Let $I \supseteq \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \leqslant a < b < \infty$. If $|f'|^q$ is (α, m)-convex on $[a, b]$ for some given numbers $m, \alpha \in (0, 1]$ and $q \geqslant 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leqslant \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q} \times \min\left\{\left[v_1|f'(a)|^q + v_2 m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}, \left[v_2 m\left|f'\left(\frac{a}{m}\right)\right|^q + v_1|f'(b)|^q\right]^{1/q}\right\},\end{aligned}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right), \quad \text{and} \quad v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right).$$

For more information and recent developments on this topic, please refer to [1, 9, 10, 12–16] and the closely related references therein.

2. A definition and a lemma

Now we introduce the notion "(α, m)-GA-convex function".

Definition 2.1. For $f : (0, b^*] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1]^2$, a function f is said to be (α, m)-GA-convex on I, if

$$f\left(x^t y^{m(1-t)}\right) \leqslant t^\alpha f(x) + m(1-t^\alpha)f(y),$$

for all $x, y \in (0, b^*]$ and $t \in [0, 1]$.

To establish some new Hermite–Hadamard type inequalities for (α, m)-GA-convex functions, we need the following lemma.

Lemma 2.2 ([8]). *Let* $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ *be a differentiable function on* I° *and* $a, b \in I^\circ$ *with* $0 < a < b$. *If* $f' \in L_1([a,b])$, *then*

$$\frac{f(a)+4f(\sqrt{ab})+f(b)}{6} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,dx = \frac{\ln b - \ln a}{4}\int_0^1 \left(t-\frac{1}{3}\right)\left[a^{1-t/2}b^{t/2}f'\left(a^{1-t/2}b^{t/2}\right) - a^{t/2}b^{1-t/2}f'\left(a^{t/2}b^{1-t/2}\right)\right] dt.$$

3. Some new integral inequalities of Hermite–Hadamard type

Now we are in a position to establish some integral inequalities of the Hermite–Hadamard type for (α, m)-GA-convex functions.

Theorem 3.1. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be differentiable on* $\mathbb{R}_+$, $a, b \in \mathbb{R}_+$ *with* $a < b$, *and* $f' \in L_1([a,b])$. *If* $|f'|^q$ *is* (α, m)-*GA-convex on* $\left(0, \max\{b, b^{1/m}\}\right]$ *for* $(\alpha, m) \in (0,1]^2$ *and* $q \geqslant 1$, *then*

$$\begin{aligned}\left|\frac{f(a)+4f(\sqrt{ab})+f(b)}{6} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,dx\right| &\leqslant \frac{\ln b - \ln a}{4}\left[\frac{1}{2^{\alpha+2}3^{\alpha+4}(\alpha+1)(\alpha+2)(\alpha+3)}\right]^{1/q} \\ &\quad\times\left\{M^{(q-1)/q}(a,b)\left[N_1(a,b)\left|f'(b)\right|^q + mN_2(a,b)\left|f'\left(a^{1/m}\right)\right|^q\right]^{1/q}\right. \\ &\quad\left.+ M^{(q-1)/q}(b,a)\left[N_1(b,a)\left|f'(a)\right|^q + mN_2(b,a)\left|f'\left(b^{1/m}\right)\right|^q\right]^{1/q}\right\},\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}M(u,v) &= \frac{2\left[u^{5/6}L\left(u^{1/6}, v^{1/6}\right) + u^{1/2}\left(2v^{1/2} - u^{1/2}\right) - 2u^{1/2}v^{1/6}L\left(u^{1/3}, v^{1/3}\right)\right]}{3(\ln v - \ln u)}, \\ N_1(u,v) &= 12(5u+v)\alpha + 12(17u+v) + 6\times 3^{\alpha+2}[(u+v)(2\alpha^2+3) + (9u+5v)\alpha], \\ N_2(u,v) &= 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(61u+29v) - 6(10u+2v)\alpha - 12(17u+v) \\ &\quad - 6\times 3^{\alpha+2}[(u+v)(2\alpha^2+3) + (9u+5v)\alpha],\end{aligned}$$

and $L(u,v) = \frac{u-v}{\ln u - \ln v}$ *is called the logarithmic mean.*

Proof. From Lemma 2.2 and the well-known Hölder integral inequality, it follows that

