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# Generalizations of Hu-type inequalities and their applications

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## Abstract

In this paper, we present some new generalizations of Hu-type inequalities, and then we obtain some new generalizations and refinements of Hölder's inequality. 

Keywords: Hölder's inequality, Hu-type inequality, generalization, refinement.
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## 1. Introduction

The classical Hölder's inequality states that if $a_k \geqslant 0,\ b_k \geqslant 0$ $(k = 1, 2, \cdots, n)$, $p > 0,\ q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}. \tag{1.1}$$

The inequality (1.1) is reversed for $p < 1$ $(p \neq 0)$; (For $p < 1$, we assume that $a_k, b_k > 0$).

Hölder's inequality plays a very important role in both theory and applications. This classical inequality has been widely studied by many authors, and it has motivated a large number of research papers involving different proofs, various generalizations, variations and applications (see e.g., [1, 6–14, 16–18] and the references therein).

Among various refinements of (1.1), Hu in [4] established the following interesting theorems.

**Theorem 1.1.** *Let* $p \geqslant q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *let* $A_k,\ B_k \geqslant 0$ $(k = 1, 2, \cdots, n)$, *and let* $1 - e_r + e_s \geqslant 0$ $(r, s = 1, 2, \cdots, n)$. *Then*

$$\begin{aligned}\sum_{k=1}^{n} A_k B_k \leqslant \left(\sum_{k=1}^{n} B_k^q\right)^{\frac{1}{q}-\frac{1}{p}} \Bigg\{ \Bigg[ \left(\sum_{k=1}^{n} B_k^q\right)\left(\sum_{k=1}^{n} A_k^p\right)\Bigg]^2 \\ - \Bigg[\left(\sum_{k=1}^{n} B_k^q e_k\right)\left(\sum_{k=1}^{n} A_k^p\right) - \left(\sum_{k=1}^{n} B_k^q\right)\left(\sum_{k=1}^{n} A_k^p e_k\right)\Bigg]^2 \Bigg\}^{\frac{1}{2p}}.\end{aligned} \tag{1.2}$$

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The integral form is as follows:

**Theorem 1.2.** *Let* $f(x), g(x), e(x)$ *be integrable functions defined on* $[a, b]$ *and* $f(x), g(x) \geqslant 0$, $1 - e(x) + e(y) \geqslant 0$ *for all* $x, y \in [a, b]$, *and let* $p \geqslant q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_a^b f(x)g(x)dx \leqslant \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx\right)^2 - \left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx\right)^2\right]^{\frac{1}{2p}}. \tag{1.3}$$

Later, Tian in [7] gave the reversed versions of Hu's inequalities (1.2) and (1.3).

**Theorem 1.3.** *Let* $p < 0$, $q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *let* $A_k > 0$, $B_k \geqslant 0$ $(k = 1, 2, \cdots, n)$, *and let* $1 - e_r + e_s \geqslant 0$ $(r, s = 1, 2, \cdots, n)$. *Then*

$$\sum_{k=1}^n A_k B_k \geqslant \left(\sum_{k=1}^n B_k^q\right)^{\frac{1}{q}-\frac{1}{p}} \left\{\left[\left(\sum_{k=1}^n B_k^q\right)\left(\sum_{k=1}^n A_k^p\right)\right]^2 - \left[\left(\sum_{k=1}^n B_k^q e_k\right)\left(\sum_{k=1}^n A_k^p\right) - \left(\sum_{k=1}^n B_k^q\right)\left(\sum_{k=1}^n A_k^p e_k\right)\right]^2\right\}^{\frac{1}{2p}}. \tag{1.4}$$

The integral form is as follows:

**Theorem 1.4.** *Let* $f(x), g(x), e(x)$ *be integrable functions defined on* $[a, b]$ *and* $f(x) > 0$, $g(x) \geqslant 0$, $1 - e(x) + e(y) \geqslant 0$ *for all* $x, y \in [a, b]$, *and let* $p < 0$, $q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_a^b f(x)g(x)dx \geqslant \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx\right)^2 - \left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx\right)^2\right]^{\frac{1}{2p}}. \tag{1.5}$$

In 2007, Wu [16] presented the generalizations of Hu's results, as follows:

**Theorem 1.5.** *Let* $A_r \geqslant 0$, $B_r > 0$ $(r = 1, 2, \cdots, n)$, *let* $1 - e_r + e_s \geqslant 0$ $(r, s = 1, 2, \cdots, n)$, *and let* $p \geqslant q > 0$, $\mu = \min\{\frac{1}{p} + \frac{1}{q}, 1\}$. *Then*

$$\sum_{r=1}^n A_r B_r \leqslant n^{1-\mu}\left(\sum_{r=1}^n B_r^q\right)^{\frac{1}{q}-\frac{1}{p}} \left\{\left[\left(\sum_{r=1}^n B_r^q\right)\left(\sum_{r=1}^n A_r^p\right)\right]^2 - \left[\left(\sum_{r=1}^n B_r^q e_r\right)\left(\sum_{r=1}^n A_r^p\right) - \left(\sum_{r=1}^n B_r^q\right)\left(\sum_{r=1}^n A_r^p e_r\right)\right]^2\right\}^{\frac{1}{2p}}. \tag{1.6}$$

**Theorem 1.6.** *Let* $f(x), g(x), e(x)$ *be integrable functions defined on* $[a, b]$ *and* $f(x) \geqslant 0$, $g(x) > 0$, $1 - e(x) + e(y) \geqslant 0$ *for all* $x, y \in [a, b]$, *and let* $p \geqslant q > 0$, $\frac{1}{p} + \frac{1}{q} \leqslant 1$. *Then*

$$\int_a^b f(x)g(x)dx \leqslant (b-a)^{1-\frac{1}{p}-\frac{1}{q}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \left[\left(\int_a^b g^q(x)dx \int_a^b f^p(x)dx\right)^2 - \left(\int_a^b g^q(x)e(x)dx \int_a^b f^p(x)dx - \int_a^b g^q(x)dx \int_a^b f^p(x)e(x)dx\right)^2\right]^{\frac{1}{2p}}. \tag{1.7}$$

In 2012, Tian [8] proved the following reversed versions of inequalities (1.6) and (1.7).

**Theorem 1.7.** *Let* $A_r > 0$, $B_r > 0$ $(r = 1, 2, \cdots, n)$, *let* $1 - e_r + e_s \geqslant 0$ $(r, s = 1, 2, \cdots, n)$, *and let* $q <$

$0,\ \frac{1}{p}+\frac{1}{q}\geqslant 0,\ \mu=\max\{\frac{1}{p}+\frac{1}{q},1\}, \lambda=\max\{\frac{1}{q},-1\}$. *Then*

$$\sum_{r=1}^{n} A_rB_r \leqslant n^{1-\mu}\left(\sum_{r=1}^{n} A_r^p\right)^{\frac{1}{p}}\left(\sum_{r=1}^{n} B_r^q\right)^{\frac{1}{q}}\left[1-\left(\frac{\sum_{r=1}^{n} A_rB_re_r}{\sum_{r=1}^{n} A_rB_r}-\frac{\sum_{r=1}^{n} B_r^qe_r}{\sum_{r=1}^{n} B_r^q}\right)^2\right]^{\frac{\lambda}{2}}.$$

**Theorem 1.8.** *Let* $f(x), g(x), e(x)$ *be integrable functions defined on* $[a, b]$ *and* $f(x) > 0,\ g(x) > 0,\ 1-e(x)+e(y)\geqslant 0$ *for all* $x, y \in [a, b]$, *and let* $q < 0,\ \frac{1}{p}+\frac{1}{q}\geqslant 1,\ \lambda=\max\{-1,\frac{1}{q}\}$. *Then*

$$\int_a^b f(x)g(x)dx \geqslant (b-a)^{1-\frac{1}{p}-\frac{1}{q}}\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \times\left[1-\left(\frac{\int_a^b f(x)g(x)e(x)dx}{\int_a^b f(x)g(x)dx}-\frac{\int_a^b g^q(x)e(x)dx}{\int_a^b g^q(x)dx}\right)^2\right]^{\frac{\lambda}{2}}.$$

Later, in 2013, Tian and Hu [13] presented another reversed versions of inequalities (1.6) and (1.7).

