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Composite relaxed extragradient method for triple hierarchical variational inequalities with constraints of systems of variational inequalities

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Abstract

In this paper, we introduce and analyze a composite relaxed extragradient viscosity algorithm for solving the triple hierarchical variational inequality problem with the constraint of general system of variational inequalities in a real Hilbert space. Strong convergence of the iteration sequences generated by the algorithm is established under some suitable conditions. Our results improve and extend the corresponding results in the earlier and recent literature.

Keywords: Composite relaxed extragradient algorithm, triple hierarchical variational inequality, general system of variational inequalities, inverse-strongly monotone mapping.
2010 MSC: 49J30, 47H09, 47J20, 49M05.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by **R** the set of all real numbers. A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geqslant 0$ such that

$$\|Sx - Sy\| \leqslant L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in [0, 1)$ then S is called a contraction.

Let $\mathcal{A} : C \to H$ be a nonlinear mapping on C. The classical variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle \mathcal{A}x, y - x\rangle \geqslant 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, \mathcal{A})$.

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We now recall that the metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

If A is a strongly monotone and Lipschitz-continuous mapping on C, then VIP (1.1) has a unique solution. In order to solve (1.1), Korpelevich [16] proposed the following extragradient algorithm in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_k = P_C(x_k - \tau A x_k), \\ x_{k+1} = P_C(x_k - \tau A y_k), \quad \forall k \geqslant 0. \end{cases}$$

The VIP and Korpelevich's extragradient method have received so much attention, see e.g., [2, 6–8, 29, 30] and references therein.

Let $A : C \to H$ and $B : H \to H$ be two mappings. Consider the following bilevel variational inequality problem (BVIP).

Problem 1.1. Find $x^* \in VI(C, B)$ such that

$$\langle Ax^*, x - x^* \rangle \geqslant 0, \quad \forall x \in VI(C, B),$$

where $VI(C, B)$ is the set of solutions of the VIP of finding $y^* \in C$ such that

$$\langle By^*, y - y^* \rangle \geqslant 0, \quad \forall y \in C.$$

Note that Anh et al. [1] studied the above BVIP with $H = \mathbf{R}^n$. BVIP includes the classes of mathematical programs with equilibrium constraints ([18]), bilevel minimization problems ([23]), variational inequalities ([3, 31, 32]) and complementarity problems as special cases. It is worth pointing out that the BVIP is quite different from other types of variational inequality problems considered in the very recent literature, see e.g., [9, 10, 21, 22].

In what follows, suppose that A and B satisfy the following conditions:

(C1) B is pseudomonotone on H and A is β-strongly monotone on C;
(C2) A is L_1-Lipschitz continuous on C;
(C3) B is L_2-Lipschitz continuous on H;
(C4) $VI(C, B) \neq \emptyset$.

In 2012, Anh et al. [1] introduced the following extragradient iterative algorithm for solving the above bilevel variational inequality.

Algorithm 1.2 ([1]). Initialization. Choose $u \in \mathbf{R}^n$, $x_0 \in C, 0 < \lambda \leqslant \frac{2\beta}{L_1^2}$, positive sequences $\{\delta_k\}$, $\{\lambda_k\}$, $\{\alpha_k\}$,$\{\beta_k\}$, $\{\gamma_k\}$, and $\{\bar{\epsilon}_k\}$ such that $\lim_{k\to\infty} \delta_k = 0, \sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty$, $\alpha_k + \beta_k + \gamma_k = 1$ $\forall k \geqslant 0, \sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0, \lim_{k\to\infty} \beta_k = \xi \in (0, \frac{1}{2}]$, $\lim_{k\to\infty} \lambda_k = 0$ and $\lambda_k \leqslant \frac{1}{L_2}$ for all $k \geqslant 0$.

Step 1. Compute $y_k := P_C(x_k - \lambda_k B x_k)$ and $z_k := P_C(x_k - \lambda_k B y_k)$.

Step 2. Inner loop $j = 0, 1, \ldots$. Compute

$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_C(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} + \gamma_j P_C(x_{k,j} - \delta_j B y_{k,j}). \\ \text{If } \|x_{k,j+1} - P_{VI(C,B)} x_{k,0}\| \leqslant \bar{\epsilon}_k \text{ then set } h_k := x_{k,j+1} \text{ and go to Step 3.} \\ \text{Otherwise, increase } j \text{ by 1 and repeat the inner loop Step 2.} \end{cases}$$

Step 3. Set $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. Then increase k by 1 and go to Step 1.

Furthermore, in [13, 14], Iiduka introduced the following three-stage variational inequality problem, that is, the following monotone variational inequality with variational inequality constraint over the fixed point set of a nonexpansive mapping.

Problem 1.3 ([14, Problem 3.1])**.** Assume that

(i) $T : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$;
(ii) $A_1 : H \to H$ is α-inverse strongly monotone;
(iii) $A_2 : H \to H$ is β-strongly monotone and L-Lipschitz continuous;
(iv) $\mathrm{VI}(\mathrm{Fix}(T), A_1) \neq \emptyset$.

Then the objective is to

$$\text{find } x^* \in \mathrm{VI}(\mathrm{VI}(\mathrm{Fix}(T), A_1), A_2) := \{x^* \in \mathrm{VI}(\mathrm{Fix}(T), A_1) : \langle A_2 x^*, v - x^* \rangle \geqslant 0, \forall v \in \mathrm{VI}(\mathrm{Fix}(T), A_1)\}.$$

Since this problem has a triple structure in contrast with bilevel programming problems ([18, 20]) or hierarchical constrained optimization problems or hierarchical fixed point problem, it is referred to as a triple hierarchical variational inequality problem (THVIP). Very recently, some authors continued the study of Iiduka's THVIP (i.e., Problem 1.3 and its variant and extension; see e.g., [6, 33]).

For solving Problem 1.3, Iiduka presented the following algorithm.

Algorithm 1.4 ([14])**.** Let $T : H \to H$ and $A_i : H \to H$ $(i = 1, 2)$ satisfy the assumptions (i)-(iv) in Problem 1.3.

Step 0. Take $\{\alpha_k\}_{k=0}^{\infty}, \{\lambda_k\}_{k=0}^{\infty} \subset (0, \infty)$, and $\mu > 0$, choose $x_0 \in H$ arbitrarily, and let $k := 0$.

Step 1. Given $x_k \in H$, compute $x_{k+1} \in H$ as

$$\begin{cases} y_k := T(x_k - \lambda_k A_1 x_k), \\ x_{k+1} := y_k - \mu \alpha_k A_2 y_k. \end{cases}$$

Update $k := k + 1$ and go to Step 1.

On the other hand, let $F_1, F_2 : C \to H$ be two mappings. Consider the following general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^* \rangle \geqslant 0, & \forall x \in C, \\ \langle \nu_2 F_2 x^* + y^* - x^*, y - y^* \rangle \geqslant 0, & \forall y \in C, \end{cases} \tag{1.2}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. The solution set of GSVI (1.2) is denoted by $\mathrm{GSVI}(C, F_1, F_2)$. Recently, many authors have been devoting the study of the GSVI (1.2); see e.g., [3, 7, 27] and the references therein.

In particular, if $F_1 = F_2 = \mathcal{A}$, then the GSVI (1.2) reduces to the new system of variational inequalities (NSVI), which was defined by Verma [25]. Further, if $x^* = y^*$ additionally, then the NSVI reduces to the classical VIP (1.1). In 2008, Ceng et al. [7] transformed the GSVI (1.2) into the fixed point problem of the mapping $G = P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2)$, that is, $Gx^* = x^*$, where $y^* = P_C(I - \nu_2 F_2)x^*$. Throughout this paper, the fixed point set of the mapping G is denoted by $\mathrm{GSVI}(G)$.

In 2010, Yao et al. [27] introduced a relaxed extragradient algorithm for finding a common element of the solution set of the GSVI (1.2) and the fixed point set of a strictly pseudocontractive mapping $T : C \to C$, and derived the strong convergence of the proposed algorithm to a common element under some mild conditions.

In this paper, we introduce and analyze a composite relaxed extragradient viscosity algorithm for solving the triple hierarchical variational inequality problem (THVIP) with the constraint of general system of variational inequalities in a real Hilbert space. The proposed algorithm is based on Korpelevich's extragradient method [16], Mann's iteration method [2] and composite viscosity approximation method [5]. Under some suitable conditions, the strong convergence of the iteration sequences generated by the algorithm is established. Our results improve and extend the corresponding results announced by some others, e.g., Iiduka [14], Zeng et al. [33], Anh et al. [1], and Yao et al. [27].

2. Preliminaries

Throughout, denoted the weak ω-limit set of the sequence $\{x_k\}$ by $\omega_w(x_k)$, i.e.,

$$\omega_w(x_k) := \{x \in H : x_{k_i} \rightharpoonup x \text{ for some subsequence } \{x_{k_i}\} \text{ of } \{x_k\}\}.$$

Definition 2.1. Recall that a mapping $A : C \to H$ is called

(i) monotone if $\langle Ax - Ay, x - y\rangle \geqslant 0, \quad \forall x, y \in C$;
(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that $\langle Ax - Ay, x - y\rangle \geqslant \eta\|x - y\|^2, \quad \forall x, y \in C$;
(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geqslant \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.2 ([26]). *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \leqslant 0,\ \forall y \in C$;
(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \leqslant \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;
(iii) $\langle P_C x - P_C y, x - y\rangle \geqslant \|P_C x - P_C y\|^2,\ \forall y \in H$.

Consequently, P_C is nonexpansive and monotone.

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda A)u - (I - \lambda A)v\|^2 \leqslant \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \tag{2.1}$$

So, if $\lambda \leqslant 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Definition 2.3. A mapping $T : H \to H$ is said to be:

(a) nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in H$;
(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism), $\langle x - y, Tx - Ty\rangle \geqslant \|Tx - Ty\|^2, \quad \forall x, y \in H$; alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = \frac{1}{2}(I + S)$, where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Proposition 2.4 ([12]). *Let $T : C \to C$ be a nonexpansive mapping. Then the followings hold:*

(i) $\mathrm{Fix}(T)$ *is closed and convex;*
(ii) $\mathrm{Fix}(T) \neq \emptyset$ *when C is bounded.*

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.5. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$$

Recall that, a mapping $A : C \to H$ is called hemicontinuous if for all $x, y \in C$, the mapping $g : [0,1] \to H$, defined by $g(t) := A(tx + (1-t)y)$, is continuous. Some properties of the solution set of the monotone variational inequality are mentioned in the following result.

