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Positive properties of the Green function for two-term fractional differential equations and its application

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Abstract

In this paper, we study the positive properties of the Green function for the following two-term fractional differential equation

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$

where $1 < \alpha < 2$, $b > 0$, D_{0+}^{α} is the standard Riemann-Liouville derivative. As an application, the existence and uniqueness of positive solution are obtained under the singular conditions. Moreover, an iterative scheme is established to approximate the unique positive solution.

Keywords: Multi-term fractional differential equation, Green function, iterative solution, boundary value problems.
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1. Introduction

It has been proved that fractional-order models are more accurate than integer-order models as fractional order models allow more degrees of freedom. For example, fractional derivatives in the sense of Caputo type can be used to describe the anomalous behaviors of diffusive phenomena [25–30]. Fractional differential equations (FDE) serve as an excellent instrument for the description of memory and hereditary properties of various materials and processes. Recently, much attention has been paid to the study of boundary value problems (BVP) of fractional differential equation, such as the singular BVP [17, 21, 33, 34], nonlocal BVP [2, 5, 20, 24], semipositone BVP [19, 22, 23] and resonant BVP [3, 4, 18, 32].

Multi-term fractional differential equations have been used to model various types of visco-elastic damping [1, 14]. The proposed model equations are almost always linear. Many authors focused on equations of the linear form:

$$[D^{\alpha_N} + b_{N-1}D^{\alpha_{N-1}} + \cdots + b_1 D^{\alpha_1} + b_0 D^0]y(t) = g(t),$$

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where $b_i \in \mathbb{R}$ $(i = 0, 1, \cdots, N-1)$, equipped with initial conditions [6, 8–10]. In [9], the authors investigated the Endolymph equation:

$$D^2x(t) + a_1Dx(t) + a_2D^{\frac{1}{2}}x(t) + a_3x(t) = -g(t),$$

which can be used to describe model for the response of the semicircular canals to the angular acceleration. By using the Laplace transformation, an exact solution was obtained for the equation of motion. In [8], by using the method of separation of variables, the authors investigated the following multi-term of fractional diffusion-wave equation along with the homogeneous/non-homogeneous boundary conditions:

$$P(D)u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + q(t), \quad 0 < x < \pi, \quad t > 0,$$

where

$$P(D) = D_t^{\mu} - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i}, \quad 0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leqslant 2.$$

It should be noted that the solution is not necessarily non-negative, and hence does not represent anomalous diffusion of any kind.

Since only positive solutions are meaningful in most practical problems, some work has been done to study the existence of positive solutions for fractional boundary value problems (FBVP) by using the techniques of nonlinear analysis such as fixed point theorems, Leray-Schauder theory, etc. We refer to the references [7, 11, 16, 31, 35]. It is well-known that the cone plays a very important role in seeking positive solutions of FBVP. Moreover, the cone is usually derived from the positive properties of the Green function. In [12], the authors discussed some positive properties of the Green function for Dirichlet-type FBVP, and obtained the existence of positive solutions by using the Krasnosel'skii fixed-point theorem. In [24], the authors investigated the following fractional differential equation:

$$\begin{cases} -D_{0+}^{\alpha}u(t) = f(t,u(t)) + e(t), \quad 0 < t < 1, \\ u(0) = 0, D_{0+}^{\beta}u(1) = aD_{0+}^{\beta}u(\xi), \end{cases}$$

where $1 < \alpha \leqslant 2$, $0 < \beta \leqslant \alpha - 1$, $0 < \xi < 1$, $0 \leqslant a \leqslant 1$ and $a\xi^{\alpha-\beta-2} \leqslant 1 - \beta$. The authors obtained some properties of the Green function. But they failed to obtain the positive properties similar to that of [12] in form, and given an open problem about positive properties of the Green function, that is, [24, Remark 2.1]. Wang et al. [21, 22] established some new positive properties of the corresponding Green function, and solved the open problem of [24]. As application, the existence of positive solutions were obtained for a class of fractional m-point BVPs.