$$\begin{aligned}\left|\frac{f(a)+4f(\sqrt{ab})+f(b)}{6} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,dx\right| &\leqslant \frac{\ln b - \ln a}{4}\int_0^1\left|t-\frac{1}{3}\right|\left[a^{1-t/2}b^{t/2}\left|f'\left(a^{1-t/2}b^{t/2}\right)\right| + a^{t/2}b^{1-t/2}\left|f'\left(a^{t/2}b^{1-t/2}\right)\right|\right] dt \\ &\leqslant \frac{\ln b - \ln a}{4}\left\{\left(\int_0^1\left|t-\frac{1}{3}\right|a^{1-t/2}b^{t/2}\,dt\right)^{1-1/q}\left[\int_0^1\left|t-\frac{1}{3}\right|a^{1-t/2}b^{t/2}\left|f'\left(a^{1-t/2}b^{t/2}\right)\right|^q dt\right]^{1/q}\right. \\ &\quad\left.+\left(\int_0^1\left|t-\frac{1}{3}\right|a^{t/2}b^{1-t/2}\,dt\right)^{1-1/q}\left[\int_0^1\left|t-\frac{1}{3}\right|a^{t/2}b^{1-t/2}\left|f'\left(a^{t/2}b^{1-t/2}\right)\right|^q dt\right]^{1/q}\right\},\end{aligned} \tag{3.2}$$

where

$$\int_0^1\left|t-\frac{1}{3}\right|a^{1-t/2}b^{t/2}\,dt = M(a,b) \quad\text{and}\quad \int_0^1\left|t-\frac{1}{3}\right|a^{t/2}b^{1-t/2}\,dt = M(b,a). \tag{3.3}$$

Since $|f'|^q$ is (α, m)-GA-convex on $\left(0, \max\{b, b^{1/m}\}\right]$, by the well-known GA-inequality, we have

$$\begin{aligned}
&\int_0^1 \left|t-\frac{1}{3}\right| a^{1-t/2}b^{t/2}\left|f'\left(a^{1-t/2}b^{t/2}\right)\right|^q \,dt \\
&\leqslant \int_0^1 \left|t-\frac{1}{3}\right| a^{1-t/2}b^{t/2}\left[\frac{t^\alpha}{2^\alpha}|f'(b)|^q + m\left(1-\frac{t^\alpha}{2^\alpha}\right)\left|f'\left(a^{1/m}\right)\right|^q\right] dt \\
&\leqslant \int_0^1 \left|t-\frac{1}{3}\right| \left[\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right]\left[\frac{t^\alpha}{2^\alpha}|f'(b)|^q + m\left(1-\frac{t^\alpha}{2^\alpha}\right)\left|f'\left(a^{1/m}\right)\right|^q\right] dt \\
&= \frac{N_1(a,b)|f'(b)|^q + mN_2(a,b)\left|f'\left(a^{1/m}\right)\right|^q}{2^{\alpha+2}\times 3^{\alpha+4}(\alpha+1)(\alpha+2)(\alpha+3)},
\end{aligned}$$

and

$$\int_0^1 \left|t-\frac{1}{3}\right| a^{t/2}b^{1-t/2}\left|f'\left(a^{t/2}b^{1-t/2}\right)\right|^q \,dt = \frac{N_1(b,a)|f'(a)|^q + mN_2(b,a)\left|f'\left(b^{1/m}\right)\right|^q}{2^{\alpha+2}\times 3^{\alpha+4}(\alpha+1)(\alpha+2)(\alpha+3)}. \tag{3.4}$$

Putting the equalities (3.3) and (3.4) into the inequality (3.2) leads to the inequality (3.1). The proof of Theorem 3.1 is complete. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, if* $q = 1$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+4f\left(\sqrt{ab}\right)+f(b)}{6} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,dx\right| \\
&\leqslant \frac{\ln b - \ln a}{6^{4+\alpha}(\alpha+1)(\alpha+2)(\alpha+3)} \\
&\times \left\{\left[N_1(b,a)|f'(a)| + N_1(a,b)|f'(b)|\right] + m\left[N_2(a,b)\left|f'\left(a^{1/m}\right)\right| + N_2(b,a)\left|f'\left(b^{1/m}\right)\right|\right]\right\}.
\end{aligned}$$