Lemma 2.6 ([15, 24]). *Let $A : C \to H$ be a monotone and hemicontinuous mapping. Then the following hold:*

(i) $\mathrm{VI}(C, A)$ *is equivalent to* $\mathrm{MVI}(C, A) := \{x^* \in C : \langle Ay, y - x^*\rangle \geqslant 0, \forall y \in C\}$;
(ii) $\mathrm{VI}(C, A) \neq \emptyset$ *when C is bounded;*
(iii) $\mathrm{VI}(C, A) = \mathrm{Fix}(P_C(I - \lambda A))$ *for all $\lambda > 0$, where I is the identity mapping on H;*
(iv) $\mathrm{VI}(C, A)$ *consists of only one point, if A is strongly monotone and Lipschitz continuous.*

Lemma 2.7 ([11]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C with $\mathrm{Fix}(S) \neq \emptyset$. Then $I - S$ is demiclosed. That is, whenever $\{x_k\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-S)x_k\}$ strongly converges to some y, it follows that $(I-S)x = y$. Here I is the identity operator of H.*

Recall that, a mapping $T : C \to C$ is called a ζ-strictly pseudocontractive mapping (or a ζ-strict pseudocontraction) if there exists a constant $\zeta \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \zeta\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C.$$

Lemma 2.8 ([19]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a mapping.*

(i) *If T is a ζ-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \leqslant \frac{1+\zeta}{1-\zeta}\|x - y\|, \quad \forall x, y \in C.$$

(ii) *If T is a ζ-strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-T)x_n \to 0$, then $(I-T)\tilde{x} = 0$.*
(iii) *If T is ζ-(quasi-)strict pseudocontraction, then the fixed-point set $\mathrm{Fix}(T)$ of T is closed and convex so that the projection $P_{\mathrm{Fix}(T)}$ is well-defined.*

Lemma 2.9 ([27]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a ζ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)\zeta \leqslant \gamma$. Then*

$$\|\gamma(x - y) + \delta(Tx - Ty)\| \leqslant (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.10. *Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, 1)$. Then $I - f$ is $(1-\rho)$-strongly monotone, that is,*

$$\langle (I-f)x - (I-f)y, x - y\rangle \geqslant (1-\rho)\|x - y\|^2, \quad \forall x, y \in C.$$

Lemma 2.11 ([7]). *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the GSVI (1.2) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x, \quad \forall x \in C,$$

where $y^* = P_C(I - \nu_2 F_2)x^*$.

In particular, if the mapping $F_i : C \to H$ is ζ_i-inverse-strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\nu_i \in (0, 2\zeta_i]$ for $i = 1, 2$. We denote by $\mathrm{GSVI}(G)$ the fixed point set of the mapping G.

Lemma 2.12 ([17]). *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{k+1} \leqslant (1-s_k)a_k + s_k t_k + r_k, \quad \forall k \geqslant 0,$$

where $\{s_k\}, \{t_k\}$, and $\{r_k\}$ are sequences of real numbers such that

(i) $\{s_k\} \subset [0,1]$ *and* $\sum_{k=0}^{\infty} s_k = \infty$;
(ii) *either* $\limsup_{k\to\infty} t_k \leqslant 0$, *or* $\sum_{k=0}^{\infty} |s_k t_k| < \infty$;
(iii) $\sum_{k=0}^{\infty} r_k < \infty$ *with* $r_k \geqslant 0, \ \forall k \geqslant 0$.

Then, $\lim_{k\to\infty} a_k = 0$.

Lemma 2.13 ([11]). *Let H be a real Hilbert space. Then the followings hold:*

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H$;
(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0,1]$ *with* $\lambda + \mu = 1$;
(c) *if $\{x_k\}$ is a sequence in H such that $x_k \rightharpoonup x$, it follows that*

$$\limsup_{k\to\infty} \|x_k - y\|^2 = \limsup_{k\to\infty} \|x_k - x\|^2 + \|x-y\|^2, \quad \forall y \in H.$$

Lemma 2.14 ([4]). *Let $\{a_k\}_{k=0}^{\infty}$ be a bounded sequence of nonnegative real numbers and $\{b_k\}_{k=0}^{\infty}$ be a sequence of real numbers such that $\limsup_{k\to\infty} b_k \leqslant 0$. Then, $\limsup_{k\to\infty} a_k b_k \leqslant 0$.*

3. Main results

Let H be a real Hilbert space. In this section, we always assume the followings.

- $F_i : H \to H$ is ζ_i-inverse strongly monotone for $i = 1,2$ and $T : H \to H$ is a ζ-strictly pseudocontractive mapping;
- $G : H \to C$ is a mapping defined by $Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x$ with $0 < \nu_i < 2\zeta_i$ for $i = 1,2$;
- $f : H \to H$ is a ρ-contraction mapping with $\rho \in [0,1)$;
- $A : H \to H$ and $B : H \to H$ are two mappings such that the hypotheses (H1)-(H4) hold:
  (H1) B is monotone on H,
  (H2) A is β-inverse-strongly monotone on H,
  (H3) B is L-Lipschitz continuous on H,
  (H4) $\Omega := \mathrm{VI}(\mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B), A) \neq \emptyset$.

Next, we introduce the following triple hierarchical variational inequality problem (THVIP) defined over the common solution set of the GSVI (1.2) and the fixed point problem of a strictly pseudocontractive mapping T.

Problem 3.1. The objective is to

$$\begin{aligned} &\text{find } x^* \in \Omega := \mathrm{VI}(\mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B), A) \\ &\qquad := \{x^* \in \mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B) : \langle Ax^*, x - x^*\rangle \geqslant 0, \forall x \in \mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B)\}. \end{aligned}$$

That is, the Ω is the solution set of the THVIP of finding $x^* \in \mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B)$ such that

$$\langle Ax^*, x - x^*\rangle \geqslant 0, \quad \forall x \in \mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B), \tag{3.1}$$

where $\mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B)$ denotes the set of solutions of the VIP of finding $y^* \in \mathrm{GSVI}(G) \cap \mathrm{Fix}(T)$ such that

$$\langle By^*, y - y^*\rangle \geqslant 0, \quad \forall y \in \mathrm{GSVI}(G) \cap \mathrm{Fix}(T).$$

It is worth pointing out that Problem 3.1 is very different from Problem 1.3 because the solution set of Problem 3.1 may not be a singleton but the solution set of Problem 1.3 must be a singleton.

Algorithm 3.2. Choose $u \in H$, $x_0 \in H$, $k = 0$, $0 < \lambda \leqslant 2\beta$, positive sequences $\{\delta_k\}, \{\lambda_k\}, \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$, and $\{\bar{\epsilon}_k\}$ such that $\lim_{k\to\infty} \delta_k = 0, \sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty$, $\alpha_k + \beta_k + \gamma_k = 1, \sum_{k=0}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0, \lim_{k\to\infty} \beta_k = \xi \in (\zeta, \frac{1}{2}]$, $\lim_{k\to\infty} \lambda_k = 0$, and $\lambda_k \leqslant \frac{1}{L}$ for all $k \geqslant 0$.

Step 1. Compute

$$\begin{cases} u_k := \alpha_k f(x_k) + (1-\alpha_k)Gx_k, \\ v_k := \alpha_k u_k + \beta_k x_k + \gamma_k[\mu G x_k + (1-\mu)TGx_k], \\ y_k := P_{GSVI(G)\cap Fix(T)}(v_k - \lambda_k B v_k), \\ z_k := P_{GSVI(G)\cap Fix(T)}(v_k - \lambda_k B y_k). \end{cases}$$

Step 2. Inner loop $j = 0, 1, \ldots$. Compute

$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_{GSVI(G)\cap Fix(T)}(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} + \gamma_j P_{GSVI(G)\cap Fix(T)}(x_{k,j} - \delta_j B y_{k,j}). \\ \text{If } \|x_{k,j+1} - P_{VI(GSVI(G)\cap Fix(T),B)} x_{k,0}\| \leqslant \bar{\epsilon}_k \text{ then set } h_k := x_{k,j+1} \text{ and go to Step 3.} \\ \text{Otherwise, increase } j \text{ by 1 and repeat the inner loop Step 2.} \end{cases}$$

Step 3. Set $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. Then increase k by 1 and go to Step 1.

Let C be a nonempty closed convex subset of H, $B : C \to H$ be monotone and L-Lipschitz continuous on C, and $S : C \to C$ be a nonexpansive mapping such that $VI(C, B) \cap Fix(S) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_k = P_C(x_k - \delta_k B x_k), \\ x_{k+1} = \alpha_k x_0 + \beta_k x_k + \gamma_k S P_C(x_k - \delta_k B y_k), \quad \forall k \geqslant 0, \end{cases}$$

where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$, and $\{\delta_k\}$ satisfy the following conditions: $\delta_k > 0, \lim_{k\to\infty} \delta_k = 0$, $\alpha_k + \beta_k + \gamma_k = 1$, $\sum_{k=0}^{\infty} \alpha_k = \infty, \lim_{k\to\infty} \alpha_k = 0$, and $0 < \liminf_{k\to\infty} \beta_k \leqslant \limsup_{k\to\infty} \beta_k < 1$ for all $k \geqslant 0$. Under these conditions, Yao et al. [28] proved that the sequences $\{x_k\}$ and $\{y_k\}$ converge strongly to the same point $P_{VI(C,B)\cap Fix(S)} x_0$.

Applying these iteration sequences with S being the identity mapping, we have the following lemma.

Lemma 3.3. *Suppose that the hypotheses* (H1)-(H4) *hold. Then the sequence* $\{x_{k,j}\}$ *generated by Algorithm* 3.2 *converges strongly to the point* $P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda A z_k)$ *as* $j \to \infty$. *Consequently, we have*

$$\|h_k - P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda A z_k)\| \leqslant \bar{\epsilon}_k, \quad \forall k \geqslant 0.$$

In the sequel, we always suppose that the inner loop in Algorithm 3.2 terminates after a finite number of steps. This assumption, by Lemma 3.3, is satisfied when B is monotone on $GSVI(G) \cap Fix(T)$.