**Theorem 1.9.** *Let* $A_r \geqslant 0,\ B_r > 0\ (r = 1, 2, \cdots, n)$, *let* $1-e_r+e_s\geqslant 0\ (r, s = 1, 2, \cdots, n)$, *and let* $q < 0,\ p > 0,\ \rho=\max\{\frac{1}{p}+\frac{1}{q},1\}$. *Then*

$$\sum_{r=1}^{n} A_rB_r \geqslant n^{1-\rho}\left(\sum_{r=1}^{n} A_r^p\right)^{\frac{1}{p}-\frac{1}{q}}\left\{\left[\left(\sum_{r=1}^{n} A_r^p\right)\left(\sum_{r=1}^{n} B_r^q\right)\right]^2 -\left[\left(\sum_{r=1}^{n} A_r^pe_r\right)\left(\sum_{r=1}^{n} B_r^q\right)-\left(\sum_{r=1}^{n} A_r^p\right)\left(\sum_{r=1}^{n} B_r^qe_r\right)\right]^2\right\}^{\frac{1}{2q}}. \tag{1.8}$$

**Theorem 1.10.** *Let* $f(x), g(x), e(x)$ *be integrable functions defined on* $[a, b]$ *and* $f(x),\ g(x) > 0, 1-e(x)+e(y)\geqslant 0$ *for all* $x, y \in [a, b]$, *and let* $q < 0,\ \frac{1}{p}+\frac{1}{q}\geqslant 1$. *Then*

$$\int_a^b f(x)g(x)dx \geqslant (b-a)^{1-\frac{1}{p}-\frac{1}{q}}\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}-\frac{1}{q}}\left[\left(\int_a^b f^p(x)dx\int_a^b g^q(x)dx\right)^2 -\left(\int_a^b f^p(x)e(x)dx\int_a^b g^q(x)dx-\int_a^b f^p(x)dx\int_a^b g^q(x)e(x)dx\right)^2\right]^{\frac{1}{2q}}. \tag{1.9}$$

In 2011, Tian in [7] gave the following generalizations of inequalities (1.2) and (1.3).

**Theorem 1.11.** *Let* $A_{nj}\geqslant 0,\ \sum_n A_{nj}^{\lambda_j}<\infty\ (j=1,2,\cdots,k),\ \lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k>0,\ \sum_{j=1}^{k}\frac{1}{\lambda_j}=1$, *and let* $1-e_n+e_m\geqslant 0,\ \sum_n |e_n|<\infty$. *If* $k$ *is even number, then*

$$\sum_n\prod_{j=1}^{k} A_{nj} \leqslant \prod_{j=1}^{\frac{k}{2}}\left\{\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\right)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\left[\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\right)\left(\sum_n A_{n(2j)}^{\lambda_{2j}}\right)\right)^2 -\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}e_n\right)\left(\sum_n A_{n(2j)}^{\lambda_{2j}}\right)-\sum_n\left(A_{n(2j-1)}^{\lambda_{2j-1}}\right)\left(\sum_n A_{n(2j)}^{\lambda_{2j}}e_n\right)\right)^2\right]^{\frac{1}{\lambda_{2j}}}\right\}. \tag{1.10}$$

*If* $k$ *is odd number, then*

$$\sum_n\prod_{j=1}^{k} A_{nj} \leqslant \left(\sum_n A_{nk}^{\lambda_k}\right)^{\frac{2}{\lambda_k}}\prod_{j=1}^{\frac{k-1}{2}}\left\{\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\right)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}} \times\left[\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}\right)\left(\sum_n A_{n(2j)}^{\lambda_{2j}}\right)\right)^2-\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}e_n\right)\left(\sum_n A_{n(2j)}^{\lambda_{2j}}\right) -\sum_n\left(A_{n(2j-1)}^{\lambda_{2j-1}}\right)\left(\sum_n A_{n(2j)}^{\lambda_{2j}}e_n\right)\right)^2\right]^{\frac{1}{\lambda_{2j}}}\right\}. \tag{1.11}$$

**Theorem 1.12.** *Let* $A_{k1} \geqslant 0,\ A_{kj} > 0,\ (j = 1, 2, \cdots, m,\ k = 1, 2, \cdots, n),\ \sum_{j=1}^{m} \frac{1}{\lambda_j} = 1$, *and let* $1 - e_r + e_s \geqslant 0\ (r, s = 1, 2, \cdots, n)$. *If* $\lambda_1 > 0,\ \lambda_j < 0,\ (j = 2, 3, \cdots, m)$, *then*

$$\begin{aligned}\sum_{k=1}^{n}\prod_{j=1}^{m} A_{kj} \geqslant \left(\sum_{k=1}^{n} A_{k1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}} \prod_{j=2}^{m}\Bigg\{\left[\left(\sum_{k=1}^{n} A_{k1}^{\lambda_1}\right)\left(\sum_{k=1}^{n} A_{kj}^{\lambda_j}\right)\right]^2 \\ -\left[\left(\sum_{k=1}^{n} A_{k1}^{\lambda_1}e_k\right)\left(\sum_{k=1}^{n} A_{kj}^{\lambda_j}\right)-\left(\sum_{k=1}^{n} A_{k1}^{\lambda_1}\right)\left(\sum_{k=1}^{n} A_{kj}^{\lambda_j}e_k\right)\right]^2\Bigg\}^{\frac{1}{2\lambda_j}}.\end{aligned} \tag{1.12}$$

The classic Hölder inequality is an important cornerstone in different branches of modern mathematics such as classical real and complex analysis, probability and statistics, numerical analysis, qualitative theory of differential equations. It is also a bridge to help solve problems into depth. The Hu's inequality (1.2), which was put forward by Hu in [4], improves the Hölder inequality exquisitely. The mathematical reviews [5] calls it "an extraordinary, outstanding and new inequality". The classic Hölder inequality is playing a basic role in mathematics and can be applied in a wide range of areas, while the function of the Hu's inequality is the same.

The purpose of this work is to give some new generalizations of the above Hu-type inequalities (1.2), (1.3), (1.4), (1.5), (1.6), (1.7) and (1.8), (1.9), (1.10), (1.11), (1.12). Moreover, the obtained results will be applied to improve Hölder's inequality and Popoviciu-type inequality which is due to Wu and Debnath.

## 2. Main results

We begin this section with some lemmas, which will be used in the sequel.

**Lemma 2.1** ([2]). *If* $x > -1$, $\alpha > 1$ *or* $\alpha < 0$, *then*

$$(1 + x)^{\alpha} \geqslant 1 + \alpha x.$$

*The inequality is reversed for* $0 < \alpha < 1$.

**Lemma 2.2** ([3]). *If* $x_i \geqslant 0$, $\lambda_i > 0$, $i = 1, 2, \cdots, n$, $0 < p \leqslant 1$, *then*

$$\sum_{i=1}^{n} \lambda_i x_i^p \leqslant \left(\sum_{i=1}^{n} \lambda_i\right)^{1-p}\left(\sum_{i=1}^{n} \lambda_i x_i\right)^{p}. \tag{2.1}$$

*The inequality is reversed for* $p \geqslant 1$ *or* $p < 0$.

**Lemma 2.3** (Generalized Hölder's inequality [15]).

(a) *Let* $A_{ij} \geqslant 0\ (i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$, *and let* $\lambda_j > 0$ *with* $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$. *Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} \leqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}\right)^{\frac{1}{\lambda_j}}. \tag{2.2}$$

(b) *Let* $A_{ij} > 0\ (i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$, *and let* $\lambda_1 > 0$, $\lambda_j < 0\ (j = 2, 3, \cdots, m)$ *with* $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$. *Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} \geqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}\right)^{\frac{1}{\lambda_j}}. \tag{2.3}$$

(c) *Let* $A_{ij} > 0\ (i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$, *and let* $\lambda_j < 0\ (j = 1, 2, \cdots, m)$. *Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} \geqslant \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}\right)^{\frac{1}{\lambda_j}}.$$

Next, we generalize the inequalities (1.2), (1.3), (1.6), (1.7), (1.10) and (1.11) as follows.