Corollary 3.3. *Under the assumptions of Theorem 3.1, if* $\alpha = m = 1$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+4f\left(\sqrt{ab}\right)+f(b)}{6} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,dx\right| \\
&\leqslant \frac{\ln b - \ln a}{4}\left(\frac{1}{3\times 6^4}\right)^{1/q} \\
&\times \left\{M^{(q-1)/q}(a,b)\left[(211a+137b)|f'(b)|^q + (521a+211b)|f'(a)|^q\right]^{1/q}\right. \\
&\left. + M^{(q-1)/q}(b,a)\left[(137a+211b)|f'(a)|^q + (211a+521b)|f'(b)|^q\right]^{1/q}\right\}.
\end{aligned}$$

Corollary 3.4. *Under the assumptions of Theorem 3.1, if* $\alpha = m = q = 1$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+4f\left(\sqrt{ab}\right)+f(b)}{6} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\leqslant \frac{(\ln b - \ln a)\left[(329a+211b)|f'(a)| + (211a+329b)|f'(b)|\right]}{6^5}.
\end{aligned}$$

Theorem 3.5. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be differentiable on* $\mathbb{R}_+$, $a, b \in \mathbb{R}_+$ *with* $a < b$, *and* $f' \in L_1([a, b])$. *If* $|f'|^q$ *is* (α, m)*-GA-convex on* $\left(0, \max\{b, b^{1/m}\}\right]$ *for* $(\alpha, m) \in (0, 1]^2$ *and* $q > 1$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+4f\left(\sqrt{ab}\right)+f(b)}{6} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,dx\right| \\
&\leqslant \frac{\ln b - \ln a}{4}\left(\frac{(q-1)[2^{(2q-1)/(q-1)}+1]}{(2q-1)3^{(2q-1)/(q-1)}}\right)^{1-1/q}
\end{aligned}$$

$$\times\left[\frac{1}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right]^{1/q}$$
$$\times\Big\{\big[2[(\alpha+3)a^q+(\alpha+1)b^q]|f'(b)|^q+m\big\{[3\times 2^\alpha(\alpha+1)(\alpha+2)-2(\alpha+3)]a^q$$
$$+(\alpha+1)[2^\alpha(\alpha+2)-2]b^q\big\}\big|f'(a^{1/m})\big|^q\big]^{1/q}+\big[2\big[(\alpha+1)a^q+(\alpha+3)b^q\big]\big|f'(a)\big|^q$$
$$+m\big\{(\alpha+1)[2^\alpha(\alpha+2)-2]a^q+[3\times 2^\alpha(\alpha+1)(\alpha+2)-2(\alpha+3)]b^q\big\}\big|f'(b^{1/m})\big|^q\big]^{1/q}\Big\}.$$

Proof. Since $|f'|^q$ is an (α, m)-GA-convex function on $\big(0, \max\{b, b^{1/m}\}\big]$, by Lemma 2.2 and Hölder's integral inequality, we have