Lemma 3.4. *Let the sequences* $\{v_k\}, \{y_k\}$, *and* $\{z_k\}$ *be generated by Algorithm* 3.2, B *be* L*-Lipschitzian and monotone on* H, *and* $p \in VI(GSVI(G) \cap Fix(T), B)$. *Then, we have*

$$\|z_k - p\|^2 \leqslant \|v_k - p\|^2 - (1 - \lambda_k L)\|v_k - y_k\|^2 - (1 - \lambda_k L)\|y_k - z_k\|^2. \tag{3.2}$$

Proof. Let $p \in VI(GSVI(G) \cap Fix(T), B)$. That means

$$\langle Bp, x - p\rangle \geqslant 0, \quad \forall x \in GSVI(G) \cap Fix(T).$$

Then, for each $\lambda_k > 0$, p satisfies the fixed point equation $p = P_{GSVI(G)\cap Fix(T)}(p - \lambda_k Bp)$. Since B is monotone on H and $p \in VI(GSVI(G) \cap Fix(T), B)$, we have

$$\langle By_k, y_k - p\rangle \geqslant \langle Bp, y_k - p\rangle \geqslant 0.$$

Then, applying Proposition 2.2 (ii) with $v_k - \lambda_k By_k$ and p, we obtain

$$\begin{aligned}
\|z_k - p\|^2 &\leqslant \|v_k - \lambda_k By_k - p\|^2 - \|v_k - \lambda_k By_k - z_k\|^2 \\
&= \|v_k - p\|^2 - 2\lambda_k\langle By_k, v_k - p\rangle + \lambda_k^2\|By_k\|^2 - \|v_k - z_k\|^2 \\
&\quad - \lambda_k^2\|By_k\|^2 + 2\lambda_k\langle By_k, v_k - z_k\rangle \\
&= \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, p - z_k\rangle \\
&= \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, p - y_k\rangle + 2\lambda_k\langle By_k, y_k - z_k\rangle \\
&\leqslant \|v_k - p\|^2 - \|v_k - z_k\|^2 + 2\lambda_k\langle By_k, y_k - z_k\rangle.
\end{aligned} \tag{3.3}$$

Applying Proposition 2.2 (i) with $v_k - \lambda_k Bv_k$ and z_k, we also have

$$\langle v_k - \lambda_k Bv_k - y_k, z_k - y_k\rangle \leqslant 0.$$

Combining this inequality with (3.3) and observing that B is L-Lipschitz continuous on H, we obtain

$$\begin{aligned}
\|z_k - p\|^2 &\leqslant \|v_k - p\|^2 - \|(v_k - y_k) + (y_k - z_k)\|^2 + 2\lambda_k\langle By_k, y_k - z_k\rangle \\
&= \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 - 2\langle v_k - y_k, y_k - z_k\rangle + 2\lambda_k\langle By_k, y_k - z_k\rangle \\
&= \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 - 2\langle v_k - \lambda_k By_k - y_k, y_k - z_k\rangle \\
&= \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 - 2\langle v_k - \lambda_k Bv_k - y_k, y_k - z_k\rangle \\
&\quad + 2\lambda_k\langle Bv_k - By_k, z_k - y_k\rangle \\
&\leqslant \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 + 2\lambda_k\langle Bv_k - By_k, z_k - y_k\rangle \\
&\leqslant \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 + 2\lambda_k\|Bv_k - By_k\|\|z_k - y_k\| \\
&\leqslant \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 + 2\lambda_k L\|v_k - y_k\|\|z_k - y_k\| \\
&\leqslant \|v_k - p\|^2 - \|v_k - y_k\|^2 - \|y_k - z_k\|^2 + \lambda_k L(\|v_k - y_k\|^2 + \|z_k - y_k\|^2) \\
&\leqslant \|v_k - p\|^2 - (1 - \lambda_k L)\|v_k - y_k\|^2 - (1 - \lambda_k L)\|y_k - z_k\|^2.
\end{aligned} \tag{3.4}$$

□

Lemma 3.5. *Suppose that the hypotheses* (H1)-(H4) *hold. Then the sequence* $\{x_k\}$ *generated by Algorithm* 3.2 *is bounded.*

Proof. Since $\lim_{k\to\infty} \alpha_k = 0$, $\lim_{k\to\infty} \beta_k = \xi \in (\zeta, \frac{1}{2}]$ and $\alpha_k + \beta_k + \gamma_k = 1$, we get $\lim_{k\to\infty}(1 - \gamma_k) = \lim_{k\to\infty}(\alpha_k + \beta_k) = \xi$. Moreover, we may assume, without loss of generality, that $\{\beta_k\} \subset [a, b] \subset (\zeta, 1)$. Take an arbitrary $p \in \Omega := VI(VI(GSVI(G) \cap Fix(T), B), A)$. Putting $\sigma = 1 - \mu$ and $\mathcal{A} = I - T$, we know that $\mathcal{A}$ is $\frac{1-\zeta}{2}$-inverse-strongly monotone since T is ζ-strictly pseudocontractive. We write $\tilde{u}_k = \mu Gx_k + (1 - \mu)TGx_k$ for $k \geqslant 0$. Then we observe that $\tilde{u}_k = \mu Gx_k + (1 - \mu)TGx_k = Gx_k - (1 - \mu)(I - T)Gx_k = Gx_k - \sigma\mathcal{A}Gx_k$, which together with (2.1), yields

$$\begin{aligned}
\|\tilde{u}_k - p\|^2 &= \|Gx_k - \sigma\mathcal{A}Gx_k - (p - \sigma\mathcal{A}p)\|^2 \\
&= \|Gx_k - p - \sigma(\mathcal{A}Gx_k - \mathcal{A}p)\|^2 \\
&\leqslant \|Gx_k - p\|^2 - \sigma(1 - \zeta - \sigma)\|\mathcal{A}Gx_k - \mathcal{A}p\|^2 \\
&= \|Gx_k - p\|^2 - (1 - \mu)(\mu - \zeta)\|Gx_k - TGx_k\|^2 \\
&\leqslant \|Gx_k - p\|^2.
\end{aligned} \tag{3.5}$$

Since $p = Gp = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)p$ and F_i is ζ_i-inverse-strongly monotone with $0 < \nu_i < 2\zeta_i$ for $i = 1, 2$, we deduce that

$$\begin{aligned}
\|Gx_k-p\|^2 &= \|P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)x_k - P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)p\|^2\\
&\leqslant \|(I-\nu_1F_1)P_C(I-\nu_2F_2)x_k-(I-\nu_1F_1)P_C(I-\nu_2F_2)p\|^2\\
&= \|[P_C(I-\nu_2F_2)x_k-P_C(I-\nu_2F_2)p]-\nu_1[F_1P_C(I-\nu_2F_2)x_k-F_1P_C(I-\nu_2F_2)p]\|^2\\
&\leqslant \|P_C(I-\nu_2F_2)x_k-P_C(I-\nu_2F_2)p\|^2\\
&\quad +\nu_1(\nu_1-2\zeta_1)\|F_1P_C(I-\nu_2F_2)x_k-F_1P_C(I-\nu_2F_2)p\|^2\\
&\leqslant \|P_C(I-\nu_2F_2)x_k-P_C(I-\nu_2F_2)p\|^2\\
&\leqslant \|(I-\nu_2F_2)x_k-(I-\nu_2F_2)p\|^2\\
&= \|(x_k-p)-\nu_2(F_2x_k-F_2p)\|^2\\
&\leqslant \|x_k-p\|^2+\nu_2(\nu_2-2\zeta_2)\|F_2x_k-F_2p\|^2\\
&\leqslant \|x_k-p\|^2.
\end{aligned} \tag{3.6}$$

So, it follows that

$$\begin{aligned}
\|u_k-p\| &= \|\alpha_k(f(x_k)-f(p))+(1-\alpha_k)(Gx_k-p)+\alpha_k(f(p)-p)\|\\
&\leqslant \alpha_k\|f(x_k)-f(p)\|+(1-\alpha_k)\|Gx_k-p\|+\alpha_k\|f(p)-p\|\\
&\leqslant \alpha_k\rho\|x_k-p\|+(1-\alpha_k)\|x_k-p\|+\alpha_k\|f(p)-p\|\\
&= (1-\alpha_k(1-\rho))\|x_k-p\|+\alpha_k\|f(p)-p\|\\
&= (1-\alpha_k(1-\rho))\|x_k-p\|+\alpha_k(1-\rho)\frac{\|f(p)-p\|}{1-\rho}\\
&\leqslant \max\{\|x_k-p\|,\frac{\|f(p)-p\|}{1-\rho}\}.
\end{aligned} \tag{3.7}$$

Thus, from (3.1) and (3.5), (3.6), and (3.7) we get

$$\begin{aligned}
\|v_k-p\| &= \|\alpha_k(u_k-p)+\beta_k(x_k-p)+\gamma_k[\mu Gx_k+(1-\mu)TGx_k-p]\|\\
&= \|\alpha_k(u_k-p)+\beta_k(x_k-p)+\gamma_k(\tilde{u}_k-p)\|\\
&\leqslant \alpha_k\|u_k-p\|+\beta_k\|x_k-p\|+\gamma_k\|\tilde{u}_k-p\|\\
&\leqslant \alpha_k\max\{\|x_k-p\|,\frac{\|f(p)-p\|}{1-\rho}\}+\beta_k\|x_k-p\|+\gamma_k\|Gx_k-p\|\\
&\leqslant \alpha_k\max\{\|x_k-p\|,\frac{\|f(p)-p\|}{1-\rho}\}+\beta_k\|x_k-p\|+\gamma_k\|x_k-p\|\\
&= \alpha_k\max\{\|x_k-p\|,\frac{\|f(p)-p\|}{1-\rho}\}+(1-\alpha_k)\|x_k-p\|\\
&\leqslant \max\{\|x_k-p\|,\frac{\|f(p)-p\|}{1-\rho}\}.
\end{aligned} \tag{3.8}$$

On the other hand, for $p \in \Omega$, we have

$$\langle Ap, x-p\rangle \geqslant 0, \quad \forall x \in VI(GSVI(G)\cap Fix(T), B),$$

which implies $p = P_{VI(GSVI(G)\cap Fix(T),B)}(p-\lambda Ap)$. Then, from (2.1), Proposition 2.2 (iii), β-inverse strong monotonicity of A, and $0<\lambda\leqslant 2\beta$, it follows that

$$\begin{aligned}
&\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)-p\|^2\\
&= \|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)-P_{VI(GSVI(G)\cap Fix(T),B)}(p-\lambda Ap)\|^2\\
&\leqslant \|(I-\lambda A)z_k-(I-\lambda A)p\|^2\\
&\leqslant \|z_k-p\|^2+\lambda(\lambda-2\beta)\|Az_k-Ap\|^2\\
&\leqslant \|z_k-p\|^2.
\end{aligned} \tag{3.9}$$

Utilizing (3.4), (3.8), (3.9) and the assumptions $0<\lambda\leqslant 2\beta$, $\sum_{k=0}^{\infty}\bar{\epsilon}_k<\infty$ we obtain that

$$\begin{aligned}
\|x_{k+1}-p\| &= \|\alpha_k u+\beta_k x_k+\gamma_k h_k-p\| \\
&\leqslant \alpha_k\|u-p\|+\beta_k\|x_k-p\|+\gamma_k\|h_k-p\| \\
&\leqslant \alpha_k\|u-p\|+\beta_k\|x_k-p\|+\gamma_k\|h_k-P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda A z_k)\| \\
&\quad +\gamma_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda A z_k)-p\| \\
&\leqslant \alpha_k\|u-p\|+\beta_k\|x_k-p\|+\gamma_k\bar{\varepsilon}_k+\gamma_k\|z_k-p\| \\
&\leqslant \alpha_k\|u-p\|+\beta_k\|x_k-p\|+\gamma_k\bar{\varepsilon}_k+\gamma_k\|v_k-p\| \\
&\leqslant \alpha_k\|u-p\|+\beta_k\|x_k-p\|+\gamma_k\bar{\varepsilon}_k+\gamma_k\max\{\|x_k-p\|,\frac{\|f(p)-p\|}{1-\rho}\} \\
&\leqslant \alpha_k\|u-p\|+(\beta_k+\gamma_k)\max\{\|x_k-p\|,\frac{\|f(p)-p\|}{1-\rho}\}+\gamma_k\bar{\varepsilon}_k \\
&= \alpha_k\|u-p\|+(1-\alpha_k)\max\{\|x_k-p\|,\frac{\|f(p)-p\|}{1-\rho}\}+\gamma_k\bar{\varepsilon}_k \\
&\leqslant \max\{\|x_k-p\|,\|u-p\|,\frac{\|f(p)-p\|}{1-\rho}\}+\bar{\varepsilon}_k \\
&\leqslant \max\{\|x_0-p\|,\|u-p\|,\frac{\|f(p)-p\|}{1-\rho}\}+\sum_{j=0}^{k}\bar{\varepsilon}_j \\
&\leqslant \max\{\|x_0-p\|,\|u-p\|,\frac{\|f(p)-p\|}{1-\rho}\}+\sum_{k=0}^{\infty}\bar{\varepsilon}_k \\
&< \infty,
\end{aligned}$$

which shows that the sequence $\{x_k\}$ is bounded, and so are the sequences $\{u_k\},\{\tilde{u}_k\},\{v_k\},\{y_k\}$, and $\{z_k\}$. □