**Theorem 2.4.** *Let* $A_{rj} \geqslant 0$, $(r = 1, 2, \cdots, n,\ j = 1, 2, \cdots, k)$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$, *let* $\rho = \min\{\sum_{j=1}^{k} \frac{1}{\lambda_j}, 1\}$, *and let* $1 - e_r + e_s \geqslant 0$, $(s = 1, 2, \cdots, n)$. *If* $k$ *is even, then*

$$\sum_{r=1}^{n}\prod_{j=1}^{k} A_{rj} \leqslant n^{1-\rho}\prod_{j=1}^{\frac{k}{2}}\left\{\left(\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\left[\left(\left(\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}\right)\left(\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}}\right)\right)^2 - \left(\left(\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} e_r\right)\left(\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}}\right) - \left(\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}\right)\left(\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} e_r\right)\right)^2\right]^{\frac{1}{2\lambda_{2j}}}\right\}. \tag{2.4}$$

*If* $k$ *is odd, then*

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{k} A_{rj} \leqslant{}& n^{1-\rho}\left(\sum_{r=1}^{n} A_{rk}^{\lambda_k}\right)^{\frac{1}{\lambda_k}}\prod_{j=1}^{\frac{k-1}{2}}\left\{\left(\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\right.\\ &\times\left[\left(\left(\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}\right)\left(\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}}\right)\right)^2 - \left(\left(\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} e_r\right)\left(\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}}\right)\right.\right.\\ &\left.\left.\left.- \left(\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}\right)\left(\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} e_r\right)\right)^2\right]^{\frac{1}{2\lambda_{2j}}}\right\}.\end{aligned} \tag{2.5}$$

*Proof.* Preforming some simple computations, we have

$$\begin{aligned}&\sum_{r=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\sum_{s=1}^{n}\left(\prod_{i=1}^{k} A_{si}\right)(1 - e_r + e_s)\\ &\quad= \sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\left(\prod_{i=1}^{k} A_{si}\right) - \sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\left(\prod_{i=1}^{k} A_{si}\right)e_r\\ &\qquad+ \sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\left(\prod_{i=1}^{k} A_{si}\right)e_s\\ &\quad= \left(\sum_{r=1}^{n}\prod_{j=1}^{k} A_{rj}\right)^2.\end{aligned} \tag{2.6}$$

Case 1. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$.

Subcase (1): When $k$ is even, and $\sum_{j=1}^{k}\frac{1}{\lambda_j} \geqslant 1$. From Lemma 2.2, we have

$$\begin{aligned}&\sum_{r=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\sum_{s=1}^{n}\left(\prod_{i=1}^{k} A_{si}\right)(1 - e_r + e_s)^{\sum_{j=1}^{k}\frac{1}{\lambda_j}}\\ &= \sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\left(\prod_{i=1}^{k} A_{si}\right)(1 - e_r + e_s)^{\sum_{j=1}^{k}\frac{1}{\lambda_j}}\\ &\geqslant \left[\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\left(\prod_{i=1}^{k} A_{si}\right)\right]^{1-\sum_{j=1}^{k}\frac{1}{\lambda_j}}\left[\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\left(\prod_{i=1}^{k} A_{si}\right)(1 - e_r + e_s)\right]^{\sum_{j=1}^{k}\frac{1}{\lambda_j}}\\ &= \left[\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\left(\prod_{i=1}^{k} A_{si}\right)\right]^{1-\sum_{j=1}^{k}\frac{1}{\lambda_j}}\left[\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k} A_{rj}\right)\left(\prod_{i=1}^{k} A_{si}\right)\right]^{\sum_{j=1}^{k}\frac{1}{\lambda_j}}\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&= \sum_{r=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{k}A_{rj}\right)\left(\prod_{i=1}^{k}A_{si}\right)\\
&= \left[\sum_{r=1}^{n}\left(\prod_{j=1}^{k}A_{rj}\right)\right]^{2}.
\end{aligned}
$$

Moreover, in view of $\sum_{j=1}^{k}\frac{1}{\lambda_j}\geqslant 1$, from inequality (2.2), we deduce

$$
\begin{aligned}
&\sum_{r=1}^{n}\left(\prod_{j=1}^{k}A_{rj}\right)\sum_{s=1}^{n}\left(\prod_{i=1}^{k}A_{si}\right)(1-e_r+e_s)^{\sum_{j=1}^{k}\frac{1}{\lambda_j}}\\
&\quad= \sum_{r=1}^{n}\left(\prod_{j=1}^{k}A_{rj}\right)\sum_{s=1}^{n}\prod_{i=1}^{k}A_{si}(1-e_r+e_s)^{\frac{1}{\lambda_i}}\\
&\quad\leqslant \sum_{r=1}^{n}\left(\prod_{j=1}^{k}A_{rj}\right)\left[\prod_{i=1}^{k}\left(\sum_{s=1}^{n}A_{si}^{\lambda_i}(1-e_r+e_s)\right)^{\frac{1}{\lambda_i}}\right] \qquad (2.8)\\
&\quad= \sum_{r=1}^{n}\left\{\prod_{j=1}^{\frac{k}{2}}\left[\left(A_{r(2j-1)}^{\lambda_{2j-1}}\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\left(A_{r(2j-1)}^{\lambda_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\lambda_{2j}}(1-e_r+e_s)\right)^{\frac{1}{\lambda_{2j}}}\right.\right.\\
&\qquad\left.\left.\times\left(A_{r(2j)}^{\lambda_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{(2j-1)}}(1-e_r+e_s)\right)^{\frac{1}{\lambda_{2j}}}\right]\right\}.
\end{aligned}
$$

Hence, according to $(\frac{1}{\lambda_1}-\frac{1}{\lambda_2})+\frac{1}{\lambda_2}+\frac{1}{\lambda_2}+(\frac{1}{\lambda_3}-\frac{1}{\lambda_4})+\frac{1}{\lambda_4}+\frac{1}{\lambda_4}+\cdots+(\frac{1}{\lambda_{k-1}}-\frac{1}{\lambda_k})+\frac{1}{\lambda_k}+\frac{1}{\lambda_k}\geqslant 1$, applying inequality (2.2) on the right side of (2.8), we have

$$
\begin{aligned}
&\sum_{r=1}^{n}\left(\prod_{j=1}^{k}A_{rj}\right)\sum_{s=1}^{n}\left(\prod_{i=1}^{k}A_{si}\right)(1-e_r+e_s)^{\sum_{j=1}^{k}\frac{1}{\lambda_j}}\\
&\quad\leqslant \prod_{j=1}^{\frac{k}{2}}\left[\left(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\right.\\
&\qquad\left.\times\left(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\lambda_{2j}}(1-e_r+e_s)\right)^{\frac{1}{\lambda_{2j}}}\left(\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\right)^{\frac{1}{\lambda_{2j}}}\right]\\
&\quad= \prod_{j=1}^{\frac{k}{2}}\left\{\left(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\right)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\left[\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}A_{s(2j)}^{\lambda_{2j}}(1-e_r+e_s)\right)\right.\right.\\
&\qquad\left.\left.\times\left(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j)}^{\lambda_{2j}}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\right)\right]^{\frac{1}{\lambda_{2j}}}\right\} \qquad (2.9)\\
&\quad= \prod_{j=1}^{\frac{k}{2}}\left\{\left(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\right)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\left[\left(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\lambda_{2j}}\right.\right.\right.\\
&\qquad\left.-\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}e_r\sum_{s=1}^{n}A_{s(2j)}^{\lambda_{2j}}+\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\lambda_{2j}}e_s\right)\\
&\qquad\left.\left.\times\left(\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}-\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}e_r\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}+\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}e_s\right)\right]^{\frac{1}{\lambda_{2j}}}\right\}
\end{aligned}
$$

$$= \prod_{j=1}^{\frac{k}{2}} \Bigg\{ \Big( \sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} \Big)^{\frac{2}{\lambda_{2j-1}} - \frac{2}{\lambda_{2j}}} \Bigg[ \Bigg( \Big( \sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} \Big) \Big( \sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} \Big) \Bigg)^2 - \Bigg( \Big( \sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} e_r \Big) \Big( \sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} \Big) - \Big( \sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} \Big) \Big( \sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} e_r \Big) \Bigg)^2 \Bigg]^{\frac{1}{\lambda_{2j}}} \Bigg\}.$$

Combining inequalities (2.7) and (2.9) yields inequality (2.4).