Inspired by the above works, in this paper, we aim to deduce some positive properties of the Green function for the following Dirichlet-type FBVP

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) = f(t,u(t)), \quad 0 < t < 1, \\ u(0) = 0, u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $b > 0$, D_{0+}^{α} is the standard Riemann-Liouville derivative. The paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we establish some positive properties of the Green function. In Section 4, we discuss the existence and uniqueness of positive solution for FBVP (1.1) under the singular conditions, that is, $f(t,x)$ may be singular at $t = 0, 1$, and $x = 0$. Moreover, we establish an iterative scheme to approximate the unique positive solution.

2. Basic definitions and preliminaries

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Denote

$$g(t) = \frac{\alpha-2}{\Gamma(\alpha-1)} + \sum_{k=1}^{+\infty}\frac{t^k}{\Gamma((k+1)\alpha-2)}.$$

It is easy to check that

$$g(0) = \frac{\alpha-2}{\Gamma(\alpha-1)} < 0, \quad g'(t) > 0, \quad \text{on} \quad [0, +\infty),$$

and

$$\lim_{t\to+\infty} g(t) = +\infty.$$

Therefore, there exists a unique $b^* > 0$ such that

$$g(b^*) = 0.$$

Throughout this paper, we always suppose that the parameter b in (1.1) satisfies

(H_1) $b \in (0, b^*]$.

Denote

$$G(t) = t^{\alpha-1}E_{\alpha,\alpha}(bt^{\alpha}), \tag{2.1}$$

where

$$E_{\alpha,\alpha}(x) = \sum_{k=0}^{+\infty}\frac{x^k}{\Gamma((k+1)\alpha)},$$

is the Mittag-Leffler function [13, 15]. Set

$$K(t,s) = \frac{1}{G(1)}\begin{cases} G(t)G(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ G(t)G(1-s) - G(t-s)G(1), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Lemma 2.3. *Suppose that* (H_1) *holds, and* $y \in L[0,1]$. *Then the problem*

$$\begin{cases} -D_{0+}^{\alpha}u(t) + bu(t) = y(t), & 0 < t < 1, \\ u(0) = 0, u(1) = 0, \end{cases} \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 K(t,s)y(s)ds.$$

Proof. As argued in [13, 15], the general solution of (2.2) can be expressed by

$$u(t) = -\int_0^t G(t-s)y(s)ds + c_1G(t) + c_2G'(t).$$

Since $u(0) = 0$, we have $c_2 = 0$.

On the other hand,

$$u(1) = -\int_0^1 G(1-s)y(s)ds + c_1 G(1).$$

So,

$$c_1 = \frac{\int_0^1 G(1-s)y(s)ds}{G(1)}.$$

Therefore, the solution of (2.2) is

$$\begin{aligned} u(t) &= -\int_0^t G(t-s)y(s)ds + \frac{\int_0^1 G(1-s)y(s)ds}{G(1)}G(t) \\ &= \frac{\int_0^1 G(t)G(1-s)y(s)ds - \int_0^t G(1)G(t-s)y(s)ds}{G(1)} \\ &= \int_0^1 K(t,s)y(s)ds. \end{aligned}$$

□

3. Main results

Theorem 3.1. *Suppose that* (H_1) *holds. Then the function* $K(t,s)$ *has the following properties:*

(1) $K(t,s) > 0, \quad \forall\, t,s \in (0,1);$

(2) $K(t,s) = K(1-s, 1-t), \quad \forall\, t,s \in [0,1];$

(3) $K(t,s) \leqslant G(1)s(1-s)^{\alpha-1}t^{\alpha-2}, \quad \forall\, t,s \in [0,1];$

(4) $K(t,s) \geqslant Ms(1-s)^{\alpha-1}(1-t)t^{\alpha-1}$, $\forall\, t,s \in [0,1]$, *where*

$$M = \min\left\{\frac{1}{G(1)[\Gamma(\alpha)]^2},\ G(1)(\alpha-1)^2\right\}.$$

Proof. We only need to prove that (3) and (4) hold.