Lemma 3.6. *Suppose that the hypotheses* (H1)-(H4) *hold. Assume that the sequences* $\{v_k\}$ *and* $\{z_k\}$ *are generated by Algorithm* 3.2. *Then, we have*

$$\|z_{k+1}-z_k\| \leqslant (1+\lambda_{k+1}L)\|v_{k+1}-v_k\|+\lambda_k\|By_k\|+\lambda_{k+1}(\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|). \tag{3.10}$$

Proof. Taking into account the L-Lipschitzian property of B, for each $x, y \in H$ we have

$$\|(I-\lambda_k B)x-(I-\lambda_k B)y\| = \|x-y-\lambda_k(Bx-By)\| \leqslant \|x-y\|+\lambda_k\|Bx-By\| \leqslant (1+\lambda_k L)\|x-y\|.$$

Combining this inequality with Proposition 2.2 (iii), we have

$$\begin{aligned}
\|z_{k+1}-z_k\| &= \|P_{GSVI(G)\cap Fix(T)}(v_{k+1}-\lambda_{k+1}By_{k+1})-P_{GSVI(G)\cap Fix(T)}(v_k-\lambda_k By_k)\| \\
&\leqslant \|(v_{k+1}-\lambda_{k+1}By_{k+1})-v_k+\lambda_k By_k\| \\
&= \|(v_{k+1}-\lambda_{k+1}Bv_{k+1})-(v_k-\lambda_{k+1}Bv_k)+\lambda_{k+1}(Bv_{k+1}-By_{k+1}-Bv_k)+\lambda_k By_k\| \\
&\leqslant (1+\lambda_{k+1}L)\|v_{k+1}-v_k\|+\lambda_k\|By_k\|+\lambda_{k+1}(\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|).
\end{aligned}$$

This completes the proof. □

Proposition 3.7. *Let* $\{x_k\}$ *and* $\{y_k\}$ *be two bounded sequences in a real Banach space* X. *Let* $\{\beta_k\}$ *be a sequence in* $[0,1]$. *Suppose that* $0 < \liminf_{k\to\infty}\beta_k \leqslant \limsup_{k\to\infty}\beta_k < 1$, $x_{k+1} = (1-\beta_k)y_k+\beta_k x_k$ *and* $\limsup_{k\to\infty}(\|y_{k+1}-y_k\|-\|x_{k+1}-x_k\|) \leqslant 0$. *Then,* $\lim_{k\to\infty}\|y_k-x_k\| = 0$.

Lemma 3.8. *Suppose that the hypotheses* (H1)-(H4) *hold. Assume that the sequence* $\{x_k\}$ *is generated by Algorithm* 3.2. *Then,* $\lim_{k\to\infty}\|x_{k+1}-x_k\| = 0$.

Proof. Since $\lim_{k\to\infty}\alpha_k = 0$, $\lim_{k\to\infty}\beta_k = \xi \in (\zeta, \frac{1}{2}]$, and $\alpha_k+\beta_k+\gamma_k = 1$, we get $\lim_{k\to\infty}\gamma_k = \lim_{k\to\infty}(1-\alpha_k-\beta_k) = 1-\xi \in [1/2, 1-\zeta)$. Now, we write $x_{k+1} = (1-\beta_k)w_k + \beta_k x_k$ for all $k \geqslant 0$. Then, we have

$$\begin{aligned} w_{k+1}-w_k &= \frac{\alpha_{k+1}u+\gamma_{k+1}h_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k u+\gamma_k h_k}{1-\beta_k} \\ &= \Big(\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}\Big)u + \Big(\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k}\Big)h_k + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(h_{k+1}-h_k). \end{aligned} \tag{3.11}$$

Note that, for $0 < \lambda \leqslant 2\beta$, we have from (2.1) that

$$\begin{aligned} &\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_{k+1}-\lambda Az_{k+1}) - P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\|^2 \\ &\leqslant \|(I-\lambda A)z_{k+1} - (I-\lambda A)z_k\|^2 \\ &\leqslant \|z_{k+1}-z_k\|^2 + \lambda(\lambda-2\beta)\|Az_{k+1}-Az_k\|^2 \\ &\leqslant \|z_{k+1}-z_k\|^2. \end{aligned}$$

Then, utilizing (3.10) and (3.11) we get

$$\begin{aligned} &\|w_{k+1}-w_k\| \\ &\leqslant \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}\Big|\|u\| + \Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k}\Big|(\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\| + \bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}}{1-\beta_{k+1}}\|z_{k+1}-z_k\| + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1}+\bar{\epsilon}_k) \\ &\leqslant \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}\Big|\|u\| + \Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}} - \frac{\gamma_k}{1-\beta_k}\Big|(\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\| + \bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}(1+\lambda_{k+1}L)}{1-\beta_{k+1}}\|v_{k+1}-v_k\| + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(\bar{\epsilon}_{k+1}+\bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}}{1-\beta_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\|) \\ &= \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}\Big|\|u\| + \Big|\frac{\alpha_{k+1}}{1-\beta_{k+1}} - \frac{\alpha_k}{1-\beta_k}\Big|(\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\| + \bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}(1+\lambda_{k+1}L)}{\alpha_{k+1}+\gamma_{k+1}}\|v_{k+1}-v_k\| + \frac{\gamma_{k+1}}{\alpha_{k+1}+\gamma_{k+1}}(\bar{\epsilon}_{k+1}+\bar{\epsilon}_k) \\ &\quad + \frac{\gamma_{k+1}}{\alpha_{k+1}+\gamma_{k+1}}(\lambda_{k+1}(\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\|) \\ &\leqslant \Big(\frac{|\alpha_{k+1}-\alpha_k|}{1-\beta_{k+1}} + \frac{\alpha_k|\beta_{k+1}-\beta_k|}{(1-\beta_{k+1})(1-\beta_k)}\Big)(\|u\| + \|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\| + \bar{\epsilon}_k) \\ &\quad + \|v_{k+1}-v_k\| + \lambda_{k+1}L\|v_{k+1}-v_k\| + \bar{\epsilon}_{k+1}+\bar{\epsilon}_k \\ &\quad + \lambda_{k+1}(\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\| \\ &\leqslant \|v_{k+1}-v_k\| + (|\alpha_{k+1}-\alpha_k|+|\beta_{k+1}-\beta_k|)\frac{\|u\|+\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\|+\bar{\epsilon}_k}{1-b} \\ &\quad + \bar{\epsilon}_{k+1}+\bar{\epsilon}_k + \lambda_{k+1}(L\|v_{k+1}-v_k\|+\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\|. \end{aligned} \tag{3.12}$$

For simplicity, we write $S = \mu I + (1-\mu)T$ for $0 \leqslant \zeta \leqslant \mu < 1$. According to Lemma 2.9 we know that S is a nonexpansive mapping. It is clear that $Fix(S) = Fix(T)$. Also, we write $v_k = \beta_k x_k + (1-\beta_k)\tilde{w}_k$ for all $k \geqslant 0$, where

$$\tilde{w}_k = \frac{v_k-\beta_k x_k}{1-\beta_k} = \frac{\alpha_k u_k + \gamma_k[\mu Gx_k + (1-\mu)TGx_k]}{1-\beta_k} = \frac{\alpha_k u_k + \gamma_k SGx_k}{1-\beta_k}.$$

Observe that

$$\begin{aligned}
&\|\tilde{w}_{k+1}-\tilde{w}_k\| \\
&=\Big\|\frac{\alpha_{k+1}u_{k+1}+\gamma_{k+1}SGx_{k+1}}{1-\beta_{k+1}}-\frac{\alpha_k u_k+\gamma_k SGx_k}{1-\beta_k}\Big\| \\
&\leqslant \Big\|\frac{\alpha_{k+1}}{1-\beta_{k+1}}u_{k+1}-\frac{\alpha_k}{1-\beta_k}u_k\Big\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|\|SGx_{k+1}\|+\frac{\gamma_k}{1-\beta_k}\|SGx_{k+1}-SGx_k\| \\
&\leqslant \frac{\alpha_{k+1}}{1-\beta_{k+1}}\|u_{k+1}\|+\frac{\alpha_k}{1-\beta_k}\|u_k\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|\|SGx_{k+1}\|+\frac{\gamma_k}{1-\beta_k}\|x_{k+1}-x_k\|.
\end{aligned}\tag{3.13}$$

Moreover, simple calculations show that

$$\begin{aligned}
v_{k+1}-v_k &= \beta_{k+1}x_{k+1}+(1-\beta_{k+1})\tilde{w}_{k+1}-\beta_k x_k-(1-\beta_k)\tilde{w}_k \\
&= (\beta_{k+1}-\beta_k)(x_{k+1}-\tilde{w}_{k+1})+\beta_k(x_{k+1}-x_k)+(1-\beta_k)(\tilde{w}_{k+1}-\tilde{w}_k),
\end{aligned}$$

which together with (3.13), leads to

$$\begin{aligned}
\|v_{k+1}-v_k\| &\leqslant |\beta_{k+1}-\beta_k|\|x_{k+1}-\tilde{w}_{k+1}\|+\beta_k\|x_{k+1}-x_k\|+(1-\beta_k)\|\tilde{w}_{k+1}-\tilde{w}_k\| \\
&\leqslant |\beta_{k+1}-\beta_k|\|x_{k+1}-\tilde{w}_{k+1}\|+\beta_k\|x_{k+1}-x_k\|+(1-\beta_k)\Big[\frac{\alpha_{k+1}}{1-\beta_{k+1}}\|u_{k+1}\| \\
&\quad +\frac{\alpha_k}{1-\beta_k}\|u_k\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|\|SGx_{k+1}\|+\frac{\gamma_k}{1-\beta_k}\|x_{k+1}-x_k\|\Big] \\
&= |\beta_{k+1}-\beta_k|\|x_{k+1}-\tilde{w}_{k+1}\|+(\beta_k+\gamma_k)\|x_{k+1}-x_k\| \\
&\quad +(1-\beta_k)\Big[\frac{\alpha_{k+1}}{1-\beta_{k+1}}\|u_{k+1}\|+\frac{\alpha_k}{1-\beta_k}\|u_k\|+\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|\|SGx_{k+1}\|\Big] \\
&\leqslant |\beta_{k+1}-\beta_k|\|x_{k+1}-\tilde{w}_{k+1}\|+\|x_{k+1}-x_k\|+\frac{\alpha_{k+1}}{1-\beta_{k+1}}\|u_{k+1}\|+\frac{\alpha_k}{1-\beta_k}\|u_k\| \\
&\quad +\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|\|SGx_{k+1}\|.
\end{aligned}\tag{3.14}$$