Subcase (2): When $k$ is even, and $\sum_{j=1}^{k} \frac{1}{\lambda_j} < 1$. Let $\sum_{j=1}^{k} \frac{1}{\lambda_j} = \beta$ $(0 < \beta < 1)$, which implies that $\sum_{j=1}^{k} \frac{1}{\beta\lambda_j} = 1$. From inequality (2.2) we find

$$\begin{aligned} &\sum_{r=1}^{n} \Big( \prod_{j=1}^{k} A_{rj} \Big) \sum_{s=1}^{n} \Big( \prod_{i=1}^{k} A_{si} \Big) (1 - e_r + e_s) \\ &= \sum_{r=1}^{n} \Big( \prod_{j=1}^{k} A_{rj} \Big) \sum_{s=1}^{n} \prod_{i=1}^{k} A_{si} (1 - e_r + e_s)^{\frac{1}{\beta\lambda_i}} \\ &\leqslant \sum_{r=1}^{n} \Big( \prod_{j=1}^{k} A_{rj} \Big) \Bigg[ \prod_{i=1}^{k} \Big( \sum_{s=1}^{n} A_{si}^{\beta\lambda_i} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_i}} \Bigg] \\ &= \sum_{r=1}^{n} \Bigg\{ \prod_{j=1}^{\frac{k}{2}} \Bigg[ \Big( A_{r(2j-1)}^{\beta\lambda_{2j-1}} \sum_{s=1}^{n} A_{s(2j-1)}^{\beta\lambda_{2j-1}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j-1}} - \frac{1}{\beta\lambda_{2j}}} \\ &\quad \times \Big( A_{r(2j-1)}^{\beta\lambda_{2j-1}} \sum_{s=1}^{n} A_{s(2j)}^{\beta\lambda_{2j}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j}}} \Big( A_{r(2j)}^{\beta\lambda_{2j}} \sum_{s=1}^{n} A_{s(2j-1)}^{\beta\lambda_{2j-1}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j}}} \Bigg] \Bigg\}. \end{aligned} \tag{2.10}$$

Consequently, in view of $\left(\frac{1}{\beta\lambda_1} - \frac{1}{\beta\lambda_2}\right) + \frac{1}{\beta\lambda_2} + \frac{1}{\beta\lambda_2} + \left(\frac{1}{\beta\lambda_3} - \frac{1}{\beta\lambda_4}\right) + \frac{1}{\beta\lambda_4} + \frac{1}{\beta\lambda_4} + \cdots + \left(\frac{1}{\beta\lambda_{k-1}} - \frac{1}{\beta\lambda_k}\right) + \frac{1}{\beta\lambda_k} + \frac{1}{\beta\lambda_k} = 1$, and applying inequality (2.2) on the right side of (2.10), we have

$$\begin{aligned} &\sum_{r=1}^{n} \Big( \prod_{j=1}^{k} A_{rj} \Big) \sum_{s=1}^{n} \Big( \prod_{i=1}^{k} A_{si} \Big) (1 - e_r + e_s) \\ &\leqslant \prod_{j=1}^{\frac{k}{2}} \Bigg[ \Big( \sum_{r=1}^{n} A_{r(2j-1)}^{\beta\lambda_{2j-1}} \sum_{s=1}^{n} A_{s(2j-1)}^{\beta\lambda_{2j-1}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j-1}} - \frac{1}{\beta\lambda_{2j}}} \\ &\quad \times \Big( \sum_{r=1}^{n} A_{r(2j-1)}^{\beta\lambda_{2j-1}} \sum_{s=1}^{n} A_{s(2j)}^{\beta\lambda_{2j}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j}}} \Big( \sum_{r=1}^{n} A_{r(2j)}^{\beta\lambda_{2j}} \sum_{s=1}^{n} A_{s(2j-1)}^{\beta\lambda_{2j-1}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j}}} \Bigg] \\ &= \prod_{j=1}^{\frac{k}{2}} \Bigg[ \Big( \sum_{r=1}^{n} \sum_{s=1}^{n} A_{r(2j-1)}^{\beta\lambda_{2j-1}} A_{s(2j-1)}^{\beta\lambda_{2j-1}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j-1}} - \frac{1}{\beta\lambda_{2j}}} \\ &\quad \times \Big( \sum_{r=1}^{n} \sum_{s=1}^{n} A_{r(2j-1)}^{\beta\lambda_{2j-1}} A_{s(2j)}^{\beta\lambda_{2j}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j}}} \Big( \sum_{r=1}^{n} \sum_{s=1}^{n} A_{r(2j)}^{\beta\lambda_{2j}} A_{s(2j-1)}^{\beta\lambda_{2j-1}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j}}} \Bigg]. \end{aligned} \tag{2.11}$$

Additionally, applying Lemma 2.2 on the right side of (2.11), we find

$$\prod_{j=1}^{\frac{k}{2}} \Bigg[ \Big( \sum_{r=1}^{n} \sum_{s=1}^{n} A_{r(2j-1)}^{\beta\lambda_{2j-1}} A_{s(2j-1)}^{\beta\lambda_{2j-1}} (1 - e_r + e_s) \Big)^{\frac{1}{\beta\lambda_{2j-1}} - \frac{1}{\beta\lambda_{2j}}}$$

$$
\begin{aligned}
&\times\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j-1)}^{\beta\lambda_{2j-1}}A_{s(2j)}^{\beta\lambda_{2j}}(1-e_r+e_s)\Big)^{\frac{1}{\beta\lambda_{2j}}}\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j)}^{\beta\lambda_{2j}}A_{s(2j-1)}^{\beta\lambda_{2j-1}}(1-e_r+e_s)\Big)^{\frac{1}{\beta\lambda_{2j}}}\Bigg]\\
&\leqslant\prod_{j=1}^{\frac{k}{2}}\Bigg[\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}(1-e_r+e_s)\Big)^{(1-\beta)(\frac{1}{\beta\lambda_{2j-1}}-\frac{1}{\beta\lambda_{2j}})}\\
&\times\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\\
&\times\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}(1-e_r+e_s)\Big)^{\frac{1-\beta}{\beta\lambda_{2j}}}\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}A_{s(2j)}^{\lambda_{2j}}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_{2j}}}\Big)\\
&\times\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}(1-e_r+e_s)\Big)^{\frac{1-\beta}{\beta\lambda_{2j}}}\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j)}^{\lambda_{2j}}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_{2j}}}\Bigg]\\
&=\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}(1-e_r+e_s)\Big)^{1-\beta}\prod_{j=1}^{\frac{k}{2}}\Bigg[\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\\
&\times\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}A_{s(2j)}^{\lambda_{2j}}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_{2j}}}\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j)}^{\lambda_{2j}}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_{2j}}}\Bigg] \qquad (2.12)\\
&=n^{2-2\beta}\prod_{j=1}^{\frac{k}{2}}\Bigg\{\Big(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\Big)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\\
&\times\Big[\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}A_{s(2j)}^{\lambda_{2j}}(1-e_r+e_s)\Big)\Big(\sum_{r=1}^{n}\sum_{s=1}^{n}A_{r(2j)}^{\lambda_{2j}}A_{s(2j-1)}^{\lambda_{2j-1}}(1-e_r+e_s)\Big)\Big]^{\frac{1}{\lambda_{2j}}}\Bigg\}\\
&=n^{2-2\beta}\prod_{j=1}^{\frac{k}{2}}\Bigg\{\Big(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\Big)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\Big[\Big(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\lambda_{2j}}\\
&-\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}e_r\sum_{s=1}^{n}A_{s(2j)}^{\lambda_{2j}}+\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\lambda_{2j}}e_s\Big)\\
&\times\Big(\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}-\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}e_r\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}+\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\lambda_{2j-1}}e_s\Big)\Big]^{\frac{1}{\lambda_{2j}}}\Bigg\}\\
&=n^{2-2\beta}\prod_{j=1}^{\frac{k}{2}}\Bigg\{\Big(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\Big)^{\frac{2}{\lambda_{2j-1}}-\frac{2}{\lambda_{2j}}}\Big[\Big(\Big(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\Big)\Big(\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}\Big)\Big)^{2}\\
&-\Big(\Big(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}e_r\Big)\Big(\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}\Big)-\Big(\sum_{r=1}^{n}A_{r(2j-1)}^{\lambda_{2j-1}}\Big)\Big(\sum_{r=1}^{n}A_{r(2j)}^{\lambda_{2j}}e_r\Big)\Big)^{2}\Big]^{\frac{1}{\lambda_{2j}}}\Bigg\}.
\end{aligned}
$$

Combining inequalities (2.6), (2.11), and (2.12) leads to inequality (2.4) immediately.