$$\left|\frac{f(a)+4f\big(\sqrt{ab}\big)+f(b)}{6}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,dx\right|$$
$$\leqslant\frac{\ln b-\ln a}{4}\int_0^1\left|t-\frac{1}{3}\right|\Big[a^{1-t/2}b^{t/2}\big|f'\big(a^{1-t/2}b^{t/2}\big)\big|+a^{t/2}b^{1-t/2}\big|f'\big(a^{t/2}b^{1-t/2}\big)\big|\Big]\,dt$$
$$\leqslant\frac{\ln b-\ln a}{4}\left\{\left(\int_0^1\left|t-\frac{1}{3}\right|^{q/(q-1)}dt\right)^{1-1/q}\left[\int_0^1 a^{q(1-t/2)}b^{qt/2}\big|f'\big(a^{1-t/2}b^{t/2}\big)\big|^q\,dt\right]^{1/q}\right.$$
$$\left.+\left(\int_0^1\left|t-\frac{1}{3}\right|^{q/(q-1)}dt\right)^{1-1/q}\left[\int_0^1 a^{qt/2}b^{q(1-t/2)}\big|f'\big(a^{t/2}b^{1-t/2}\big)\big|^q\,dt\right]^{1/q}\right\}$$
$$\leqslant\frac{\ln b-\ln a}{4}\left(\frac{(q-1)[2^{(2q-1)/(q-1)}+1]}{(2q-1)3^{(2q-1)/(q-1)}}\right)^{1-1/q}$$
$$\times\left\{\left[\int_0^1\left[\left(1-\frac{t}{2}\right)a^q+\frac{t}{2}b^q\right]\left(\frac{t^\alpha}{2^\alpha}|f'(b)|^q+m\left(1-\frac{t^\alpha}{2^\alpha}\right)|f'(a^{1/m})|^q\right)dt\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1\left[\frac{t}{2}a^q+\left(1-\frac{t}{2}\right)b^q\right]\left(\frac{t^\alpha}{2^\alpha}|f'(a)|^q+m\left(1-\frac{t^\alpha}{2^\alpha}\right)|f'(b^{1/m})|^q\right)dt\right]^{1/q}\right\}$$
$$=\frac{\ln b-\ln a}{4}\left(\frac{(q-1)[2^{(2q-1)/(q-1)}+1]}{(2q-1)3^{(2q-1)/(q-1)}}\right)^{1-1/q}\left(\frac{1}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\right)^{1/q}$$
$$\times\Big\{\big[2[(\alpha+3)a^q+(\alpha+1)b^q]|f'(b)|^q+m\big\{[3\times 2^\alpha(\alpha+1)(\alpha+2)-2(\alpha+3)]a^q$$
$$+(\alpha+1)[2^\alpha(\alpha+2)-2]b^q\big\}\big|f'(a^{1/m})\big|^q\big]^{1/q}+\big[2\big[(\alpha+1)a^q+(\alpha+3)b^q\big]\big|f'(a)\big|^q$$
$$+m\big\{(\alpha+1)[2^\alpha(\alpha+2)-2]a^q+[3\times 2^\alpha(\alpha+1)(\alpha+2)-2(\alpha+3)]b^q\big\}\big|f'(b^{1/m})\big|^q\big]^{1/q}\Big\}.$$

Theorem 3.5 is thus proved. □

Corollary 3.6. *Under the assumptions of Theorem 3.5, if $\alpha = m = 1$, then*

$$\left|\frac{f(a)+4f\big(\sqrt{ab}\big)+f(b)}{6}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,dx\right|$$
$$\leqslant\frac{\ln b-\ln a}{4}\left(\frac{1}{48}\right)^{1/q}\left(\frac{(q-1)[2^{(2q-1)/(q-1)}+1]}{(2q-1)3^{(2q-1)/(q-1)}}\right)^{1-1/q}$$
$$\times\Big\{\big[(8a^q+4b^q)\big|f'(b)\big|^q+(28a^q+8b^q)\big|f'(a)\big|^q\big]^{1/q}$$
$$+\big[(4a^q+8b^q)\big|f'(a)\big|^q+(8a^q+28b^q)\big|f'(b)\big|^q\big]^{1/q}\Big\}.$$

Theorem 3.7. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be differentiable on $\mathbb{R}_+$, $a, b\in\mathbb{R}_+$ with $a<b$, and $f'\in L_1([a,b])$. If $|f'|^q$ is (α, m)-GA-convex on $\big(0,\max\{b, b^{1/m}\}\big]$ for $(\alpha, m)\in(0,1]^2$ and $q>1$, then*

$$\left|\frac{f(a)+4f\big(\sqrt{ab}\big)+f(b)}{6}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,dx\right|$$