Combining (3.12) and (3.14), we obtain

$$\begin{aligned}
\|w_{k+1}-w_k\| &\leqslant \|v_{k+1}-v_k\|+(|\alpha_{k+1}-\alpha_k|+|\beta_{k+1}-\beta_k|)\frac{\|u\|+\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\|+\bar{\epsilon}_k}{1-b} \\
&\quad +\bar{\epsilon}_{k+1}+\bar{\epsilon}_k+\lambda_{k+1}(L\|v_{k+1}-v_k\|+\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\| \\
&\leqslant |\beta_{k+1}-\beta_k|\|x_{k+1}-\tilde{w}_{k+1}\|+\|x_{k+1}-x_k\|+\frac{\alpha_{k+1}}{1-\beta_{k+1}}\|u_{k+1}\|+\frac{\alpha_k}{1-\beta_k}\|u_k\| \\
&\quad +\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|\|SGx_{k+1}\|+(|\alpha_{k+1}-\alpha_k|+|\beta_{k+1}-\beta_k|) \\
&\quad \times\frac{\|u\|+\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\|+\bar{\epsilon}_k}{1-b}+\bar{\epsilon}_{k+1}+\bar{\epsilon}_k \\
&\quad +\lambda_{k+1}(L\|v_{k+1}-v_k\|+\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\|,
\end{aligned}$$

which immediately yields

$$\begin{aligned}
\|w_{k+1}-w_k\|-\|x_{k+1}-x_k\| &\leqslant |\beta_{k+1}-\beta_k|\|x_{k+1}-\tilde{w}_{k+1}\|+\frac{\alpha_{k+1}}{1-\beta_{k+1}}\|u_{k+1}\|+\frac{\alpha_k}{1-\beta_k}\|u_k\| \\
&\quad +\Big|\frac{\gamma_{k+1}}{1-\beta_{k+1}}-\frac{\gamma_k}{1-\beta_k}\Big|\|SGx_{k+1}\|+(|\alpha_{k+1}-\alpha_k|+|\beta_{k+1}-\beta_k|) \\
&\quad \times\frac{\|u\|+\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)\|+\bar{\epsilon}_k}{1-b}+\bar{\epsilon}_{k+1}+\bar{\epsilon}_k \\
&\quad +\lambda_{k+1}(L\|v_{k+1}-v_k\|+\|Bv_{k+1}\|+\|By_{k+1}\|+\|Bv_k\|)+\lambda_k\|By_k\|.
\end{aligned}$$

Since $\lim_{k\to\infty}\alpha_k = 0$, $\lim_{k\to\infty}\beta_k = \xi \in (\zeta, \frac{1}{2}]$, $\alpha_k+\beta_k+\gamma_k = 1$, $\lim_{k\to\infty}\bar{\epsilon}_k = 0$, and $\lim_{k\to\infty}\lambda_k = 0$, we conclude from the boundedness of the sequences $\{u_k\},\{v_k\},\{x_k\},\{y_k\},\{z_k\}$, and $\{\tilde{w}_k\}$ that

$$\limsup_{k\to\infty}(\|w_{k+1}-w_k\| - \|x_{k+1}-x_k\|) \leqslant 0.$$

Therefore, by Proposition 3.1 we have

$$\lim_{k\to\infty}\|w_k - x_k\| = 0,$$

which together with $x_{k+1} = \beta_k x_k + (1-\beta_k)w_k$, implies that

$$\lim_{k\to\infty}\|x_{k+1}-x_k\| = \lim_{k\to\infty}(1-\beta_k)\|w_k - x_k\| = 0.$$

□

Lemma 3.9. *Suppose that the hypotheses* (H1)-(H4) *hold. Then for any* $p \in \Omega := \mathrm{VI}(\mathrm{VI}(\mathrm{GSVI}(G)\cap \mathrm{Fix}(T), B), A)$ *we have*

$$\begin{aligned}\|x_{k+1}-p\|^2 \leqslant{}& \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|v_k-p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| \\ &+\gamma_k\bar{\epsilon}_k^2 - \gamma_k(1-\lambda_k L)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2).\end{aligned}$$

Moreover, we have

$$\lim_{k\to\infty}\|P_{\mathrm{VI}(\mathrm{GSVI}(G)\cap\mathrm{Fix}(T),B)}(z_k-\lambda_k A z_k) - z_k\| = 0$$

and

$$\lim_{k\to\infty}\|P_{\mathrm{VI}(\mathrm{GSVI}(G)\cap\mathrm{Fix}(T),B)}(y_k-\lambda_k A y_k) - y_k\| = 0.$$

Proof. By Lemma 3.3, we know that

$$\lim_{j\to\infty} x_{k,j} = P_{\mathrm{VI}(\mathrm{GSVI}(G)\cap\mathrm{Fix}(T),B)}(z_k - \lambda A z_k),$$

which together with $0<\lambda\leqslant 2\beta$, inequality (3.2), $\lim_{k\to\infty}\beta_k = \xi\in(\zeta,\frac{1}{2}]$, and

$$p\in\Omega := \mathrm{VI}(\mathrm{VI}(\mathrm{GSVI}(G)\cap\mathrm{Fix}(T),B),A),$$

implies that

$$\begin{aligned}\|x_{k+1}-p\|^2 &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - p\|^2 \\ &\leqslant \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|h_k-p\|^2 \\ &\leqslant \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k(\|P_{\mathrm{VI}(\mathrm{GSVI}(G)\cap\mathrm{Fix}(T),B)}(z_k-\lambda A z_k) - p\| + \bar{\epsilon}_k)^2 \\ &= \gamma_k(\|P_{\mathrm{VI}(\mathrm{GSVI}(G)\cap\mathrm{Fix}(T),B)}(z_k-\lambda A z_k) - P_{\mathrm{VI}(\mathrm{GSVI}(G)\cap\mathrm{Fix}(T),B)} \\ &\quad + \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2(p-\lambda A p)\| + \bar{\epsilon}_k)^2 \\ &\leqslant \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k(\|(I-\lambda A)z_k - (I-\lambda A)p\| + \bar{\epsilon}_k)^2 \\ &\leqslant \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k(\|z_k-p\| + \bar{\epsilon}_k)^2 \\ &= \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|z_k-p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| + \gamma_k\bar{\epsilon}_k^2 \\ &\leqslant \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| + \gamma_k\bar{\epsilon}_k^2 \\ &\quad + \gamma_k(\|v_k-p\|^2 - (1-\lambda_k L)\|v_k-y_k\|^2 - (1-\lambda_k L)\|y_k-z_k\|^2) \\ &= \alpha_k\|u-p\|^2 + \beta_k\|x_k-p\|^2 + \gamma_k\|v_k-p\|^2 + 2\gamma_k\bar{\epsilon}_k\|z_k-p\| + \gamma_k\bar{\epsilon}_k^2 \\ &\quad - \gamma_k(1-\lambda_k L)(\|v_k-y_k\|^2 + \|y_k-z_k\|^2).\end{aligned} \tag{3.15}$$

Next we claim that $\|x_k - v_k\| \to 0$ as $n\to\infty$. Indeed, for simplicity, we write $q_k = Gx_k$, $\tilde{x}_k = P_C(I-\nu_2 F_2)x_k$ and $\tilde{p} = P_C(I-\nu_2F_2)p$. Then $q_k = P_C(I-\nu_1F_1)\tilde{x}_k$. Utilizing Algorithm 3.2, we obtain from (3.5) and (3.6) that

$$\begin{aligned}\|v_k-p\|^2 &= \|\alpha_k(u_k-p)+\beta_k(x_k-p)+\gamma_k(\tilde{u}_k-p)\|^2\\ &\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|\tilde{u}_k-p\|^2\\ &\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k[\|Gx_k-p\|^2-(1-\mu)(\mu-\zeta)\|Gx_k-TGx_k\|^2]\\ &\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k[\|x_k-p\|^2+\nu_2(\nu_2-2\zeta_2)\|F_2x_k-F_2p\|^2\\ &\quad+\nu_1(\nu_1-2\zeta_1)\|F_1\tilde{x}_k-F_1\tilde{p}\|^2-(1-\mu)(\mu-\zeta)\|Gx_k-TGx_k\|^2]\\ &= \alpha_k\|u_k-p\|^2+(\beta_k+\gamma_k)\|x_k-p\|^2+\gamma_k[\nu_2(\nu_2-2\zeta_2)\|F_2x_k-F_2p\|^2\\ &\quad+\nu_1(\nu_1-2\zeta_1)\|F_1\tilde{x}_k-F_1\tilde{p}\|^2-(1-\mu)(\mu-\zeta)\|Gx_k-TGx_k\|^2]\\ &\leqslant \alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\gamma_k[\nu_2(2\zeta_2-\nu_2)\|F_2x_k-F_2p\|^2\\ &\quad+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_k-F_1\tilde{p}\|^2+(1-\mu)(\mu-\zeta)\|Gx_k-TGx_k\|^2].\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) we get

$$\begin{aligned}\|x_{k+1}-p\|^2 &\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|v_k-p\|^2+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2\\ &\quad-\gamma_k(1-\lambda_kL)(\|v_k-y_k\|^2+\|y_k-z_k\|^2)\\ &\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\{\alpha_k\|u_k-p\|^2+\|x_k-p\|^2\\ &\quad-\gamma_k[\nu_2(2\zeta_2-\nu_2)\|F_2x_k-F_2p\|^2+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_k-F_1\tilde{p}\|^2\\ &\quad+(1-\mu)(\mu-\zeta)\|Gx_k-TGx_k\|^2]\}+2\gamma_k\bar{\epsilon}_k\|z_k-p\|\\ &\quad+\gamma_k\bar{\epsilon}_k^2-\gamma_k(1-\lambda_kL)(\|v_k-y_k\|^2+\|y_k-z_k\|^2)\\ &\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+(\beta_k+\gamma_k)\|x_k-p\|^2\\ &\quad-\gamma_k^2[\nu_2(2\zeta_2-\nu_2)\|F_2x_k-F_2p\|^2+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_k-F_1\tilde{p}\|^2\\ &\quad+(1-\mu)(\mu-\zeta)\|Gx_k-TGx_k\|^2]+2\bar{\epsilon}_k\|z_k-p\|\\ &\quad+\bar{\epsilon}_k^2-\gamma_k(1-\lambda_kL)(\|v_k-y_k\|^2+\|y_k-z_k\|^2)\\ &\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\gamma_k^2[\nu_2(2\zeta_2-\nu_2)\|F_2x_k-F_2p\|^2\\ &\quad+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_k-F_1\tilde{p}\|^2+(1-\mu)(\mu-\zeta)\|Gx_k-TGx_k\|^2]\\ &\quad+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2-\gamma_k(1-\lambda_kL)(\|v_k-y_k\|^2+\|y_k-z_k\|^2),\end{aligned}$$

which immediately yields

$$\begin{aligned}&\gamma_k(1-\lambda_kL)(\|v_k-y_k\|^2+\|y_k-z_k\|^2)+\gamma_k^2[\nu_2(2\zeta_2-\nu_2)\|F_2x_k-F_2p\|^2\\ &\quad+\nu_1(2\zeta_1-\nu_1)\|F_1\tilde{x}_k-F_1\tilde{p}\|^2+(1-\mu)(\mu-\zeta)\|Gx_k-TGx_k\|^2]\\ &\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\|x_{k+1}-p\|^2+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2\\ &\leqslant \alpha_k(\|u-p\|^2+\|u_k-p\|^2)+\|x_k-x_{k+1}\|(\|x_k-p\|+\|x_{k+1}-p\|)+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2.\end{aligned}$$