Subcase (3): When $k$ is odd, and $\sum_{j=1}^{k}\frac{1}{\lambda_j}\geqslant 1$. By the same method as in the above Subcase (1), we have the inequality (2.5).

Subcase (4): When $k$ is odd, and $\sum_{j=1}^{k}\frac{1}{\lambda_j}<1$. By the same method as in the above Subcase (2), we have the inequality (2.5).

Case 2. When $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k>0$, at least one of the equalities holds. By the same way as in Case 1, we can obtain the desired results. The proof of Theorem 2.4 is completed. □

From Theorem 2.4 and Lemma 2.1, we obtain the generalizations and refinements of the generalized Hölder's inequality (2.2) as follows.

**Corollary 2.5.** *Let $A_{rj}, \lambda_j, e_r$ be as in Theorem 2.4, let $\rho = \min\{\sum_{j=1}^{k} \frac{1}{\lambda_j}, 1\}$, and let $\sum_{r=1}^{n} A_{rj}^{\lambda_j} \neq 0$. If $k$ is even, then*

$$\sum_{r=1}^{n}\prod_{j=1}^{k} A_{rj} \leqslant n^{1-\rho}\left[\prod_{j=1}^{k}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]\left\{\prod_{j=1}^{\frac{k}{2}}\left[1-\frac{1}{2\lambda_{2j}}\left(\frac{\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} e_r}{\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}}-\frac{\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} e_r}{\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}}}\right)^2\right]\right\}. \quad (2.13)$$

*If $k$ is odd, then*

$$\sum_{r=1}^{n}\prod_{j=1}^{k} A_{rj} \leqslant n^{1-\rho}\left[\prod_{j=1}^{k}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]\left\{\prod_{j=1}^{\frac{k-1}{2}}\left[1-\frac{1}{2\lambda_{2j}}\left(\frac{\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} e_r}{\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}}-\frac{\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} e_r}{\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}}}\right)^2\right]\right\}. \quad (2.14)$$

*Proof.* We only need to prove the inequality (2.13). The proof of inequality (2.14) is similar. From inequality (2.4), we obtain

$$\sum_{r=1}^{n}\prod_{j=1}^{k} A_{rj} \leqslant n^{1-\rho}\left[\prod_{j=1}^{k}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]\left\{\prod_{j=1}^{\frac{k}{2}}\left[1-\left(\frac{\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} e_r}{\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}}-\frac{\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} e_r}{\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}}}\right)^2\right]^{\frac{1}{2\lambda_{2j}}}\right\}. \quad (2.15)$$

Furthermore, preforming some simple computations, we have

$$\left|\frac{\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}} e_r}{\sum_{r=1}^{n} A_{r(2j-1)}^{\lambda_{2j-1}}}-\frac{\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}} e_r}{\sum_{r=1}^{n} A_{r(2j)}^{\lambda_{2j}}}\right| < 1. \quad (2.16)$$

Consequently, from Lemma 2.1 and the inequalities (2.15) and (2.16), we have the desired inequality (2.13). The proof of Corollary 2.5 is complete. □

**Theorem 2.6.** *Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$, $\sum_{j=1}^{k} \frac{1}{\lambda_j} \leqslant 1$, let $F_j(x)$, $e(x)$ be nonnegative integrable functions defined on $[a, b]$, and let $1 - e(x) + e(y) \geqslant 0$. If $k$ is even, then*

$$\begin{aligned}\int_a^b \prod_{j=1}^{k} F_j(x)dx \leqslant (b-a)^{1-\sum_{j=1}^{k}\frac{1}{\lambda_j}} \prod_{j=1}^{\frac{k}{2}}\Bigg\{&\left(\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)dx\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\Bigg[\left(\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)dx\int_a^b F_{2j}^{\lambda_{2j}}(x)dx\right)^2\\ &-\left(\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)e(x)dx\int_a^b F_{2j}^{\lambda_{2j}}(x)dx-\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)dx\int_a^b F_{2j}^{\lambda_{2j}}(x)e(x)dx\right)^2\Bigg]^{\frac{1}{2\lambda_{2j}}}\Bigg\}.\end{aligned} \quad (2.17)$$

*If $k$ is odd, then*

$$\begin{aligned}\int_a^b \prod_{j=1}^{k} F_j(x)dx \leqslant\ & (b-a)^{1-\sum_{j=1}^{k}\frac{1}{\lambda_j}}\left(\int_a^b F_k^{\lambda_k}(x)dx\right)^{\frac{1}{\lambda_k}}\\ &\times\prod_{j=1}^{\frac{k-1}{2}}\Bigg\{\left(\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)dx\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\Bigg[\left(\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)dx\int_a^b F_{2j}^{\lambda_{2j}}(x)dx\right)^2\\ &-\left(\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)e(x)dx\int_a^b F_{2j}^{\lambda_{2j}}(x)dx-\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)dx\int_a^b F_{2j}^{\lambda_{2j}}(x)e(x)dx\right)^2\Bigg]^{\frac{1}{2\lambda_{2j}}}\Bigg\}.\end{aligned} \quad (2.18)$$

*Proof.* For any positive integer $n$, we choose an equidistant partition of $[a,b]$ as

$$a < a+\frac{b-a}{n} < \cdots < a+\frac{b-a}{n}r < \cdots < a+\frac{b-a}{n}(n-1) < b,$$

$$x_r = a+\frac{b-a}{n}r, \quad \Delta x_r = \frac{b-a}{n}, \quad r=1,2,\cdots,n.$$

Case (a). When $k$ is even, by the inequality (2.4) we have

$$\sum_{r=1}^{n}\prod_{j=1}^{k}F_j(x_r) \leqslant n^{1-\sum_{j=1}^{k}\frac{1}{\lambda_j}}\prod_{j=1}^{\frac{k}{2}}\left\{\left(\sum_{r=1}^{n}F_{2j-1}^{\lambda_{2j-1}}(x_r)\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\left[\left(\left(\sum_{r=1}^{n}F_{2j-1}^{\lambda_{2j-1}}(x_r)\right)\left(\sum_{r=1}^{n}F_{2j}^{\lambda_{2j}}(x_r)\right)\right)^2 - \left(\left(\sum_{r=1}^{n}F_{2j-1}^{\lambda_{2j-1}}(x_r)e_r\right)\left(\sum_{r=1}^{n}F_{2j}^{\lambda_{2j}}(x_r)\right)-\left(\sum_{r=1}^{n}F_{2j-1}^{\lambda_{2j-1}}(x_r)\right)\left(\sum_{r=1}^{n}F_{2j}^{\lambda_{2j}}(x_r)e_r\right)\right)^2\right]^{\frac{1}{2\lambda_{2j}}}\right\}.$$