By (2.1), we have

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \leqslant G(t) = t^{\alpha-1}\sum_{k=0}^{+\infty}\frac{b^k t^{\alpha k}}{\Gamma((k+1)\alpha)} \leqslant t^{\alpha-1}G(1), \quad t \in [0,1], \tag{3.1}$$

$$G'(t) = \sum_{k=0}^{+\infty}\frac{b^k t^{(k+1)\alpha-2}}{\Gamma((k+1)\alpha-1)} > 0, \quad t \in (0,1],$$

and

$$\begin{aligned} G''(t) =& t^{\alpha-3}\left[\frac{\alpha-2}{\Gamma(\alpha-1)} + \sum_{k=1}^{+\infty}\frac{b^k t^{k\alpha}}{\Gamma((k+1)\alpha)-2}\right] \\ =& t^{\alpha-3}g(bt^\alpha) < t^{\alpha-3}g(b) \leqslant t^{\alpha-3}g(b^*) = 0, \quad t \in (0,1), \end{aligned}$$

which implies that $G(t)$ is strictly increasing on $[0,1]$, and $G'(t)$ is strictly decreasing on $(0,1]$. Moreover, it is easy to see that $G''(t)$ is strictly increasing on $(0,1]$.

(i) For $0 < t \leqslant s < 1$. By (3.1), we have

$$\begin{aligned} K(t,s) &= \frac{G(t)G(1-s)}{G(1)} \leqslant \frac{G(t)G(1-s)}{G(1)}\frac{1-t}{1-s} \\ &\leqslant G(1)(1-s)^{\alpha-1}t^{\alpha-1}\frac{1-t}{1-s} \\ &= G(1)(1-t)t^{\alpha-1}(1-s)^{\alpha-2}, \end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
K(t,s) &= \frac{G(t)G(1-s)}{G(1)} \geqslant \frac{G(t)G(1-s)(1-t)s}{G(1)} \\
&\geqslant \frac{s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}}{G(1)[\Gamma(\alpha)]^2}.
\end{aligned}
\tag{3.3}
$$

(ii) For $0 \leqslant s < t \leqslant 1$. Following the monotonicity of $G(t)$, $G'(t)$ and $G''(t)$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}K(t,s) &= \frac{G'(t)G(1-s) - G'(t-s)G(1)}{G(1)} \\
&< \frac{G'(t)G(1) - G'(t-s)G(1)}{G(1)} < 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}K(t,s) &= \frac{G''(t)G(1-s) - G''(t-s)G(1)}{G(1)} \\
&> \frac{G''(t)G(1) - G''(t-s)G(1)}{G(1)} \\
&= G''(t) - G''(t-s) > 0,
\end{aligned}
$$

which implies $K(t,s) \geqslant K(1,s) = 0$, and $\frac{\partial}{\partial t}K(t,s)$ is strictly increasing with respect to t on $(s,1]$. Therefore,

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[\frac{K(t,s)}{1-t}\right] &= \frac{(1-t)\frac{\partial}{\partial t}K(t,s) + K(t,s)}{(1-t)^2} \\
&= \frac{\frac{\partial}{\partial t}K(t,s) - \frac{K(1,s)-K(t,s)}{(1-t)}}{(1-t)} \\
&= \frac{\frac{\partial}{\partial t}K(t,s) - \frac{\partial}{\partial t}K(\xi,s)}{(1-t)},
\end{aligned}
$$

where $t < \xi < 1$. By the monotonicity of $\frac{\partial}{\partial t}K(t,s)$, we have $\frac{\partial}{\partial t}\left[\frac{K(t,s)}{1-t}\right] \leqslant 0$. Thus,

$$
\begin{aligned}
\frac{K(t,s)}{1-t} &\leqslant \frac{K(s,s)}{1-s} = \frac{G(1-s)G(s)}{G(1)(1-s)} \leqslant \frac{G(1-s)G(t)}{G(1)(1-s)} \\
&\leqslant \frac{G(1)t^{\alpha-1}(1-s)^{\alpha-1}}{1-s} = G(1)t^{\alpha-1}(1-s)^{\alpha-2},
\end{aligned}
$$

which implies that

$$
K(t,s) \leqslant G(1)(1-t)t^{\alpha-1}(1-s)^{\alpha-2}. \tag{3.4}
$$

On the other hand, by the monotonicity of $G'(t)$, we have

$$
\begin{aligned}
\frac{\partial}{\partial s}K(t,s) &= \frac{G'(t-s)G(1) - G(t)G'(1-s)}{G(1)} \\
&\geqslant \frac{[G(1) - G(t)]G'(1-s)}{G(1)}.
\end{aligned}
\tag{3.5}
$$