Since $\alpha_k+\beta_k+\gamma_k=1,\ \alpha_k\to0,\ \beta_k\to\xi\in(\zeta,\frac{1}{2}],\ \bar{\epsilon}_k\to0,\ \lambda_k\to0$, and $\|x_{k+1}-x_k\|\to0$ (due to Lemma 3.8), we deduce from the boundedness of $\{x_k\},\{u_k\}$ and $\{z_k\}$ that

$$\begin{cases}\lim\limits_{k\to\infty}\|F_2x_k-F_2p\|=\lim\limits_{k\to\infty}\|F_1\tilde{x}_k-F_1\tilde{p}\|=0,\\ \lim\limits_{k\to\infty}\|Gx_k-TGx_k\|=\lim\limits_{k\to\infty}\|v_k-y_k\|=\lim\limits_{k\to\infty}\|y_k-z_k\|=0.\end{cases} \tag{3.17}$$

On the other hand, in terms of the firm nonexpansivity of P_C and the ζ_i-inverse strong monotonicity of F_i for $i=1,2$, we obtain from $\nu_i\in(0,2\zeta_i), i=1,2$ and (3.6) that

$$\|\tilde{x}_k-\tilde{p}\|^2=\|P_C(I-\nu_2F_2)x_k-P_C(I-\nu_2F_2)p\|^2$$

$$
\begin{aligned}
&\leqslant \langle (I-\nu_2F_2)x_k-(I-\nu_2F_2)p, \tilde{x}_k-\tilde{p}\rangle \\
&= \frac{1}{2}[\|(I-\nu_2F_2)x_k-(I-\nu_2F_2)p\|^2+\|\tilde{x}_k-\tilde{p}\|^2-\|(I-\nu_2F_2)x_k-(I-\nu_2F_2)p-(\tilde{x}_k-\tilde{p})\|^2] \\
&\leqslant \frac{1}{2}[\|x_k-p\|^2+\|\tilde{x}_k-\tilde{p}\|^2-\|(x_k-\tilde{x}_k)-\nu_2(F_2x_k-F_2p)-(p-\tilde{p})\|^2] \\
&= \frac{1}{2}[\|x_k-p\|^2+\|\tilde{x}_k-\tilde{p}\|^2-\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2 \\
&\quad +2\nu_2\langle (x_k-\tilde{x}_k)-(p-\tilde{p}), F_2x_k-F_2p\rangle-\nu_2^2\|F_2x_k-F_2p\|^2],
\end{aligned}
$$

and

$$
\begin{aligned}
\|q_k-p\|^2 &= \|P_C(I-\nu_1F_1)\tilde{x}_k-P_C(I-\nu_1F_1)\tilde{p}\|^2 \\
&\leqslant \langle (I-\nu_1F_1)\tilde{x}_k-(I-\nu_1F_1)\tilde{p}, q_k-p\rangle \\
&= \frac{1}{2}[\|(I-\nu_1F_1)\tilde{x}_k-(I-\nu_1F_1)\tilde{p}\|^2+\|q_k-p\|^2-\|(I-\nu_1F_1)\tilde{x}_k-(I-\nu_1F_1)\tilde{p}-(q_k-p)\|^2] \\
&\leqslant \frac{1}{2}[\|\tilde{x}_k-\tilde{p}\|^2+\|q_k-p\|^2-\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2+2\nu_1\langle F_1\tilde{x}_k-F_1\tilde{p}, (\tilde{x}_k-q_k)+(p-\tilde{p})\rangle \\
&\quad -\nu_1^2\|F_1\tilde{x}_k-F_1\tilde{p}\|^2] \\
&\leqslant \frac{1}{2}[\|x_k-p\|^2+\|q_k-p\|^2-\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2+2\nu_1\langle F_1\tilde{x}_k-F_1\tilde{p}, (\tilde{x}_k-q_k)+(p-\tilde{p})\rangle].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|\tilde{x}_k-\tilde{p}\|^2 \leqslant \|x_k-p\|^2-\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2+2\nu_2\langle (x_k-\tilde{x}_k)-(p-\tilde{p}), F_2x_k-F_2p\rangle \\
-\nu_2^2\|F_2x_k-F_2p\|^2,
\end{aligned}
\tag{3.18}
$$

and

$$
\|q_k-p\|^2 \leqslant \|x_k-p\|^2-\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|. \tag{3.19}
$$

In the meantime, utilizing (3.16) and (3.18) we obtain

$$
\begin{aligned}
\|v_k-p\|^2 &\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|Gx_k-p\|^2 \\
&\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|\tilde{x}_k-p\|^2 \\
&\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k[\|x_k-p\|^2-\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2 \\
&\quad +2\nu_2\langle (x_k-\tilde{x}_k)-(p-\tilde{p}), F_2x_k-F_2p\rangle-\nu_2^2\|F_2x_k-F_2p\|^2] \\
&\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k[\|x_k-p\|^2-\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2 \\
&\quad +2\nu_2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|\|F_2x_k-F_2p\|] \\
&\leqslant \alpha_k\|u_k-p\|^2+(\beta_k+\gamma_k)\|x_k-p\|^2-\gamma_k\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2 \\
&\quad +2\nu_2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|\|F_2x_k-F_2p\| \\
&\leqslant \alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\gamma_k\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2+2\nu_2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|\|F_2x_k-F_2p\|,
\end{aligned}
$$

which together with (3.15), leads to

$$
\begin{aligned}
\|x_{k+1}-p\|^2 &\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|v_k-p\|^2+2\gamma_k\bar{e}_k\|z_k-p\|+\gamma_k\bar{e}_k^2 \\
&\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k[\alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\gamma_k\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2 \\
&\quad +2\nu_2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|\|F_2x_k-F_2p\|]+2\bar{e}_k\|z_k-p\|+\bar{e}_k^2 \\
&\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+(\beta_k+\gamma_k)\|x_k-p\|^2-\gamma_k^2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2 \\
&\quad +2\nu_2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|\|F_2x_k-F_2p\|+2\bar{e}_k\|z_k-p\|+\bar{e}_k^2 \\
&\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\gamma_k^2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2 \\
&\quad +2\nu_2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|\|F_2x_k-F_2p\|+2\bar{e}_k\|z_k-p\|+\bar{e}_k^2.
\end{aligned}
$$

So, it follows that

$$\begin{aligned}\gamma_k^2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|^2 &\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\|x_{k+1}-p\|^2\\&\quad+2\nu_2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|\|F_2x_k-F_2p\|+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2\\&\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+\|x_k-x_{k+1}\|(\|x_k-p\|+\|x_{k+1}-p\|)\\&\quad+2\nu_2\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|\|F_2x_k-F_2p\|+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2.\end{aligned}$$

Since $\alpha_k+\beta_k+\gamma_k=1$, $\alpha_k\to 0$, $\beta_k\to\xi\in(\zeta,\frac{1}{2}]$, $\bar{\epsilon}_k\to 0$, $\|F_2x_k-F_2p\|\to 0$ (due to (3.17)), and $\|x_{k+1}-x_k\|\to 0$ (due to Lemma 3.8), we deduce from the boundedness of $\{x_k\},\{\tilde{x}_k\},\{u_k\}$, and $\{z_k\}$ that

$$\lim_{k\to\infty}\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|=0. \tag{3.20}$$

Also, utilizing (3.16) and (3.19) we obtain

$$\begin{aligned}\|v_k-p\|^2 &\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|Gx_k-p\|^2\\&\leqslant \alpha_k\|u_k-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k[\|x_k-p\|^2-\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2\\&\quad+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|]\\&\leqslant \alpha_k\|u_k-p\|^2+(\beta_k+\gamma_k)\|x_k-p\|^2-\gamma_k\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2\\&\quad+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|\\&\leqslant \alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\gamma_k\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|,\end{aligned}$$

which together with (3.15), leads to

$$\begin{aligned}\|x_{k+1}-p\|^2 &\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k\|v_k-p\|^2+2\gamma_k\bar{\epsilon}_k\|z_k-p\|+\gamma_k\bar{\epsilon}_k^2\\&\leqslant \alpha_k\|u-p\|^2+\beta_k\|x_k-p\|^2+\gamma_k[\alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\gamma_k\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2\\&\quad+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|]+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2\\&\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+(\beta_k+\gamma_k)\|x_k-p\|^2-\gamma_k^2\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2\\&\quad+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2\\&\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\gamma_k^2\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2\\&\quad+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2.\end{aligned}$$

So, it follows that

$$\begin{aligned}\gamma_k^2\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|^2 &\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+\|x_k-p\|^2-\|x_{k+1}-p\|^2\\&\quad+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2\\&\leqslant \alpha_k\|u-p\|^2+\alpha_k\|u_k-p\|^2+\|x_k-x_{k+1}\|(\|x_k-p\|+\|x_{k+1}-p\|)\\&\quad+2\nu_1\|F_1\tilde{x}_k-F_1\tilde{p}\|\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|+2\bar{\epsilon}_k\|z_k-p\|+\bar{\epsilon}_k^2.\end{aligned}$$

Since $\alpha_k+\beta_k+\gamma_k=1$, $\alpha_k\to 0$, $\beta_k\to\xi\in(\zeta,\frac{1}{2}]$, $\bar{\epsilon}_k\to 0$, $\|F_1\tilde{x}_k-F_1\tilde{p}\|\to 0$ (due to (3.17)), and $\|x_{k+1}-x_k\|\to 0$ (due to Lemma 3.8), we deduce from the boundedness of $\{x_k\},\{\tilde{x}_k\},\{u_k\}$, and $\{z_k\}$ that

$$\lim_{k\to\infty}\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|=0. \tag{3.21}$$

Note that

$$\|x_k-q_k\|\leqslant\|(x_k-\tilde{x}_k)-(p-\tilde{p})\|+\|(\tilde{x}_k-q_k)+(p-\tilde{p})\|.$$