Hence, we have

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{k}F_j(x_r)\frac{b-a}{n} \leqslant (b-a)^{1-\sum_{j=1}^{k}\frac{1}{\lambda_j}}\prod_{j=1}^{\frac{k}{2}}&\left\{\left(\sum_{r=1}^{n}F_{2j-1}^{\lambda_{2j-1}}(x_r)\frac{b-a}{n}\right)^{\frac{1}{\lambda_{2j-1}}-\frac{1}{\lambda_{2j}}}\right.\\ &\times\left[\left(\left(\sum_{r=1}^{n}F_{2j-1}^{\lambda_{2j-1}}(x_r)\frac{b-a}{n}\right)\left(\sum_{r=1}^{n}F_{2j}^{\lambda_{2j}}(x_r)\frac{b-a}{n}\right)\right)^2\right.\\ &-\left(\left(\sum_{r=1}^{n}F_{2j-1}^{\lambda_{2j-1}}(x_r)e(x_r)\frac{b-a}{n}\right)\left(\sum_{r=1}^{n}F_{2j}^{\lambda_{2j}}(x_r)\frac{b-a}{n}\right)\right.\\ &\left.\left.\left.-\left(\sum_{r=1}^{n}F_{2j-1}^{\lambda_{2j-1}}(x_r)\frac{b-a}{n}\right)\left(\sum_{r=1}^{n}F_{2j}^{\lambda_{2j}}(x_r)e(x_r)\frac{b-a}{n}\right)\right)^2\right]^{\frac{1}{2\lambda_{2j}}}\right\}.\end{aligned} \tag{2.19}$$

In view of the hypotheses that $F_j(x), e(x)$ are positive Riemann integrable functions on $[a,b]$, we conclude that $F_j^{\lambda_j}(x)$, $F_j^{\lambda_j}(x)e(x)$ are also integrable on $[a,b]$. Passing the limit as $n\to\infty$ in both sides of inequality (2.19), we obtain inequality (2.17).

Case (b). When $k$ is odd, by the same method as in the above Case (a), we have the inequality (2.18). The proof of Theorem 2.6 is completed. □

From Theorem 2.6 we obtain the following generalizations and refinements of the generalized Hölder's inequality.

**Corollary 2.7.** *Let $F_j(x), \lambda_j, e(x)$ be as in Theorem 2.6, and let $\int_a^b F_j^{\lambda_j}(x)dx \neq 0$. If $k$ is even, then*

$$\begin{aligned}\int_a^b\prod_{j=1}^{k}F_j(x)dx \leqslant (b-a)^{1-\sum_{j=1}^{k}\frac{1}{\lambda_j}}&\left[\prod_{j=1}^{k}\left(\int_a^b F_j^{\lambda_j}(x)dx\right)^{\frac{1}{\lambda_j}}\right]\\ &\times\left\{\prod_{j=1}^{\frac{k}{2}}\left[1-\frac{1}{2\lambda_{2j}}\left(\frac{\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)e(x)dx}{\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)dx}-\frac{\int_a^b F_{2j}^{\lambda_{2j}}(x)e(x)dx}{\int_a^b F_{2j}^{\lambda_{2j}}(x)dx}\right)^2\right]\right\}.\end{aligned}$$

*If $k$ is odd, then*

$$\int_a^b\prod_{j=1}^{k}F_j(x)dx \leqslant (b-a)^{1-\sum_{j=1}^{k}\frac{1}{\lambda_j}}\left[\prod_{j=1}^{k}\left(\int_a^b F_j^{\lambda_j}(x)dx\right)^{\frac{1}{\lambda_j}}\right]$$

$$\times\left\{\prod_{j=1}^{\frac{k-1}{2}}\left[1-\frac{1}{2\lambda_{2j}}\left(\frac{\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)e(x)dx}{\int_a^b F_{2j-1}^{\lambda_{2j-1}}(x)dx}-\frac{\int_a^b F_{2j}^{\lambda_{2j}}(x)e(x)dx}{\int_a^b F_{2j}^{\lambda_{2j}}(x)dx}\right)^2\right]\right\}.$$

Now, we present the following generalizations of inequalities (1.4), (1.5), (1.8), (1.9) and (1.12).

**Theorem 2.8.** *Let* $A_{rj}>0$, $(r=1,2,\cdots,n,\ j=1,2,\cdots,m)$, $\lambda_1>0$, $\lambda_i<0$ $(i=2,3,\cdots,m)$, *and* $\sum_{j=1}^m\frac{1}{\lambda_j}>0$, *let* $\tau=\max\{\sum_{j=1}^m\frac{1}{\lambda_j},1\}$, *and let* $1-e_r+e_s\geqslant 0$ $(s=1,2,\cdots,n)$. *Then*

$$\begin{aligned}\sum_{r=1}^n\prod_{j=1}^m A_{rj}\geqslant n^{1-\tau}\left(\sum_{r=1}^n A_{r1}^{\lambda_1}\right)^{\frac{1}{\lambda_1}-\sum_{j=2}^m\frac{1}{\lambda_j}}\prod_{j=2}^m\Bigg\{&\left[\left(\sum_{r=1}^n A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^n A_{rj}^{\lambda_j}\right)\right]^2\\&-\left[\left(\sum_{r=1}^n A_{r1}^{\lambda_1}e_r\right)\left(\sum_{r=1}^n A_{rj}^{\lambda_j}\right)-\left(\sum_{r=1}^n A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^n A_{rj}^{\lambda_j}e_r\right)\right]^2\Bigg\}^{\frac{1}{2\lambda_j}}.\end{aligned}\tag{2.20}$$

*Proof.* We first consider the case (I) $\sum_{j=1}^m\frac{1}{\lambda_j}\geqslant 1$. Let $\sum_{j=1}^m\frac{1}{\lambda_j}=t$ $(t\geqslant 1)$, which implies $\sum_{j=1}^m\frac{1}{t\lambda_j}=1$. Preforming some simple computations, we have

$$\begin{aligned}&\sum_{r=1}^n\left(\prod_{j=1}^k A_{rj}\right)\sum_{s=1}^n\left(\prod_{i=1}^k A_{si}\right)(1-e_r+e_s)\\&=\sum_{r=1}^n\sum_{s=1}^n\left(\prod_{j=1}^k A_{rj}\right)\left(\prod_{i=1}^k A_{si}\right)-\sum_{r=1}^n\sum_{s=1}^n\left(\prod_{j=1}^k A_{rj}\right)\left(\prod_{i=1}^k A_{si}\right)e_r\\&\quad+\sum_{r=1}^n\sum_{s=1}^n\left(\prod_{j=1}^k A_{rj}\right)\left(\prod_{i=1}^k A_{si}\right)e_s\\&=\left(\sum_{r=1}^n\prod_{j=1}^k A_{rj}\right)^2.\end{aligned}\tag{2.21}$$

From inequality (2.3), we have

$$\begin{aligned}&\sum_{s=1}^n\left(\prod_{i=1}^m A_{si}\right)\sum_{r=1}^n\left(\prod_{j=1}^m A_{rj}\right)(1-e_r+e_s)\\&=\sum_{s=1}^n\left(\prod_{i=1}^m A_{si}\right)\sum_{r=1}^n\prod_{j=1}^m A_{rj}(1-e_r+e_s)^{\frac{1}{t\lambda_j}}\\&\geqslant\sum_{s=1}^n\left(\prod_{i=1}^m A_{si}\right)\left[\prod_{j=1}^m\left(\sum_{r=1}^n A_{rj}^{t\lambda_j}(1-e_r+e_s)\right)^{\frac{1}{t\lambda_j}}\right]\\&=\sum_{s=1}^n\Bigg\{\left(A_{s1}^{t\lambda_1}\sum_{r=1}^n A_{r1}^{t\lambda_1}(1-e_r+e_s)\right)^{\frac{1}{t\lambda_1}-\sum_{j=2}^m\frac{1}{t\lambda_j}}\left[\prod_{j=2}^m\left(A_{s1}^{t\lambda_1}\sum_{r=1}^n A_{rj}^{t\lambda_j}(1-e_r+e_s)\right)^{\frac{1}{t\lambda_j}}\right]\\&\quad\times\left[\prod_{j=2}^m\left(A_{sj}^{t\lambda_j}\sum_{r=1}^n A_{r1}^{t\lambda_1}(1-e_r+e_s)\right)^{\frac{1}{t\lambda_j}}\right]\Bigg\}.\end{aligned}\tag{2.22}$$