$$
\begin{aligned}
&\leqslant \frac{\ln b-\ln a}{4}\left(\frac{5}{18}\right)^{1-1/q} \\
&\times\left(\frac{1}{2^{\alpha+2}\times 3^{\alpha+4}(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{1/q}\Big\{\Big[\Big\{\big[12(5\alpha+17)+2\times 3^{\alpha+3}(2\alpha^2+9\alpha+3)\big]a^q \\
&+6\times\big[2(\alpha+1)+3^{\alpha+2}(2\alpha^2+5\alpha+3)\big]b^q\Big\}|f'(b)|^q+m\Big\{\big[61\times 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3) \\
&-12(5\alpha+17)-6\times 3^{\alpha+2}(2\alpha^2+9\alpha+3)\big]a^q+\big[29\times 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3)-12(\alpha+1) \\
&-6\times 3^{\alpha+2}(2\alpha^2+5\alpha+3)\big]b^q\Big\}|f'(a^{1/m})|^q\Big]^{1/q}+\Big[\Big\{\big[12(5\alpha+17)+6\times 3^{\alpha+2}(2\alpha^2+9\alpha+3)\big]b^q \\
&+6\times\big[2(\alpha+1)+(2\alpha^2+5\alpha+3)3^{\alpha+2}\big]a^q\Big\}|f'(a)|^q+m\Big\{\big[61\times 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3) \\
&-12(5\alpha+17)-6\times 3^{\alpha+2}(2\alpha^2+9\alpha+3)\big]b^q+\big[29\times 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3) \\
&-12(\alpha+1)-6\times 3^{\alpha+2}(2\alpha^2+5\alpha+3)\big]a^q\Big\}|f'(b^{1/m})|^q\Big]^{1/q}\Big\}.
\end{aligned}
$$

Proof. Since $|f'|^q$ is an (α, m)-GA-convex function on $\big(0, \max\{b, b^{1/m}\}\big]$, by Lemma 2.2 and Hölder's integral inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+4f\big(\sqrt{ab}\big)+f(b)}{6}-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,dx\right| \\
&\leqslant \frac{\ln b-\ln a}{4}\int_0^1\left|t-\frac{1}{3}\right|\big[a^{1-t/2}b^{t/2}\big|f'\big(a^{1-t/2}b^{t/2}\big)\big|+a^{t/2}b^{1-t/2}\big|f'\big(a^{t/2}b^{1-t/2}\big)\big|\big]\,dt \\
&\leqslant \frac{\ln b-\ln a}{4}\left\{\left(\int_0^1\left|t-\frac{1}{3}\right|dt\right)^{1-1/q}\left[\int_0^1\left|t-\frac{1}{3}\right|a^{q(1-t/2)}b^{qt/2}\big|f'\big(a^{1-t/2}b^{t/2}\big)\big|^q\,dt\right]^{1/q}\right. \\
&\left.+\left(\int_0^1\left|t-\frac{1}{3}\right|dt\right)^{1-1/q}\left[\int_0^1\left|t-\frac{1}{3}\right|a^{qt/2}b^{q(1-t/2)}\big|f'\big(a^{t/2}b^{1-t/2}\big)\big|^q\,dt\right]^{1/q}\right\} \\
&\leqslant \frac{\ln b-\ln a}{4}\left(\frac{5}{18}\right)^{1-1/q} \\
&\times\left\{\left[\int_0^1\left|t-\frac{1}{3}\right|\left[\left(1-\frac{t}{2}\right)a^q+\frac{t}{2}b^q\right]\left(\frac{t^\alpha}{2^\alpha}|f'(b)|^q+m\left(1-\frac{t^\alpha}{2^\alpha}\right)|f'(a^{1/m})|^q\right)dt\right]^{1/q}\right. \\
&\left.+\left[\int_0^1\left|t-\frac{1}{3}\right|\left[\frac{t}{2}a^q+\left(1-\frac{t}{2}\right)b^q\right]\left(\frac{t^\alpha}{2^\alpha}|f'(a)|^q+m\left(1-\frac{t^\alpha}{2^\alpha}\right)|f'(b^{1/m})|^q\right)dt\right]^{1/q}\right\} \\
&=\frac{\ln b-\ln a}{4}\left(\frac{5}{18}\right)^{1-1/q}\left[\frac{1}{2^{\alpha+2}\times 3^{\alpha+4}(\alpha+1)(\alpha+2)(\alpha+3)}\right]^{1/q}\Big\{\Big[\Big\{\big[12(5\alpha+17)+2\times 3^{\alpha+3}(2\alpha^2 \\
&+9\alpha+3)\big]a^q+6\times\big[2(\alpha+1)+3^{\alpha+2}(2\alpha^2+5\alpha+3)\big]b^q\Big\}|f'(b)|^q \\
&+m\Big\{\big[61\times 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3) \\
&-12(5\alpha+17)-6\times 3^{\alpha+2}(2\alpha^2+9\alpha+3)\big]a^q+\big[29\times 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3)-12(\alpha+1) \\
&-6\times 3^{\alpha+2}(2\alpha^2+5\alpha+3)\big]b^q\Big\}|f'(a^{1/m})|^q\Big]^{1/q}+\Big[\Big\{\big[12(5\alpha+17)+6\times 3^{\alpha+2}(2\alpha^2+9\alpha+3)\big]b^q \\
&+6\times\big[2(\alpha+1)+(2\alpha^2+5\alpha+3)3^{\alpha+2}\big]a^q\Big\}|f'(a)|^q+m\Big\{\big[61\times 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3) \\
&-12(5\alpha+17)-6\times 3^{\alpha+2}(2\alpha^2+9\alpha+3)\big]b^q+\big[29\times 6^\alpha(\alpha+1)(\alpha+2)(\alpha+3)-12(\alpha+1) \\
&-6\times 3^{\alpha+2}(2\alpha^2+5\alpha+3)\big]a^q\Big\}|f'(b^{1/m})|^q\Big]^{1/q}\Big\}.
\end{aligned}
$$