Hence from (3.20) and (3.21) we get

$$\lim_{k\to\infty}\|x_k-Gx_k\|=\lim_{k\to\infty}\|x_k-q_k\|=0. \tag{3.22}$$

Also, observe that

$$\begin{aligned}\|\tilde{u}_k - x_k\| &\leqslant \mu\|Gx_k - x_k\| + (1-\mu)\|TGx_k - x_k\| \\ &\leqslant \mu\|Gx_k - x_k\| + (1-\mu)(\|TGx_k - Gx_k\| + \|Gx_k - x_k\|) \\ &= \|Gx_k - x_k\| + (1-\mu)\|TGx_k - Gx_k\| \\ &\leqslant \|Gx_k - x_k\| + \|TGx_k - Gx_k\|,\end{aligned}$$

So, from (3.17) and (3.22) we know that

$$\lim_{k\to\infty} \|\tilde{u}_k - x_k\| = 0. \tag{3.23}$$

Taking into consideration that

$$\|u_k - x_k\| \leqslant \alpha_k\|f(x_k) - x_k\| + (1-\alpha_k)\|Gx_k - x_k\| \leqslant \alpha_k\|f(x_k) - x_k\| + \|Gx_k - x_k\|,$$

and

$$\|v_k - x_k\| \leqslant \alpha_k\|u_k - x_k\| + \gamma_k\|\tilde{u}_k - x_k\| \leqslant \alpha_k\|u_k - x_k\| + \|\tilde{u}_k - x_k\|,$$

we conclude from (3.17), (3.23), and $\alpha_k \to 0$ that

$$\lim_{k\to\infty} \|x_k - u_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|x_k - v_k\| = 0. \tag{3.24}$$

So, it follows from (3.17) that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|x_k - z_k\| = 0. \tag{3.25}$$

Since A is β-inverse-strongly monotone, it is known that A is L_1-Lipschitzian with $L_1 = 1/\beta$. Again by Proposition 2.2 (iii) and Lemma 3.3 we have

$$\begin{aligned}&\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k) - x_{k+1}\| \\ &\leqslant \|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k) - P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k)\| \\ &\quad + \|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k) - x_{k+1}\| \\ &\leqslant (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k) - u\| \\ &\quad + \beta_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k) - x_k\| + \bar{\epsilon}_k \\ &\leqslant (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k) - u\| + \bar{\epsilon}_k \\ &\quad + \beta_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k) - P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k)\| \\ &\quad + \beta_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k) - y_k\| + \beta_k\|y_k - x_k\| \\ &\leqslant (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k) - u\| + \bar{\epsilon}_k \\ &\quad + \beta_k(1+\lambda L_1)\|z_k - y_k\| + \beta_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k) - y_k\| + \beta_k\|y_k - x_k\|.\end{aligned} \tag{3.26}$$

Consequently, from (3.26), we have

$$\begin{aligned}&\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k) - y_k\| \\ &\leqslant \|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k) - x_{k+1}\| + \|x_{k+1} - x_k\| + \|x_k - y_k\| \\ &\leqslant (1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k) - u\| + \bar{\epsilon}_k \\ &\quad + \beta_k(1+\lambda L_1)\|z_k - y_k\| + \beta_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k) - y_k\| + \beta_k\|y_k - x_k\| \\ &\quad + \|x_{k+1} - x_k\| + \|x_k - y_k\| \\ &= (1+\beta_k)(1+\lambda L_1)\|y_k - z_k\| + \alpha_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda Az_k) - u\| + \bar{\epsilon}_k \\ &\quad + \beta_k\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda Ay_k) - y_k\| + (1+\beta_k)\|y_k - x_k\| + \|x_{k+1} - x_k\|,\end{aligned}$$

which immediately yields

$$\begin{aligned}
&\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k-\lambda Ay_k)-y_k\|\\
&\leqslant \frac{1+\beta_k}{1-\beta_k}(1+\lambda L_1)\|y_k-z_k\|+\frac{\alpha_k}{1-\beta_k}\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)-u\|+\frac{\bar{\epsilon}_k}{1-\beta_k}\\
&\quad+\frac{1+\beta_k}{1-\beta_k}\|y_k-x_k\|+\frac{1}{1-\beta_k}\|x_{k+1}-x_k\|.
\end{aligned}$$

Since $\alpha_k+\beta_k+\gamma_k=1$, $\alpha_k\to 0$, $\beta_k\to\xi\in(\zeta,\frac{1}{2}]$, $\bar{\epsilon}_k\to 0$, $\|y_k-z_k\|\to 0$, $\|x_k-y_k\|\to 0$, and $\|x_{k+1}-x_k\|\to 0$ (due to Lemma 3.8, (3.17), and (3.25)), we conclude that

$$\lim_{k\to\infty}\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k-\lambda Ay_k)-y_k\|=0. \tag{3.27}$$

From Proposition 2.2 (iii), it follows that

$$\begin{aligned}
&\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)-z_k\|\\
&\leqslant \|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)-P_{VI(GSVI(G)\cap Fix(T),B)}(y_k-\lambda Ay_k)\|\\
&\quad+\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k-\lambda Ay_k)-y_k\|+\|y_k-z_k\|\\
&\leqslant (1+\lambda L_1)\|z_k-y_k\|+\|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k-\lambda Ay_k)-y_k\|+\|y_k-z_k\|\\
&\leqslant \|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k-\lambda Ay_k)-y_k\|+(2+\lambda L_1)\|y_k-z_k\|.
\end{aligned}$$

Utilizing the last inequality we obtain from (3.17) and (3.27) that

$$\lim_{k\to\infty}\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k-\lambda Az_k)-z_k\|=0.$$

□

Theorem 3.10. *Suppose that the hypotheses* (H1)-(H4) *hold. Then the two sequences* $\{x_k\}$ *and* $\{z_k\}$ *in Algorithm 3.2 converge strongly to the same point* $x^*\in\Omega:=VI(VI(GSVI(G)\cap Fix(T),B),A)$ *provided* $\|x_k-v_k\|=o(\alpha_k^2)$, *which is a unique solution to the VIP*

$$\langle (I-f)x^*,p-x^*\rangle\geqslant 0,\quad \forall p\in\Omega.$$

Equivalently, $x^*=P_\Omega f(x^*)$.

Proof. Note that Lemma 3.5 shows the boundedness of $\{x_k\}$. Since H is reflexive, there is at least a weak convergence subsequence of $\{x_k\}$. First, let us assert that $\omega_w(x_k)\subset\Omega$. As a matter of fact, take an arbitrary $w\in\omega_w(x_k)$. Then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i}\rightharpoonup w$. From (3.25), we know that $y_{k_i}\rightharpoonup w$. It is easy to see that the mapping $P_{VI(GSVI(G)\cap Fix(T),B)}(I-\lambda A):H\to VI(GSVI(G)\cap Fix(T),B)\subset H$ is nonexpansive because $P_{VI(GSVI(G)\cap Fix(T),B)}$ is nonexpansive and $I-\lambda A$ is nonexpansive for β-inverse strongly monotone mapping A with $0<\lambda\leqslant 2\beta$. So, utilizing Lemma 2.7 and (3.27), we obtain

$$w=P_{VI(GSVI(G)\cap Fix(T),B)}(w-\lambda Aw),$$

which leads to $w\in VI(VI(GSVI(G)\cap Fix(T),B),A)=:\Omega$. Thus, the assertion is valid.

Also, note that

$$\langle (I-f)x-(I-f)y,x-y\rangle\geqslant(1-\rho)\|x-y\|^2,\quad \forall x,y\in H.$$

Hence, it follows from $0\leqslant\rho<1$ that $I-f$ is $(1-\rho)$-strongly monotone. In the meantime, it is clear that $I-f$ is Lipschitzian with constant $1+\rho>0$. Thus, by Lemma 2.6 (iv) we know that there exists a unique solution $x^*\in\Omega:=VI(VI(GSVI(G)\cap Fix(T),B),A)$ to the VIP

$$\langle (I-f)x^*,p-x^*\rangle\geqslant 0,\quad \forall p\in\Omega. \tag{3.28}$$

Equivalently, $x^*=P_\Omega f(x^*)$.

Next, let us show that $x_k \rightharpoonup x^*$. Indeed, take an arbitrary $p \in \Omega := \mathrm{VI}(\mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B), A)$. Then, from Algorithm 3.2, Lemma 2.5, (3.5), and (3.6), we have

$$\begin{aligned}
\|u_k - p\|^2 &= \|\alpha_k(f(x_k) - f(p)) + (1-\alpha_k)(Gx_k - p) + \alpha_k(f(p) - p)\|^2 \\
&\leqslant \|\alpha_k(f(x_k) - f(p))\|^2 + (1-\alpha_k)(Gx_k - p)\|^2 + 2\alpha_k\langle f(p) - p, u_k - p\rangle \\
&\leqslant \alpha_k\|f(x_k) - f(p)\|^2 + (1-\alpha_k)\|Gx_k - p\|^2 + 2\alpha_k\langle f(p) - p, u_k - p\rangle \\
&\leqslant \alpha_k\rho\|x_k - p\|^2 + (1-\alpha_k)\|x_k - p\|^2 + 2\alpha_k\langle f(p) - p, u_k - p\rangle \\
&= (1 - \alpha_k(1-\rho))\|x_k - p\|^2 + 2\alpha_k\langle f(p) - p, u_k - p\rangle,
\end{aligned}$$

and hence

$$\begin{aligned}
\|v_k - p\|^2 &= \|\alpha_k u_k + \beta_k x_k + \gamma_k \tilde{u}_k - p\|^2 \\
&\leqslant \alpha_k\|u_k - p\|^2 + \beta_k\|x_k - p\|^2 + \gamma_k\|\tilde{u}_k - p\|^2 \\
&\leqslant \alpha_k[(1 - \alpha_k(1-\rho))\|x_k - p\|^2 + 2\alpha_k\langle f(p) - p, u_k - p\rangle] + \beta_k\|x_k - p\|^2 + \gamma_k\|Gx_k - p\|^2 \\
&\leqslant \alpha_k[(1 - \alpha_k(1-\rho))\|x_k - p\|^2 + 2\alpha_k\langle f(p) - p, u_k - p\rangle] + \beta_k\|x_k - p\|^2 + \gamma_k\|x_k - p\|^2 \\
&= (1 - \alpha_k^2(1-\rho))\|x_k - p\|^2 + 2\alpha_k^2\langle f(p) - p, u_k - p\rangle \\
&\leqslant \|x_k - p\|^2 + 2\alpha_k^2\langle f(p) - p, u_k - p\rangle,
\end{aligned} \tag{3.29}$$

which immediately leads to

$$\begin{aligned}
0 &\leqslant \|x_k - p\|^2 - \|v_k - p\|^2 + 2\alpha_k^2\langle f(p) - p, u_k - p\rangle \\
&\leqslant \|x_k - v_k\|(\|x_k - p\| + \|v_k - p\|) + 2\alpha_k^2(\langle f(p) - p, u_k - x_k\rangle + \langle f(p) - p, x_k - p\rangle) \\
&\leqslant \|x_k - v_k\|(\|x_k - p\| + \|v_k - p\|) + 2\alpha_k^2(\|f(p) - p\|\|u_k - x_k\| + \langle f(p) - p, x_k - p\rangle).
\end{aligned}$$