Consequently, in view of $\left(\frac{1}{t\lambda_1}-\sum_{j=2}^m\frac{1}{t\lambda_j}\right)+\frac{1}{t\lambda_2}+\frac{1}{t\lambda_3}+\cdots+\frac{1}{t\lambda_m}+\frac{1}{t\lambda_2}+\frac{1}{t\lambda_3}+\cdots+\frac{1}{t\lambda_m}=1$, applying inequality (2.3) on the right side of (2.22), we find

$$
\begin{aligned}
&\sum_{s=1}^{n}\Big(\prod_{i=1}^{m}A_{si}\Big)\sum_{r=1}^{n}\Big(\prod_{j=1}^{m}A_{rj}\Big)(1-e_r+e_s)\\
&\quad\geqslant\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{t\lambda_1}A_{r1}^{t\lambda_1}(1-e_r+e_s)\Big)^{\frac{1}{t\lambda_1}-\sum_{j=2}^{m}\frac{1}{t\lambda_j}}\Big[\prod_{j=2}^{m}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{t\lambda_1}A_{rj}^{t\lambda_j}(1-e_r+e_s)\Big)^{\frac{1}{t\lambda_j}}\Big]\\
&\qquad\times\Big[\prod_{j=2}^{m}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{sj}^{t\lambda_j}A_{r1}^{t\lambda_1}(1-e_r+e_s)\Big)^{\frac{1}{t\lambda_j}}\Big].
\end{aligned}\tag{2.23}
$$

Moreover, using Lemma 2.2 together with $t\geqslant 1$, we find

$$
\begin{aligned}
&\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{t\lambda_1}A_{r1}^{t\lambda_1}(1-e_r+e_s)\Big)^{\frac{1}{t\lambda_1}-\sum_{j=2}^{m}\frac{1}{t\lambda_j}}\Big[\prod_{j=2}^{m}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{t\lambda_1}A_{rj}^{t\lambda_j}(1-e_r+e_s)\Big)^{\frac{1}{t\lambda_j}}\Big]\\
&\quad\times\Big[\prod_{j=2}^{m}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{sj}^{t\lambda_j}A_{r1}^{t\lambda_1}(1-e_r+e_s)\Big)^{\frac{1}{t\lambda_j}}\Big]\\
&\quad\geqslant\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}(1-e_r+e_s)\Big)^{(1-t)(\frac{1}{t\lambda_1}-\sum_{j=2}^{m}\frac{1}{t\lambda_j})}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{\lambda_1}A_{r1}^{\lambda_1}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}}\\
&\quad\times\Big[\prod_{j=2}^{m}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}(1-e_r+e_s)\Big)^{\frac{1-t}{t\lambda_j}}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{\lambda_1}A_{rj}^{\lambda_j}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_j}}\Big]\\
&\quad\times\Big[\prod_{j=2}^{m}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}(1-e_r+e_s)\Big)^{\frac{1-t}{t\lambda_j}}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{sj}^{\lambda_j}A_{r1}^{\lambda_1}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_j}}\Big]\\
&\quad=\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}(1-e_r+e_s)\Big)^{1-t}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{\lambda_1}A_{r1}^{\lambda_1}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}}\\
&\quad\times\Big[\prod_{j=2}^{m}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{\lambda_1}A_{rj}^{\lambda_j}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_j}}\Big]\Big[\prod_{j=2}^{m}\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{sj}^{\lambda_j}A_{r1}^{\lambda_1}(1-e_r+e_s)\Big)^{\frac{1}{\lambda_j}}\Big]\\
&\quad=n^{2-2t}\Big(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\Big)^{\frac{2}{\lambda_1}-\sum_{j=2}^{m}\frac{2}{\lambda_j}}\Big\{\prod_{j=2}^{m}\Big[\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{\lambda_1}A_{rj}^{\lambda_j}(1-e_r+e_s)\Big)\\
&\quad\times\Big(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{sj}^{\lambda_j}A_{r1}^{\lambda_1}(1-e_r+e_s)\Big)\Big]^{\frac{1}{\lambda_j}}\Big\}\\
&\quad=n^{2-2t}\Big(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\Big)^{\frac{2}{\lambda_1}-\sum_{j=2}^{m}\frac{2}{\lambda_j}}\\
&\quad\times\Big\{\prod_{j=2}^{m}\Big[\Big(\sum_{s=1}^{n}A_{s1}^{\lambda_1}\sum_{r=1}^{n}A_{rj}^{\lambda_j}-\sum_{s=1}^{n}A_{s1}^{\lambda_1}\sum_{r=1}^{n}A_{rj}^{\lambda_j}e_r+\sum_{s=1}^{n}A_{s1}^{\lambda_1}e_s\sum_{r=1}^{n}A_{rj}^{\lambda_j}\Big)\\
&\quad\times\Big(\sum_{s=1}^{n}A_{sj}^{\lambda_j}\sum_{r=1}^{n}A_{r1}^{\lambda_1}-\sum_{s=1}^{n}A_{sj}^{\lambda_j}\sum_{r=1}^{n}A_{r1}^{\lambda_1}e_r+\sum_{s=1}^{n}A_{sj}^{\lambda_j}e_s\sum_{r=1}^{n}A_{r1}^{\lambda_1}\Big)\Big]^{\frac{1}{\lambda_j}}\Big\}\\
&\quad=n^{2-2t}\Big(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\Big)^{\frac{2}{\lambda_1}-\sum_{j=2}^{m}\frac{2}{\lambda_j}}\Big\{\prod_{j=2}^{m}\Big[\Big(\Big(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\Big)\Big(\sum_{r=1}^{n}A_{rj}^{\lambda_j}\Big)\Big)^2\\
&\quad-\Big(\Big(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\Big)\Big(\sum_{r=1}^{n}A_{rj}^{\lambda_j}e_r\Big)-\Big(\sum_{r=1}^{n}A_{r1}^{\lambda_1}e_r\Big)\Big(\sum_{r=1}^{n}A_{rj}^{\lambda_j}\Big)\Big)^2\Big]^{\frac{1}{\lambda_j}}\Big\}.
\end{aligned}\tag{2.24}
$$

Combining inequalities (2.21), (2.23) and (2.24) leads to inequality (2.20) immediately.

Nextly, we consider the case (II) $0 < \sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$. On the one hand, by the inequality (2.1), we have

$$\begin{aligned}
&\sum_{s=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\sum_{r=1}^{n}\left(\prod_{j=1}^{m} A_{rj}\right)(1-e_r+e_s)^{\sum_{j=1}^{m}\frac{1}{\lambda_j}} \\
&= \sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)(1-e_r+e_s)^{\sum_{j=1}^{m}\frac{1}{\lambda_j}} \\
&\leqslant \left[\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)\right]^{1-\sum_{j=1}^{m}\frac{1}{\lambda_j}} \\
&\quad\times\left[\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)(1-e_r+e_s)\right]^{\sum_{j=1}^{m}\frac{1}{\lambda_j}} \\
&= \left[\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)\right]^{1-\sum_{j=1}^{m}\frac{1}{\lambda_j}} \\
&\quad\times\left[\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)-\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)e_r\right. \\
&\quad\left.+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)e_s\right]^{\sum_{j=1}^{m}\frac{1}{\lambda_j}} \\
&= \left[\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)\right]^{1-\sum_{j=1}^{m}\frac{1}{\lambda_j}}\left[\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right)\right]^{\sum_{j=1}^{m}\frac{1}{\lambda_j}} \\
&= \sum_{s=1}^{n}\sum_{r=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left(\prod_{j=1}^{m} A_{rj}\right) \\
&= \left[\sum_{r=1}^{n}\left(\prod_{j=1}^{m} A_{rj}\right)\right]^{2}.
\end{aligned} \tag{2.25}$$

On the other hand, by the inequality (2.3), we obtain

$$\begin{aligned}
&\sum_{s=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\sum_{r=1}^{n}\left(\prod_{j=1}^{m} A_{rj}\right)(1-e_r+e_s)^{\sum_{j=1}^{m}\frac{1}{\lambda_j}} \\
&= \sum_{s=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj}(1-e_r+e_s)^{\frac{1}{\lambda_j}} \\
&\geqslant \sum_{s=1}^{n}\left(\prod_{i=1}^{m} A_{si}\right)\left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}(1-e_r+e_s)\right)^{\frac{1}{\lambda_j}}\right] \\
&= \sum_{s=1}^{n}\left\{\left(A_{s1}^{\lambda_1}\sum_{r=1}^{n} A_{r1}^{\lambda_1}(1-e_r+e_s)\right)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}}\left[\prod_{j=2}^{m}\left(A_{s1}^{\lambda_1}\sum_{r=1}^{n} A_{rj}^{\lambda_j}(1-e_r+e_s)\right)^{\frac{1}{\lambda_j}}\right]\right. \\
&\quad\left.\times\left[\prod_{j=2}^{m}\left(A_{sj}^{\lambda_j}\sum_{r=1}^{n} A_{r1}^{\lambda_1}(1-e_r+e_s)\right)^{\frac{1}{\lambda_j}}\right]\right\}.
\end{aligned} \tag{2.26}$$