Integrate (3.5) with respect to s, we obtain

$$
\begin{aligned}
K(t,s) &\geqslant \int_0^s \frac{[G(1) - G(t)]G'(1-\tau)}{G(1)} d\tau \\
&= \frac{[G(1) - G(t)][G(1) - G(1-s)]}{G(1)}.
\end{aligned}
\tag{3.6}
$$

Since

$$\frac{d}{ds}[(\alpha-1)(1-s)+s^{\alpha-1}] = (\alpha-1)[s^{\alpha-2}-1] \geqslant 0, \quad s \in (0,1],$$

we have

$$1-s^{\alpha-1} \geqslant (\alpha-1)(1-s). \tag{3.7}$$

From (3.1), (3.6) and (3.7), we get

$$\begin{aligned} K(t,s) &\geqslant \frac{[G(1)-t^{\alpha-1}G(1)][G(1)-(1-s)^{\alpha-1}G(1)]}{G(1)} \\ &= G(1)[1-t^{\alpha-1}][1-(1-s)^{\alpha-1}] \\ &\geqslant G(1)(\alpha-1)^2(1-t)s \\ &\geqslant G(1)(\alpha-1)^2(1-t)st^{\alpha-1}(1-s)^{\alpha-1}. \end{aligned} \tag{3.8}$$

Combining (3.2) and (3.4) with $K(t,s) = K(1-s, 1-t)$, we have

$$K(t,s) \leqslant G(1)s(1-s)^{\alpha-1}t^{\alpha-2},$$

which yields (3).

It follows from (3.3) and (3.8) that (4) holds.

□

4. Applications

In this section, we consider the existence and uniqueness of positive solution for FBVP (1.1).

By Theorem 3.1, we have the following lemma:

Lemma 4.1. *The function* $K^*(t,s) =: t^{2-\alpha}K(t,s)$ *satisfies:*

(1) $K^*(t,s) > 0, \quad \forall\, t,s \in (0,1);$

(2) $K^*(t,s) \leqslant G(1)t(1-t)(1-s)^{\alpha-2}, \quad \forall\, t,s \in [0,1];$

(3) $K^*(t,s) \leqslant G(1)s(1-s)^{\alpha-1}, \quad \forall\, t,s \in [0,1];$

(4) $K^*(t,s) \geqslant Ms(1-s)^{\alpha-1}t(1-t), \quad \forall\, t,s \in [0,1].$

For convenience, we list here two more assumptions:

(H_2) $f(t,x) = g(t,x,x)$, where $g : (0,1)\times(0,+\infty)\times(0,+\infty) \to [0,+\infty)$ is continuous, $g(t,x,y)$ is nondecreasing on x, nonincreasing on y, and there exists $\mu \in (0,1)$, such that

$$g(t, rx, \frac{y}{r}) \geqslant r^{\mu}g(t,x,y), \quad \forall x,y > 0, r \in (0,1); \tag{4.1}$$

(H_3)

$$0 < \int_0^1 (1-s)^{\alpha-2}g(s,(1-s)s^{\alpha-1},(1-s)s^{\alpha-1})ds < +\infty.$$

Remark 4.2. Inequality (4.1) is equivalent to

$$g(t, \frac{x}{r}, ry) \leqslant r^{-\mu}g(t,x,y), \quad \forall x,y > 0, \;\; r \in (0,1).$$

Remark 4.3. Condition (H_2) possesses singularity, that is, $f(t,x)$ may be singular at $t = 0, 1$, and $x = 0$.

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0\leqslant t\leqslant 1} |u(t)|$. Define a cone P by

$$P = \left\{ u \in E : \exists\, l_u > 0 \text{ such that } \frac{M\|u\|t(1-t)}{G(1)} \leqslant u(t) \leqslant l_u t(1-t),\ t \in [0,1] \right\}.$$

Let

$$A(u,v)(t) = \int_0^1 K^*(t,s)g(s, s^{\alpha-2}u(s), s^{\alpha-2}v(s))ds.$$

Set $Q = P \setminus \{\theta\}$, where θ is the zero element of E. We have the following lemma.