That is,

$$0 \leqslant \frac{\|x_k - v_k\|}{2\alpha_k^2}(\|x_k - p\| + \|v_k - p\|) + \|f(p) - p\|\|u_k - x_k\| + \langle f(p) - p, x_k - p\rangle.$$

Since for any $w \in \omega_w(x_k)$ there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup w$, we deduce from (3.24), $\alpha_k \to 0$, and $\|x_k - v_k\| = o(\alpha_k^2)$ that for all $p \in \Omega := \mathrm{VI}(\mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B), A)$

$$\begin{aligned}
0 &\leqslant \lim_{i\to\infty}\{\frac{\|x_{k_i} - v_{k_i}\|}{2\alpha_{k_i}^2}(\|x_{k_i} - p\| + \|v_{k_i} - p\|) + \|f(p) - p\|\|u_{k_i} - x_{k_i}\| + \langle f(p) - p, x_{k_i} - p\rangle\} \\
&= \lim_{i\to\infty}\langle f(p) - p, x_{k_i} - p\rangle = \langle (f - I)p, w - p\rangle.
\end{aligned}$$

That is,

$$\langle (I - f)p, p - w\rangle \geqslant 0, \quad \forall p \in \Omega.$$

Consequently, by Lemma 2.6 (i) (Minty's lemma), we know that

$$\langle (I - f)w, p - w\rangle \geqslant 0, \quad \forall p \in \Omega,$$

that is, w is a solution of VIP (3.28). By the uniqueness of solutions of VIP (3.28), we get $w = x^*$, which hence implies that $\omega_w(x_k) = \{x^*\}$. Therefore, it is known that $\{x_k\}$ converges weakly to the unique solution $x^* \in \Omega := \mathrm{VI}(\mathrm{VI}(\mathrm{GSVI}(G) \cap \mathrm{Fix}(T), B), A)$ of VIP (3.28).

Finally, let us show that $\|x_k - x^*\| \to 0$ as $k \to \infty$. Indeed, in terms of Algorithm 3.2 and Lemma 2.5, we conclude from (3.4) and the β-inverse-strong monotonicity of A with $0 < \lambda \leqslant 2\beta$, that

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &= \|\alpha_k u + \beta_k x_k + \gamma_k h_k - x^*\|^2 \\
&\leqslant \|\beta_k(x_k - x^*) + \gamma_k(h_k - x^*)\|^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle
\end{aligned}$$

$$\begin{aligned}
&\leqslant \beta_k\|x_k - x^*\|^2 + \gamma_k\|h_k - x^*\|^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&\leqslant \beta_k\|x_k - x^*\|^2 + \gamma_k(\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda A z_k) - x^*\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&= \beta_k\|x_k - x^*\|^2 + \gamma_k(\|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda A z_k) \\
&\quad - P_{VI(GSVI(G)\cap Fix(T),B)}(x^* - \lambda A x^*)\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&\leqslant \beta_k\|x_k - x^*\|^2 + \gamma_k(\|(I - \lambda A)z_k - (I - \lambda A)x^*\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&\leqslant \beta_k\|x_k - x^*\|^2 + \gamma_k(\|z_k - x^*\| + \bar{\epsilon}_k)^2 + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&= \beta_k\|x_k - x^*\|^2 + \gamma_k\|z_k - x^*\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&\leqslant \beta_k\|x_k - x^*\|^2 + \gamma_k\|v_k - x^*\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&= \beta_k\|x_k - x^*\|^2 + \gamma_k\|x_k - x^* + v_k - x_k\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&= \beta_k\|x_k - x^*\|^2 + \gamma_k(\|x_k - x^*\|^2 + 2\langle x_k - x^*, v_k - x_k\rangle + \|v_k - x_k\|^2) \\
&\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &\leqslant \beta_k\|x_k - x^*\|^2 + \gamma_k(\|x_k - x^*\|^2 + \|v_k - x_k\|(2\|x_k - x^*\| + \|v_k - x_k\|) \\
&\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&\leqslant (1 - \alpha_k)\|x_k - x^*\|^2 + \|v_k - x_k\|(2\|x_k - x^*\| + \|v_k - x_k\|) \\
&\quad + \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&= (1 - \alpha_k)\|x_k - x^*\|^2 + \alpha_k[\frac{\|v_k - x_k\|}{\alpha_k}(2\|x_k - x^*\| + \|v_k - x_k\|) \\
&\quad + 2\langle u - x^*, x_{k+1} - x^*\rangle] + \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k).
\end{aligned} \tag{3.30}$$

Since $\alpha_k \to 0$, $\|x_k - v_k\| = o(\alpha_k)$, $\sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty$, and $x_k \rightharpoonup x^*$, we deduce from the boundedness of $\{x_k\}, \{v_k\}, \{z_k\}$ that $\sum_{k=0}^{\infty} \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) < \infty$ and

$$\limsup_{k\to\infty}[\frac{\|v_k - x_k\|}{\alpha_k}(2\|x_k - x^*\| + \|v_k - x_k\|) + 2\langle u - x^*, x_{k+1} - x^*\rangle] \leqslant 0.$$

Therefore, applying Lemma 2.12 to (3.29), we infer from $\sum_{k=0}^{\infty} \alpha_k = \infty$ that $\|x_k - x^*\| \to 0$ as $k \to \infty$. Utilizing (3.25) we also obtain that $\|z_k - x^*\| \to 0$ as $k \to \infty$. This completes the proof. □

Theorem 3.11. *Suppose that the hypotheses* (H1)-(H4) *hold. Then the two sequences* $\{x_k\}$ *and* $\{z_k\}$ *in Algorithm* 3.2 *converge strongly to the same point* $x^* \in \Omega := VI(VI(GSVI(G) \cap Fix(T), B), A)$, *where* $x^* = P_\Omega u$, *i.e.*, $\|u - x^*\| = \inf_{p\in\Omega} \|u - p\|$.

Proof. Assume that the hypotheses (H1)-(H4) hold and that $\|x_{k+1} - x_k\| = o(\alpha_k^2)$. In this case, it is easy to see that Lemmas 3.3-3.6, 3.8 and 3.9 hold.

Next, we divide the rest of the proof into several steps.

Step 1. Repeating the same arguments as those of (3.24) and (3.25), we can prove that

$$\lim_{k\to\infty} \|x_k - u_k\| = 0, \quad \lim_{k\to\infty} \|x_k - v_k\| = 0, \quad \lim_{k\to\infty} \|x_k - y_k\| = 0, \quad \lim_{k\to\infty} \|x_k - z_k\| = 0.$$

Step 2. We prove that $\omega_w(x_k) \subset \Omega := VI(VI(GSVI(G) \cap Fix(T), B), A)$.

Indeed, from Lemma 3.9 and $\lim_{k\to\infty} \|x_k - v_k\| = 0$, we have

$$\begin{aligned}
&\lim_{k\to\infty} \|P_{VI(GSVI(G)\cap Fix(T),B)}(z_k - \lambda_k A z_k) - z_k\| = 0, \\
&\lim_{k\to\infty} \|P_{VI(GSVI(G)\cap Fix(T),B)}(y_k - \lambda_k A y_k) - y_k\| = 0.
\end{aligned}$$

Utilizing the same argument as in the proof of Theorem 3.10, we obtain that $\omega_w(x_k) \subset \Omega$.

Step 3. We prove that $\lim_{k\to\infty} \|x_k - x^*\| = 0$ where $x^* = P_\Omega u$.

Indeed, we may assume, without loss of generality, that there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

$$\limsup_{k\to\infty}\langle u - x^*, x_k - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{k_i} - x^*\rangle$$

and $x_{k_i} \rightharpoonup w \in \Omega$. Since $x^* = P_\Omega u$ and $\|x_{k+1} - x_k\| \to 0$, we have

$$\limsup_{k\to\infty}\langle u - x^*, x_{k+1} - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{k_i} - x^*\rangle = \langle u - x^*, w - x^*\rangle \leqslant 0. \tag{3.31}$$

Utilizing the similar arguments to those of (3.29) and (3.30), we get

$$\|v_k - x^*\|^2 \leqslant \|x_k - x^*\|^2 + 2\alpha_k^2\langle f(x^*) - x^*, u_k - x^*\rangle,$$

and

$$\|x_{k+1} - x^*\|^2 \leqslant \beta_k\|x_k - x^*\|^2 + \gamma_k\|v_k - x^*\|^2 + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle.$$

Combining the last two inequalities, we get

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &\leqslant \beta_k\|x_k - x^*\|^2 + \gamma_k[\|x_k - x^*\|^2 + 2\alpha_k^2\langle f(x^*) - x^*, u_k - x^*\rangle] \\
&\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&= (\beta_k + \gamma_k)\|x_k - x^*\|^2 + 2\gamma_k\alpha_k^2\langle f(x^*) - x^*, u_k - x^*\rangle \\
&\quad + \gamma_k\bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&\leqslant (1 - \alpha_k)\|x_k - x^*\|^2 + 2\alpha_k^2\|f(x^*) - x^*\|\|u_k - x^*\| \\
&\quad + \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) + 2\alpha_k\langle u - x^*, x_{k+1} - x^*\rangle \\
&= (1 - \alpha_k)\|x_k - x^*\|^2 + \alpha_k \cdot 2(\alpha_k\|f(x^*) - x^*\|\|u_k - x^*\| \\
&\quad + \langle u - x^*, x_{k+1} - x^*\rangle) + \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k) \\
&= (1 - s_k)\|x_k - x^*\|^2 + s_k \cdot t_k + r_k,
\end{aligned} \tag{3.32}$$

where $s_k = \alpha_k$, $t_k = 2(\alpha_k\|f(x^*) - x^*\|\|u_k - x^*\| + \langle u - x^*, x_{k+1} - x^*\rangle)$ and $r_k = \bar{\epsilon}_k(2\|z_k - x^*\| + \bar{\epsilon}_k)$. Since $\alpha_k \to 0$, $\sum_{k=0}^{\infty}\alpha_k = \infty$, $\sum_{k=0}^{\infty}\bar{\epsilon}_k < \infty$, and $\limsup_{k\to\infty}\langle u - x^*, x_{k+1} - x^*\rangle \leqslant 0$ (due to (3.31)), we deduce from the boundedness of $\{x_k\}$,$\{u_k\}$,$\{z_k\}$ that $\limsup_{k\to\infty} t_k \leqslant 0$, $\sum_{k=0}^{\infty} s_k = \infty$, and $\sum_{k=0}^{\infty} r_k < \infty$. Therefore, applying Lemma 2.12 to (3.32), we obtain

$$\lim_{k\to\infty}\|x_k - x^*\| = 0.$$

This completes the proof. □

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