Consequently, according to $\left(\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}\right)+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\cdots+\frac{1}{\lambda_m}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\cdots+\frac{1}{\lambda_m}\leqslant 1$, by using the inequality (2.3) on the right side of (2.26), we observe that

$$\begin{aligned}
&\sum_{s=1}^{n}\left(\prod_{i=1}^{m}A_{si}\right)\sum_{r=1}^{n}\left(\prod_{j=1}^{m}A_{rj}\right)(1-e_r+e_s)^{\sum_{j=1}^{m}\frac{1}{\lambda_j}}\\
&\geqslant\left(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{\lambda_1}A_{r1}^{\lambda_1}(1-e_r+e_s)\right)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}}\left[\prod_{j=2}^{m}\left(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{\lambda_1}A_{rj}^{\lambda_j}(1-e_r+e_s)\right)^{\frac{1}{\lambda_j}}\right]\\
&\quad\times\left[\prod_{j=2}^{m}\left(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{sj}^{\lambda_j}A_{r1}^{\lambda_1}(1-e_r+e_s)\right)^{\frac{1}{\lambda_j}}\right]\\
&=\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)^{\frac{2}{\lambda_1}-\sum_{j=2}^{m}\frac{2}{\lambda_j}}\left\{\prod_{j=2}^{m}\left[\left(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{s1}^{\lambda_1}A_{rj}^{\lambda_j}(1-e_r+e_s)\right)\right.\right.\\
&\quad\left.\left.\times\left(\sum_{s=1}^{n}\sum_{r=1}^{n}A_{sj}^{\lambda_j}A_{r1}^{\lambda_1}(1-e_r+e_s)\right)\right]^{\frac{1}{\lambda_j}}\right\} \qquad (2.27)\\
&=\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)^{\frac{2}{\lambda_1}-\sum_{j=2}^{m}\frac{2}{\lambda_j}}\left\{\prod_{j=2}^{m}\left[\left(\sum_{s=1}^{n}A_{s1}^{\lambda_1}\sum_{r=1}^{n}A_{rj}^{\lambda_j}-\sum_{s=1}^{n}A_{s1}^{\lambda_1}\sum_{r=1}^{n}A_{rj}^{\lambda_j}e_r+\sum_{s=1}^{n}A_{s1}^{\lambda_1}e_s\sum_{r=1}^{n}A_{rj}^{\lambda_j}\right)\right.\right.\\
&\quad\left.\left.\times\left(\sum_{s=1}^{n}A_{sj}^{\lambda_j}\sum_{r=1}^{n}A_{r1}^{\lambda_1}-\sum_{s=1}^{n}A_{sj}^{\lambda_j}\sum_{r=1}^{n}A_{r1}^{\lambda_1}e_r+\sum_{s=1}^{n}A_{sj}^{\lambda_j}e_s\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)\right]^{\frac{1}{\lambda_j}}\right\}\\
&=\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)^{\frac{2}{\lambda_1}-\sum_{j=2}^{m}\frac{2}{\lambda_j}}\left\{\prod_{j=2}^{m}\left[\left(\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^{n}A_{rj}^{\lambda_j}\right)\right)^{2}\right.\right.\\
&\quad\left.\left.-\left(\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}\right)\left(\sum_{r=1}^{n}A_{rj}^{\lambda_j}e_r\right)-\left(\sum_{r=1}^{n}A_{r1}^{\lambda_1}e_r\right)\left(\sum_{r=1}^{n}A_{rj}^{\lambda_j}\right)\right)^{2}\right]^{\frac{1}{\lambda_j}}\right\}.
\end{aligned}$$

Combining inequalities (2.25) and (2.27) leads to inequality (2.20) immediately. The proof of Theorem 2.8 is completed.

□

From Theorem 2.8 and Lemma 2.1, we obtain the following generalizations and refinements of generalized Hölder's inequality (2.3).

**Corollary 2.9.** *Let* $A_{rj}$, $\lambda_j$, $e_r$ *be as in Theorem 2.8, let* $\tau=\max\{\sum_{j=1}^{m}\frac{1}{\lambda_j},1\}$, *and let* $\sum_{r=1}^{n}A_{rj}^{\lambda_j}\neq 0$. *Then*

$$\sum_{r=1}^{n}\prod_{j=1}^{m}A_{rj}\geqslant n^{1-\tau}\left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n}A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]\left\{\prod_{j=2}^{m}\left[1-\frac{1}{2\lambda_j}\left(\frac{\sum_{r=1}^{n}A_{r1}^{\lambda_1}e_r}{\sum_{r=1}^{n}A_{r1}^{\lambda_1}}-\frac{\sum_{r=1}^{n}A_{rj}^{\lambda_j}e_r}{\sum_{r=1}^{n}A_{rj}^{\lambda_j}}\right)^{2}\right]\right\}.$$

**Theorem 2.10.** *Let* $F_j(x)$, $e(x)$ *be integrable functions defined on* $[a,b]$ *and* $F_j(x)>0$, $1-e(x)+e(y)\geqslant 0$ *for all* $x,y\in[a,b]$, *and let* $\sum_{j=1}^{m}\frac{1}{\lambda_j}\geqslant 1$. *If* $\lambda_1>0$, $\lambda_j<0$ $(j=2,3,\cdots,m)$, *then*

$$\begin{aligned}
\int_a^b\prod_{j=1}^{m}F_j(x)dx\geqslant(b-a)^{1-\sum_{j=1}^{m}\frac{1}{\lambda_j}}\left(\int_a^bF_1^{\lambda_1}(x)dx\right)^{\frac{1}{\lambda_1}-\sum_{j=2}^{m}\frac{1}{\lambda_j}}\prod_{j=2}^{m}\left[\left(\int_a^bF_1^{\lambda_1}(x)dx\int_a^bF_j^{\lambda_j}(x)dx\right)^{2}\right.\\
\left.-\left(\int_a^bF_1^{\lambda_1}(x)e(x)dx\int_a^bF_j^{\lambda_j}(x)dx-\int_a^bF_1^{\lambda_1}(x)dx\int_a^bF_j^{\lambda_j}(x)e(x)dx\right)^{2}\right]^{\frac{1}{2\lambda_j}}. \qquad (2.28)
\end{aligned}$$

*Proof.* Making similar method as in the proof of Theorem 2.6 by using inequality (2.20), we have the desired inequality (2.28). □

**Corollary 2.11.** *Let $F_j(x)$, $\lambda_j$, $e(x)$ be as in Theorem 2.10, and let $\int_a^b F_j^{\lambda_j}(x)dx \neq 0$. Then, we have the following generalization and refinement of generalized Hölder's inequality (1.9).*

$$\int_a^b \prod_{j=1}^m F_j(x)dx \geqslant (b-a)^{1-\sum_{j=1}^m \frac{1}{\lambda_j}} \left[\prod_{j=1}^m \left(\int_a^b F_j^{\lambda_j}(x)dx\right)^{\frac{1}{\lambda_j}}\right] \times \left\{\prod_{j=2}^m \left[1 - \frac{1}{2\lambda_j}\left(\frac{\int_a^b F_1^{\lambda_1}(x)e(x)dx}{\int_a^b F_1^{\lambda_1}(x)dx} - \frac{\int_a^b F_j^{\lambda_j}(x)e(x)dx}{\int_a^b F_j^{\lambda_j}(x)dx}\right)^2\right]\right\}. \tag{2.29}$$

*Proof.* Making similar arguments as in the proof of Corollary 2.5, we have the desired inequality (2.29). □


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