Lemma 4.4. *Suppose that* (H_1)-(H_3) *hold. Then* $A : Q \times Q \to Q$ *is a mixed monotone operator.*

Proof. The proof is similar to that of [21, Lemma 2.7], so we omit it. □

Theorem 4.5. *Suppose that* (H_1)-(H_3) *hold. Then the BVP* (1.1) *has a unique positive solution in Q.*

Proof. The proof is similar to that of [21, Theorem 3.1], so we omit it. □

Similar to [21, Remark 3.1], we have the following results.

Remark 4.6. *The unique positive solution* y *of* (1.1) *can be approximated by the iterative schemes: for any* $w \in Q$, *choose* $r_0 \in (0,1)$ *small enough such that*

$$r_0^{1-\mu} w \leqslant A(w,w) \leqslant r_0^{-(1-\mu)} w.$$

Set

$$u_0 = r_0 w, \quad v_0 = r_0^{-1} w,$$

and $u_n = A(u_{n-1}, v_{n-1})$, $v_n = A(v_{n-1}, u_{n-1})$ $(n = 1, 2, \cdots)$, *then* $t^{\alpha-2}u_n \to y$.

Example 4.7. Consider the following problem

$$\begin{cases} -D_{0+}^{\frac{3}{2}}u(t) + \frac{1}{5}u(t) = f(t,u(t)), \quad 0 < t < 1, \\ u(0) = 0, u(1) = 0, \end{cases} \tag{4.2}$$

where

$$f(t,x) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{6}}\left[x^{\frac{1}{4}} + x^{-\frac{1}{4}}\right].$$

It is clear that $f(t,x)$ is singular at $t = 0, 1$, and $x = 0$.

Since $\Gamma(\cdot)$ is strictly increasing on $[2, +\infty)$, for any $t \in [0, +\infty)$, we have

$$\begin{aligned} g(t) &= -\frac{1}{2\sqrt{\pi}} + \sum_{k=1}^{+\infty} \frac{t^k}{\Gamma(\frac{3}{2}k - \frac{1}{2})} = -\frac{1}{2\sqrt{\pi}} + t + \sum_{k=2}^{+\infty} \frac{t^k}{\Gamma(\frac{3}{2}k - \frac{1}{2})} \\ &\leqslant -\frac{1}{2\sqrt{\pi}} + t + \sum_{k=2}^{+\infty} \frac{t^k}{\Gamma(k)} = -\frac{1}{2\sqrt{\pi}} + t\left[1 + \sum_{k=1}^{+\infty} \frac{t^k}{k!}\right] \\ &= -\frac{1}{2\sqrt{\pi}} + te^t. \end{aligned}$$

Since $\frac{1}{2\sqrt{\pi}} \approx 0.282$, $\frac{1}{5}e^{\frac{1}{5}} \approx 0.243$, we have $g(\frac{1}{5}) < 0$. Therefore $\frac{1}{5} < b^*$, which implies that (H_1) holds.

Denote

$$g(t,x,y) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{6}}\left[x^{\frac{1}{4}} + y^{-\frac{1}{4}}\right].$$

It is easy to check that (H_2) holds. Through direct calculation, we have

$$\int_0^1 (1-s)^{-\frac{1}{2}} g\left(s, (1-s)s^{\frac{1}{2}}, (1-s)s^{\frac{1}{2}}\right) ds = B(\frac{5}{8}, \frac{7}{12}) + B(\frac{3}{8}, \frac{1}{12}) = \frac{\Gamma(\frac{5}{8})\Gamma(\frac{7}{12})}{\Gamma(\frac{29}{24})} + \frac{\Gamma(\frac{3}{8})\Gamma(\frac{1}{12})}{\Gamma(\frac{11}{24})},$$

which implies (H_3) holds.

Therefore all the assumptions of Theorem 4.5 are satisfied, which implies that BVP (4.2) has a unique positive solution.

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