Theorem 3.7 is thus proved. □

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References

[1] R.-F. Bai, F. Qi, B.-Y. Xi, *Hermite-Hadamard type inequalities for the* m*- and* (α, m)*-logarithmically convex functions*, Filomat, **27** (2013), 1–7. 1

[2] S. S. Dragomir, *On some new inequalities of Hermite-Hadamard type for* m*-convex functions*, Tamkang J. Math., **33** (2002), 55–65. 1.8

[3] S. S. Dragomir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., **11** (1998), 91–95. 1.5

[4] S. S. Dragomir, G. Toader, *Some inequalities for* m*-convex functions*, Studia Univ. Babeş-Bolyai Math., **38** (1993), 21–28. 1.7

[5] M. Klaričić Bakula, M. E. Özdemir, J. Pečarić, *Hadamard type inequalities for* m*-convex and* (α, m)*-convex functions*, JIPAM. J. Inequal. Pure Appl. Math., **9** (2008), 12 pages. 1.9

[6] V. G. Miheşan, *A generalization of the convexity*, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania, (1993). 1.4

[7] C. E. M. Pearce, J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulæ*, Appl. Math. Lett., **13** (2000), 51–55. 1.6

[8] F. Qi, B.-Y. Xi, *Some integral inequalities of Simpson type for GA-*ε*-convex functions*, Georgian Math. J., **20** (2013), 775–788. 2.2

[9] F. Qi, B.-Y. Xi, *Some Hermite-Hadamard type inequalities for geometrically quasi-convex functions*, Proc. Indian Acad. Sci. Math. Sci., **124** (2014), 333–342. 1

[10] F. Qi, T.-Y. Zhang, B.-Y. Xi, *Hermite-Hadamard-type integral inequalities for functions whose first derivatives are convex*, Reprint of Ukraïn. Mat. Zh., **67** (2015), 555–567, Ukrainian Math. J., **67** (2015), 625–640. 1

[11] G. Toader, *Some generalizations of the convexity*, Proceedings of the colloquium on approximation and optimization, Cluj-Napoca, (1985), Univ. Cluj-Napoca, Cluj-Napoca, (1985), 329–338. 1.3

[12] B.-Y. Xi, R.-F. Bai, F. Qi, *Hermite-Hadamard type inequalities for the* m*- and* (α, m)*-geometrically convex functions*, Aequationes Math., **84** (2012), 261–269. 1

[13] B.-Y. Xi, F. Qi, *Hermite-Hadamard type inequalities for geometrically* r*-convex functions*, Studia Sci. Math. Hungar., **51** (2014), 530–546.

[14] B.-Y. Xi, F. Qi, *Inequalities of Hermite-Hadamard type for extended* s*-convex functions and applications to means*, J. Nonlinear Convex Anal., **16** (2015), 873–890.

[15] B.-Y. Xi, S.-H. Wang, F. Qi, *Some inequalities for* (h, m)*-convex functions*, J. Inequal. Appl., **2014** (2014), 12 pages.

[16] B.-Y. Xi, T.-Y. Zhang, F. Qi, *Some inequalities of Hermite-Hadamard type for* m*-harmonic-arithmetically convex functions*, ScienceAsia, **41** (2015), 357–361. 1